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NETWORK RESILIENCY

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OVERVIEW

- Common understanding of “network resiliency”
- Mathematical model of network resiliency
  - Delivery functions
  - Delivery importance
- Network resiliency
- Network interdiction
- Conclusion
- Next Steps

WHAT DOES "NETWORK RESILIENCY" MEAN?

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- Network continues to function "adequately" despite possible disruptions in infrastructure
  - Networks are not ends in themselves
  - They exist to perform services or functions
    - Telecom
    - Oil, gas, water, electricity distribution
    - Logistics
- Network returns "quickly" to full operational condition after disruptions in infrastructure

WHAT DOES "NETWORK RESILIENCY" MEAN?

- We will consider the first criterion in detail
- Key concepts:
  - Delivery function
  - Delivery importance

RESILIENCY AND RELIABILITY

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- Resiliency applies to the functions or services performed by the network
- Reliability often applies to network elements
  - But has been used for connectivity, etc.
- OTBE, a network with more reliable elements should be more resilient

RESILIENCY AND SURVIVABILITY

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- Survivability = provision of enough geographically diverse alternate routes so that potentially lost traffic may be carried
- Survivability studies rarely incorporate *how much* traffic is successful
- Explicit quantitative consideration in resiliency studies

FACTORS BEARING ON NETWORK RESILIENCY

- Network graph
- Link and node capacities
  - Link and node reliabilities
- Routing rules
- Protocols
  - Customer class structure

NETWORK RESILIENCY
MATHEMATICAL MODEL

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- Networks are not ends in themselves
- They exist to perform certain functions or deliver certain services
  - Oil, gas, electricity transport
  - Logistics
    - Dedicated
    - Public
  - Telecommunications
  - Advertising

NETWORK RESILIENCY
MATHEMATICAL MODEL

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- Formalize the notion of function or service provided by a network using the <u>delivery function</u>
- Examples of delivery functions
  - Volume of oil delivered from/to specified terminals during a specified time period
  - Telecommunications service reliability for a particular set of origins and destinations

DELIVERY FUNCTION

- Network with associated delivery function $(\mathcal{H}, \Psi)$
- $\mathcal{H} = (\mathcal{N}, \mathcal{L})$
  - $|\mathcal{N}| = N$
- $\Psi : \mathcal{H} \rightarrow \mathbf{R}$
  - Can also consider vector-valued delivery functions

DELIVERY IMPORTANCE
NETWORK CLASSES

- Deterministic/stochastic
  - Static/dynamic
- Continuum/discrete
- Capacitated/uncapacitated
  - Capacity can be understood broadly
    - Capacity in the usual flow network sense
    - Presence or absence of network element(s)
    - Network element reliability

DELIVERY IMPORTANCE
MAIN IDEA

- Compare $\Psi(C_0 + hA)$ to $\Psi(C_0)$
- C_0 is a nominal capacity matrix
  - So delivery importance may change depending on where you measure it
- A is a direction of increment
- $h > 0$

DELIVERY IMPORTANCE
SINGLE NETWORK ELEMENT

- $\Psi\left(c_{ij} + ha\right) - \Psi\left(c_{ij}\right) = \left.\frac{\partial \Psi}{\partial c_{ij}}\right|_{C_0} ha + o(h), \;\; h \geq 0$
- $a = \pm 1$
- $o(h)/h \to 0$ as $h \to 0^+$ if Ψ is differentiable at c_{ij}
- Define $\Omega_a(i, j; C_0) = a \left.\frac{\partial \Psi}{\partial c_{ij}}\right|_{C_0}$

EXAMPLE

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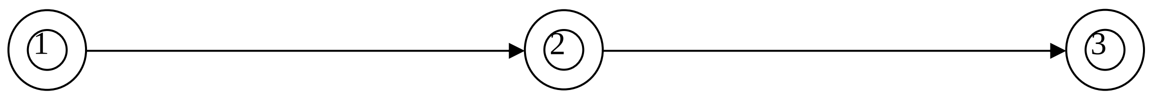

$$C = \begin{pmatrix} \infty & x & 0 \\ 0 & \infty & y \\ 0 & 0 & \infty \end{pmatrix}$$

$$\Psi(C) = \min\{x, y\} = ½(|x + y| - |x - y|),\ x, y \geq 0$$

EXAMPLE

- For $x \neq y$,

  $D_1\Psi(x, y) = I\{x < y\}$ and $D_2\Psi(x, y) = I\{x > y\}$

- $\forall C$, $\Omega_1(1, 2; C) = 1$ if $x < y$ and 0 if $x > y$
- $\forall C$, $\Omega_1(2, 3; C) = 0$ if $x < y$ and 1 if $x > y$

EXAMPLE

- When $x = y$, the derivatives do not exist
- Define $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$
- $\frac{1}{h}\left[\Psi\left(C_0 + hA\right) - \Psi\left(C_0\right)\right] = \frac{1}{h}\left[\min\left\{x + h, x\right\} - x\right] = 0$
- Same is true for link (2, 3)

EXAMPLE

- When the two links have the same capacity, they are of equal delivery importance
  - Not true in general
- If the initial link capacities are equal, making one of them larger has no effect on the delivery function
  - Not true if make one of them smaller

DELIVERY IMPORTANCE
LARGER SUBSETS

- $\mathcal{M} \subset \mathcal{N} \cup \mathcal{L}$
- $\mathbf{1}_{\mathcal{M}}$ = matrix of $I\{(i, j) \in \mathcal{M}\}$
- $\Omega_{+}(\mathcal{M}, C) = \langle \Psi'(C), \mathbf{1}_{\mathcal{M}} \rangle$
  - Need to write the matrices as long row vectors
  - $\Psi'(C) = \left(\frac{\partial \Psi}{\partial c_{11}}, \ldots, \frac{\partial \Psi}{\partial c_{1N}}, \frac{\partial \Psi}{\partial c_{21}}, \ldots, \frac{\partial \Psi}{\partial c_{2N}}, \frac{\partial \Psi}{\partial c_{31}}, \ldots, \frac{\partial \Psi}{\partial c_{NN}} \right)$
- $\Omega_{-}(\mathcal{M}, C) = \langle \Psi'(C), -\mathbf{1}_{\mathcal{M}} \rangle$

DELIVERY IMPORTANCE
LARGER SUBSETS

- From the inner product representation,

$$\Omega_{+}\left(\mathcal{M};C_{0}\right)=\sum_{(i,j)\in\mathcal{M}}\left.\frac{\partial\Psi}{\partial c_{ij}}\right|_{C_0}$$

- Eases computation of network resiliency

EXAMPLE
CONTINUED

- $\mathcal{M} = \{(1, 2), (2,3)\}$
- $\Omega_+(\mathcal{M};C_0) = I\{x < y\} + I\{y < x\} = I\{x \neq y\}$
- When $x = y$,

$$\Omega_+(\mathcal{M};C_0) = \lim_{h\to 0^+} \frac{1}{h}\left[\Psi\left(C_0 + h\mathbf{1}_{\mathcal{M}}\right) - \Psi\left(C_0\right)\right]$$

$$= \lim_{h\to 0^+} \frac{1}{h}\left[\min\{x+h, x+h\} - x\right] = 1.$$

DISCRETE NETWORKS

- $\Omega_+\left(i, j; C\right) = \Psi(C + \mathbf{1}_{ij}) - \Psi(C)$

- $\Omega_+\left(\mathcal{M}; C\right) = \Psi(C + \mathbf{1}_{\mathcal{M}}) - \Psi(C)$

STOCHASTIC NETWORKS

- C is a random matrix
  - Static
  - Dynamic (function of time)
- $\Omega_+(\mathcal{M}, C)$ is a random variable
- $E\Omega_+\left(\mathcal{M};C\right) = E\left\langle \Psi'(C), \mathbf{1}_{\mathcal{M}} \right\rangle = \int \left\langle \Psi'(C(\omega)), \mathbf{1}_{\mathcal{M}} \right\rangle P(d\omega)$

UNCAPACITATED NETWORKS

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- Presence or absence of network element(s) affects delivery function
- Deterministic
  - Assign "capacities" 0 or 1 to each network element
    - + Only certain delivery functions are possible
  - Treat as a discrete capacitated network
- Stochastic, static
  - Assign $p_{ij} = P\{(i, j) \in \mathcal{H}\}$
  - Treat as capacitated continuum network

UNCAPACITATED NETWORKS

- Stochastic, dynamic
  - Assign $p_{ij}(t) = P\{(i, j) \in \mathcal{H} \text{ at time } t\}$
  - Treat as dynamic stochastic capacitated network

NETWORK RESILIENCY
DETERMINISTIC NETWORKS

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- Proportion of subsets of the network whose absolute delivery importance values in the negative direction do not exceed a given value
- $\rho\left(\mathcal{H}, C_0; x\right) = \frac{1}{2^{|\mathcal{N} \cup \mathcal{L}|}} \sum_{\mathcal{M} \subset \mathcal{N} \cup \mathcal{L}} I\left\{\left|\Omega_{-}\left(\mathcal{M}; C_0\right)\right| \le x\right\}, \quad x \ge 0$

NETWORK RESILIENCY
DETERMINISTIC NETWORKS

- Network resiliency, considering only some subset $\mathcal{Z}$ of $\mathcal{N} \cup \mathcal{L}$:

$$\rho\left(\mathcal{H}, \mathcal{Z}, C_0; x\right) = \frac{1}{|\mathcal{Z}|} \sum_{\mathcal{M} \subset \mathcal{Z}} I\left\{\left|\Omega_-\left(\mathcal{M}; C_0\right)\right| \le x\right\}, \quad x \ge 0$$

- Network resiliency considering only *k*-element subsets of $\mathcal{N} \cup \mathcal{L}$:

$$\rho_k\left(\mathcal{H}, C_0; x\right) = \binom{|\mathcal{N} \cup \mathcal{L}|}{k}^{-1} \sum_{\mathcal{M} \subset \mathcal{Z}_k} I\left\{\left|\Omega_-\left(\mathcal{M}; C_0\right)\right| \le x\right\}, \quad x \ge 0$$

NETWORK RESILIENCY
STOCHASTIC NETWORKS

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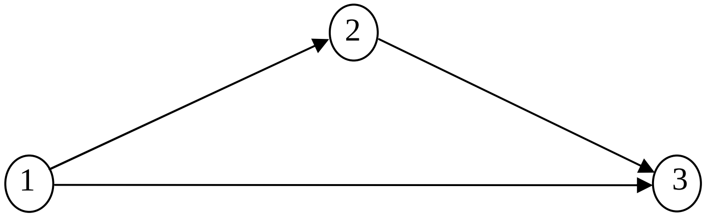
- $\Omega\_(\mathcal{M}, C)$ is a random variable
- $$\mathrm{E}\rho\left(\mathcal{H}, C_0; x\right) = \frac{1}{2^{\left|\mathcal{N} \cup \mathcal{L}\right|}} \sum_{\mathcal{M} \subset \mathcal{N} \cup \mathcal{L}} P\left\{\left|\Omega_{-}\left(\mathcal{M}; C_0\right)\right| \le x\right\}, \quad x \ge 0$$

EXAMPLE

CONNECTIVITY



$$H = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Psi = I\left\{\left\{(1,3) \in \mathcal{H}\right\} \cup \left[\left\{(1,2) \in \mathcal{H}\right\} \cap \left\{(2,3) \in \mathcal{H}\right\}\right]\right\} = c_{13} + c_{12}c_{23} - c_{12}c_{23}c_{13}$$

EXAMPLE
CONNECTIVITY

SUBSET OF $\mathcal{H}$	DELIVERY IMPORTANCE
$\{(1, 3)\}$	$c_{12}c_{23}c_{13} - c_{13}$
$\{(2, 3)\}$	$c_{12}c_{23}c_{13} - c_{12}c_{23}$
$\{(1, 2)\}$	$c_{12}c_{23}c_{13} - c_{12}c_{23}$
$\{(1, 2) \cup (2, 3)\}$	$c_{12}c_{23}c_{13} - c_{12}c_{23}$
$\{(1, 3) \cup (2, 3)\}$	$c_{12}c_{23}c_{13} - c_{12}c_{23} - c_{13} = -\Psi(\mathcal{H})$
$\{(1, 3) \cup (1, 2)\}$	$c_{12}c_{23}c_{13} - c_{12}c_{23} - c_{13}$
$\mathcal{H}$	$c_{12}c_{23}c_{13} - c_{12}c_{23} - c_{13}$

Network resiliency (at nominal capacity H) is a right-continuous step function with a jump at 0 of height 5/8 and a jump at 1 of height 3/8.

EXAMPLE
RELIABILITY

SUBSET OF $\mathcal{H}$	DELIVERY IMPORTANCE	EXPECTED DELIVERY IMPORTANCE
{(1, 3)}	$c_{12}c_{23}c_{13} - c_{13}$	$p_{12}p_{23}p_{13} - p_{13}$
{(2, 3)}	$c_{12}c_{23}c_{13} - c_{12}c_{23}$	$p_{12}p_{23}p_{13} - p_{12}p_{23}$
{(1, 2)}	$c_{12}c_{23}c_{13} - c_{12}c_{23}$	$p_{12}p_{23}p_{13} - p_{12}p_{23}$
{(1, 2) ∪ (2, 3)}	$c_{12}c_{23}c_{13} - c_{12}c_{23}$	$p_{12}p_{23}p_{13} - p_{12}p_{23}$
{(1, 3) ∪ (2, 3)}	$c_{12}c_{23}c_{13} - c_{12}c_{23} - c_{13} = -\Psi(\mathcal{H})$	$p_{12}p_{23}p_{13} - p_{12}p_{23} - p_{13}$
{(1, 3) ∪ (1, 2)}	$c_{12}c_{23}c_{13} - c_{12}c_{23} - c_{13}$	$p_{12}p_{23}p_{13} - p_{12}p_{23} - p_{13}$
$\mathcal{H}$	$c_{12}c_{23}c_{13} - c_{12}c_{23} - c_{13}$	$p_{12}p_{23}p_{13} - p_{12}p_{23} - p_{13}$

Expected network resiliency is $p_\infty - 1$

EXAMPLE
STOCHASTIC FLOW

- Oil delivery network
- Delivery function will be volume of oil sent from node A to node B during $[0, T]$
- Link and node capacities are continuous time Gaussian processes $\{X_{ij}(t) : t \geq 0\}$
- Capacity is the maximum volume of oil per unit time in each network element

EXAMPLE
STOCHASTIC FLOW

- $X(t)$ = matrix of the $X_{ij}(t)$
- $C_0 = X(0)$
- Assume oil flow is a max flow
- $\varphi_{AB}(t)$ = flow from A to B at time t
- Delivery function is $\Psi(X;T) = \int_0^T \varphi_{AB}(t)\,dt$

EXAMPLE
STOCHASTIC FLOW

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- Delivery importance of $\mathcal{M}$ is

$$\Omega_{-}(\mathcal{M}, X) = -\sum_{(i,j)\in\mathcal{M}} \int_0^T \frac{\partial \varphi_{AB}}{\partial x_{ij}}(t)\, dt$$

- If φ_{AB} is nondecreasing as a function of element capacities, then scalar network resiliency is

$$\rho^*(\mathcal{H}; X) = \max_{\mathcal{M}\subset\mathcal{N}\cup\mathcal{L}} \sum_{(i,j)\in\mathcal{M}} \int_0^T \frac{\partial \varphi_{AB}}{\partial x_{ij}}(t)\, dt$$

EXAMPLE
STOCHASTIC FLOW

- If processes $X_{ij}(t)$ are mutually independent, then

$$P\{\Psi(X;T) > V\} = \int_0^\infty \cdots \int_0^\infty P\left\{\int_0^T \varphi_{AB}(t)\,dt > V \,\middle|\, X_{ij}(t) = x_{ij}\right\} dx_{ij}$$

- The methods of Ramirez-Marquez *et al.* may be used to obtain the conditional probability

NETWORK INTERDICTION

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- How to make a network *less* resilient
- Find the network element(s) whose removal maximally disrupts the network's delivery function
- If $\mathcal{M}_0 \subset \mathcal{N} \cup \mathcal{L}$ realizes the scalar network resiliency, $\rho^*(\mathcal{H};C) = \left|\Omega_-(\mathcal{M}_0;C)\right|$, then this is the desired set

CONCLUSION

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- Far from the last word on this subject
- More practical applications needed
  - Guide future development of the theory
- Need to solidify design for resiliency principles
- Need to formulate and solve network resiliency optimization problem

NEXT STEPS

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- Evaluate resiliency of some real networks
- Incorporate current reliability of network elements
- Incorporate higher-layer variables
- Formulate Pontryagin control problem based on delivery importance variables