On Weighted Graphs Yielding Facets of the Linear Ordering Polytope

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Definition
For any finite set $Z$,

▶ for $R \subseteq Z \times Z$, the vector $x^R$ is the characteristic vector of $R$, that is,

$$x^R_{i,j} = \begin{cases} 
1 & \text{if } (i,j) \in R \\
0 & \text{otherwise}
\end{cases}$$

▶ the linear ordering polytope $P^Z_{LO} \subset \mathbb{R}^{Z \times Z}$ is

$$P^Z_{LO} = \text{conv}\{x^L : L \text{ linear order on } Z\}$$
**Definition**

For a vertex-weighted graph \((G, \mu)\) and \(S \subseteq V(G)\),

- \(\mu(S) := \sum_{v \in S} \mu(v)\) \hspace{1cm} \text{(weight of } S\text{)}
- \(w(S) := \mu(S) - |E(G[S])|\) \hspace{1cm} \text{(worth of } S\text{)}
- \(\alpha(G, \mu) := \max_{S \subseteq V(G)} w(S)\)
- \(S\) is tight if \(w(S) = \alpha(G, \mu)\)
Suppose

- \((G, \mu)\) is any weighted graph
- \(Y\) is a set s.t. \(|Y| = |V(G)|\) and \(Y \cap V(G) = \emptyset\)
- \(f : V(G) \to Y\) is a bijection
- \(Z\) is a finite set s.t. \(V(G) \cup Y \subseteq Z\)
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Definition
- The graphical inequality of $(G, \mu)$, which is valid for $P_{LO}^Z$, is
  \[ \sum_{v \in V(G)} \mu(v) \cdot x_{v, f(v)} - \sum_{\{v, w\} \in E(G)} (x_{v, f(w)} + x_{f(v), w}) \leq \alpha(G, \mu) \]
- $(G, \mu)$ is facet-defining if its graphical inequality defines a facet of $P_{LO}^Z$
Suppose

- \((G, \mu)\) is any weighted graph
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**Definition**

- The **graphical inequality** of \((G, \mu)\), which is valid for \(P^Z_{LO}\), is
  \[
  \sum_{v \in V(G)} \mu(v) \cdot x_{v, f(v)} - \sum_{\{v, w\} \in E(G)} (x_{v, f(w)} + x_{f(v), w}) \leq \alpha(G, \mu)
  \]

- \((G, \mu)\) is **facet-defining** if its graphical inequality defines a facet of \(P^Z_{LO}\)

N.B. \((G, \mu)\) being facet-defining is a property of the graph solely, i.e. it is independent of the particular choice of \(Y, f\) and \(Z\)
A characterization of facet-defining graphs

Definition

► For any tight set $T$ of $(G, \mu)$, a corresponding affine equation is defined:

$$\sum_{v \in T} y_v + \sum_{e \in E(T)} y_e = \alpha(G, \mu)$$

► The system of $(G, \mu)$ is obtained by putting all these equations together
A characterization of facet-defining graphs

Definition

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- The system of $(G, \mu)$ is obtained by putting all these equations together

Theorem (Christophe, Doignon and Fiorini, 2004)

$(G, \mu)$ is facet-defining $\iff$ the system of $(G, \mu)$ has a unique solution

- Basically rephrases the fact that the dimension of the face of $P_{LO}^Z$ defined by the graphical inequality must be high enough
- We lack a ‘good characterization’ of these graphs...
A few results
(assuming from now on that all graphs have at least 3 vertices)

Definition
\( G \) is **stability critical** if \( G \) has no isolated vertex and
\[ \alpha(G \setminus e) > \alpha(G) \]
for all \( e \in E(G) \)

Theorem (Koppen, 1995)
\((G, \mathbb{1})\) is facet-defining \( \iff \) \( G \) is connected and stability critical
A few results
(assuming from now on that all graphs have at least 3 vertices)

Definition
G is stability critical if G has no isolated vertex and
\( \alpha(G \setminus e) > \alpha(G) \) for all \( e \in E(G) \)

Theorem (Koppen, 1995)
\((G, 1)\) is facet-defining \( \iff \) G is connected and stability critical

Theorem (Christophe, Doignon and Fiorini, 2004)
\((G, \mu)\) is facet-defining \( \iff \) its 'mirror image' \((G, \text{deg} - \mu)\) is
facet-defining
Definition

- The defect of $G$ is $|V(G)| - 2\alpha(G)$

![Diagram](image.png)

a stability critical graph

$|V(G)| = 12$

$\alpha(G) = 3$

$\rightarrow$ defect $= 6$
Definition

- The defect of $G$ is $|V(G)| - 2\alpha(G)$
- The defect of $(G, \mu)$ is $\mu(V(G)) - 2\alpha(G, \mu)$

A stability critical graph

$|V(G)| = 12$
$\alpha(G) = 3$
$\rightarrow$ defect $= 6$

A facet-defining graph

$\mu(V(G)) = 7$
$\alpha(G, \mu) = 2$
$\rightarrow$ defect $= 3$
Theorem

- The defect $\delta$ of a connected stability critical graph $G$ is always positive \cite{erdos1961}.
- Moreover, $\delta \geq \deg(v) - 1$ for all $v \in V(G)$ \cite{hajnal1965}.
Theorem

- The defect $\delta$ of a connected stability critical graph $G$ is always positive (Erdős and Gallai, 1961)
- Moreover, $\delta \geq \deg(v) - 1$ for all $v \in V(G)$ (Hajnal, 1965)

Theorem (Doignon, Fiorini, J.)

- The defect $\delta$ of any facet-defining graph $(G, \mu)$ is positive
- $(G, \mu)$ and $(G, \deg - \mu)$ have the same defect
- For all $v \in V(G)$, we have
  \[ \delta \geq \deg(v) - \mu(v) \geq 1 \]

  and, because of the mirror image, also
  \[ \delta \geq \mu(v) \geq 1 \]
Odd subdivision

Here is an extension of a classical operation on stability-critical graphs:

Odd subdivision $\rightarrow$ inverse of odd subdivision

Theorem (Christophe, Doignon and Fiorini, 2004)

*The odd subdivision operation and its inverse keep both a graph facet-defining. Moreover, the defect does not change*
Lemma

An inclusionwise minimal cutset of a facet-defining graph cannot span "○" or "○—○"

Thus when we have \( \begin{array}{c}
\text{1} \quad \text{1}
\end{array} \) we can always contract both edges by using the inverse of odd subdivision operation
Lemma
An inclusionwise minimal cutset of a facet-defining graph cannot span "○" or "○---○"

Thus when we have \(\xrightarrow{1-1}\) we can always contract both edges by using the inverse of odd subdivision operation

Definition
A facet-defining graph is minimal if no two adjacent vertices have degree 2
Theorem (Lovász, 1978)

For every positive integer $\delta$, the set $S_\delta$ of minimal connected stability critical graphs with defect $\delta$ is finite.
Classification of stability critical graphs

Theorem (Lovász, 1978)

For every positive integer $\delta$, the set $S_\delta$ of minimal connected stability critical graphs with defect $\delta$ is finite

Research problem

Is there a finite number of minimal facet-defining graphs with defect $\delta$, for every $\delta \geq 1$?

- It turns out to be true for $\delta \leq 3$
  → an overview of the proofs is given in the next few slides
- The problem is wide open for $\delta \geq 4$
Notice first that the only minimal facet-defining graph with defect $\delta = 1$ is $\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}$, because $\delta \geq \mu(\nu) \geq 1$
Notice first that the only minimal facet-defining graph with defect \( \delta = 1 \) is \( \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ 
\end{array} \), because \( \delta \geq \mu(v) \geq 1 \).

Let’s look at another operation:

\[
\begin{array}{c}
1 \\
2 \\
1 \\
1 \\
1
\end{array} \quad \text{subdivision of a star} \quad \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
1
\end{array}
\]

**Theorem**

The subdivision of a star operation keeps a graph facet-defining. Moreover, the defect does not change.
Definition

\((G_1, \mu_1)\) and \((G_2, \mu_2)\) are equivalent if one can be obtained from the other by using the

- odd subdivision
- inverse of odd subdivision
- subdivision of a star

operations finitely many times.
Definition

$(G_1, \mu_1)$ and $(G_2, \mu_2)$ are equivalent if one can be obtained from the other by using the

- odd subdivision
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- subdivision of a star

operations finitely many times.

Notice

- two equivalent graphs have the same defect
- $(G, \mu)$ and $(G, \text{deg} - \mu)$ are equivalent:
Facet-defining graphs with defect 2

Recall

\[
\begin{align*}
\delta & \geq \mu(v) \geq 1 \\
\delta & \geq \deg(v) - \mu(v) \geq 1
\end{align*}
\]

for any vertex \( v \) of a facet-defining graph with defect \( \delta \)

\( \Rightarrow \) \( \deg(v) \leq 2\delta \)
Facet-defining graphs with defect 2

Recall

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for any vertex \( v \) of a facet-defining graph with defect \( \delta \)

\( \Rightarrow \deg(v) \leq 2\delta \)

**Theorem**

\( \deg(v) \leq 2\delta - 1 \) for any vertex \( v \) of a facet-defining graph with defect \( \delta \geq 2 \)
Facet-defining graphs with defect 2

Recall

\[
\begin{align*}
\delta & \geq \mu(v) \geq 1 \\
\delta & \geq \deg(v) - \mu(v) \geq 1
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\]

for any vertex \( v \) of a facet-defining graph with defect \( \delta \)

\( \Rightarrow \) \( \deg(v) \leq 2\delta \)

**Theorem**

\( \deg(v) \leq 2\delta - 1 \) for any vertex \( v \) of a facet-defining graph with defect \( \delta \geq 2 \)

Thus, every vertex of a facet-defining graph with defect 2 is either

\[ \begin{array}{c}
1 \\
\end{array} \quad \text{or} \quad \begin{array}{c}
1 \\
\end{array} \quad \text{or} \quad \begin{array}{c}
2
\end{array} \]

\( \Rightarrow \) Any facet-defining graph with defect 2 is equivalent to some stability critical graph
Theorem (Andrásfai, 1967)

The only minimal connected stability critical graph with defect 2 is

\[
\begin{array}{c}
\text{Diagram here}
\end{array}
\]
Theorem (Andrásfai, 1967)

The only minimal connected stability critical graph with defect 2 is

\[ \xymatrix{ & 1 \\
1 & 1 & 1 }
\]

→ we derive:

**Theorem**

There are exactly five minimal facet-defining graphs with defect 2:

\[ \begin{align*}
\text{Graph 1} &: \xymatrix{ & 1 \\
1 & 1 & 1 }, \\
\text{Graph 2} &: \xymatrix{ & 2 \\
2 & 2 & 2 }, \\
\text{Graph 3} &: \xymatrix{ & 1 \\
1 & 1 & 1 \\
1 & 2 & 1 }, \\
\text{Graph 4} &: \xymatrix{ & 2 \\
2 & 1 & 1 \\
1 & 1 & 1 }, \\
\text{Graph 5} &: \xymatrix{ & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 }.
\end{align*} \]
Facet-defining graphs with defect 3

By previous bounds, any vertex falls in one of these cases when \( \delta = 3 \):

![Facet-defining graphs](image-url)
Facet-defining graphs with defect 3

By previous bounds, any vertex falls in one of these cases when $\delta = 3$:

```
  1   1   1   2   2   2   3   3
```

The subdivision of a star operation is no longer sufficient!
Facet-defining graphs with defect 3

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The subdivision of a star operation is no longer sufficient!

Definition

a $(p, q)$-vertex is a vertex with weight $p$ and degree $q$
Facet-defining graphs with defect 3

By previous bounds, any vertex falls in one of these cases when $\delta = 3$:

![Diagram of vertex cases with defect 3]

The subdivision of a star operation is no longer sufficient!

**Definition**

A $(p, q)$-vertex is a vertex with weight $p$ and degree $q$.

Fix $(G, \mu)$ to be any facet-defining graph with defect 3.
Facet-defining graphs with defect 3

By previous bounds, any vertex falls in one of these cases when \( \delta = 3 \):

- \( \star \)
- \( \star \)
- \( \star \)

The subdivision of a star operation is no longer sufficient!

**Definition**

A \((p, q)\)-vertex is a vertex with weight \( p \) and degree \( q \)

Fix \((G, \mu)\) to be any facet-defining graph with defect 3

- We would like to show that the number of vertices \( v \) of \((G, \mu)\) with \( \deg(v) \geq 3 \) is bounded by some absolute constant
Facet-defining graphs with defect 3

By previous bounds, any vertex falls in one of these cases when \( \delta = 3 \):

![Vertex Cases](image)

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**Definition**

A \((p, q)\)-vertex is a vertex with weight \( p \) and degree \( q \)

Fix \((G, \mu)\) to be any facet-defining graph with defect 3

- We would like to show that the number of vertices \( \nu \) of \((G, \mu)\) with \( \text{deg}(\nu) \geq 3 \) is bounded by some absolute constant
- By the subdivision of a star operation, w.l.o.g. \( \not\exists \) \((2, 3)\)-, \((3, 4)\)-, or \((3, 5)\)-vertices in \((G, \mu)\)
Facet-defining graphs with defect 3

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- By the subdivision of a star operation, w.l.o.g. \( \not\exists (2, 3)\)-, \((3, 4)\)-, or \((3, 5)\)-vertices in \((G, \mu)\)
- Main issue: how to get rid of the \((2, 4)\)-vertices and \((2, 5)\)-vertices?
Suppose \( v \) is a (2, 4)- or (2, 5)-vertex and look at those tight sets including exactly two neighbors of \( v \) but avoiding \( v \):
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\[ \rightarrow \text{defines a graph on the neighborhood } N(v) \text{ of } v, \text{ denoted } H_v: \]
Expanding a vertex

Assume \( \exists a, b, c, d \in V(H_v) \) s. t. \( \{a, b\} \in E(H_v) \) and \( \{c, d\} \notin E(H_v) \)

Lemma

- Expanding \( v \) keeps \( (G, \mu) \) facet-defining and does not change the defect
- Any \((2, 5)\)-vertex of \( (G, \mu) \) is expandable
Expanding a vertex

Assume \( \exists a, b, c, d \in V(H_v) \) s. t. \( \{a, b\} \in E(H_v) \) and \( \{c, d\} \not\in E(H_v) \)

Lemma

- Expanding \( v \) keeps \( (G, \mu) \) facet-defining and does not change the defect
- Any \((2, 5)\)-vertex of \( (G, \mu) \) is expandable

\( \rightarrow \) w.l.o.g. \( (G, \mu) \) has no expandable vertices, as expanding a vertex increases the number of vertices with degree at least 3
Splitting a vertex

Suppose that $v$ is a $(2, 4)$-vertex and that $\{a, b\}, \{c, d\} \not\in E(H_v)$

Lemma

- Splitting $v$ keeps $(G, \mu)$ facet-defining and does not change the defect
- Every nonexpandable $(2, 4)$-vertex is splittable
Assume now that $v$ is a nonexpandable $(2, 4)$-vertex. As $v$ is splittable, $H_v$ is isomorphic to one of these 3 graphs:
Assume now that \( v \) is a nonexpandable \((2, 4)\)-vertex. As \( v \) is splittable, \( H_v \) is isomorphic to one of these 3 graphs:

- \( v \) is "thin"
- \( v \) is "thick"

**Lemma**

- \( v \) must be thin or thick, i.e. \( H_v \) cannot be isomorphic to the leftmost graph
- \((G, \mu)\) has at most 5 thick vertices
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- \((G, \mu)\) has at most 5 thick vertices

\[ \rightarrow \] it remains to show that \((G, \mu)\) has not too many thin vertices...
Key lemma

$(G, \mu)$ has at most $\frac{3}{2}N$ thin vertices, where $N$ is the number of vertices with weight 1 and degree at least 3
Key lemma

\((G, \mu)\) has at most \(\frac{3}{2}N\) thin vertices, where \(N\) is the number of vertices with weight 1 and degree at least 3

- Iteratively split every vertex of \((G, \mu)\) which is thin or thick until there are no more left
Key lemma

\((G, \mu)\) has at most \(\frac{3}{2}N\) thin vertices, where \(N\) is the number of vertices with weight 1 and degree at least 3

- Iteratively split every vertex of \((G, \mu)\) which is thin or thick until there are no more left
- The resulting graph is a connected stability graph with defect 3, with exactly \(N\) vertices of degree at least 3
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- So, the number of vertices with degree at least 3 in $(G, \mu)$ is at most $N + \frac{3}{2}N + 5 = \frac{5}{2}N + 5 \leq \frac{5}{2}c + 5$
Key lemma

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- So, the number of vertices with degree at least 3 in \((G, \mu)\) is at most \(N + \frac{3}{2}N + 5 = \frac{5}{2}N + 5 \leq \frac{5}{2}c + 5\)

Thus we obtain:

**Theorem**

*There is a finite number of minimal facet-defining graphs with defect 3*
As a (brief) conclusion

Graphical inequalities for the linear ordering polytope give rise to a new family of weighted graphs with interesting structural properties.
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Determining if the set of minimal facet-defining graphs with defect $\delta$ is finite remains an open problem for $\delta \geq 4$. 
As a (brief) conclusion

Graphical inequalities for the linear ordering polytope give rise to a new family of weighted graphs with interesting structural properties.

Determining if the set of minimal facet-defining graphs with defect $\delta$ is finite remains an open problem for $\delta \geq 4$.

Thank you!