Vertices and Facets of the Semiorder Polytope

Examples and Preliminary Results

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DIMACS Workshop on the Polyhedral Combinatorics of Random Utility
May 24, 2006

Work supported, in part, by a Prix d’Ouverture Internationale from the Universite Libre de Bruxelles

Outline

Strict Partial Ordering

A (strict) partial ordering P on a set X is a binary relation which satisfies

- Irreflexivity: $x \not\prec x$
- Asymmetry: If $x \prec y$ then $y \not\prec x$
- Transitivity: If $x \prec y$ and $y \prec z$, then $x \prec z$

If $x \not\prec y$ and $y \not\prec x$, we write $x \parallel y$

Interval Orders and Interval Representations

We seek an easier way of representing our poset than a list of comparabilities between elements.

An *Interval Representation* of a poset (X, P) assigns to each element $x \in X$ an interval I_x.

$a \prec b$ if $I_a \cap I_b = \emptyset$ and I_a is to the left of I_b.

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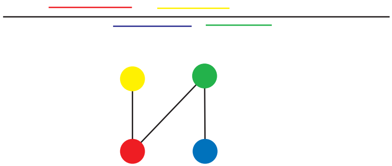

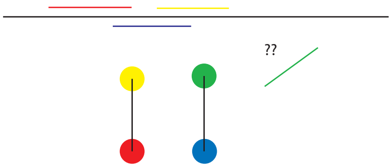

Fishburn and Mirkin Theorem

Any Interval Order may not contain a $\underline{2} + \underline{2}$,

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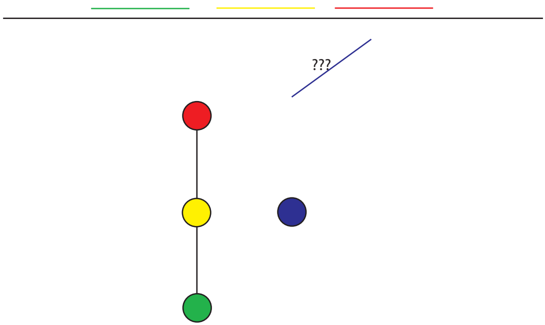
A *unit interval representation* is an interval representation in which all intervals have the same length.

A *unit interval order* (or *semiorder*) is a poset that has a unit interval representation.



Scott-Suppes Theorem

No unit interval order can contain a $\underline{3} + \underline{1}$ suborder

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Analytical Motivation-Interval Orders

We can think of an interval order as a pair of functions, $f, g : X \to \mathbb{R}$

For all $x \in X$, $f(x) \leq g(x)$

$x \prec y$ iff $g(x) < f(y)$

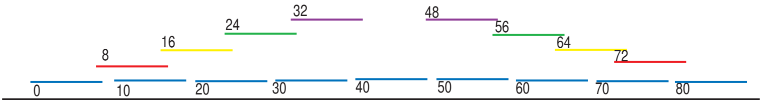
Analytical Motivation-Semiorders

For a semiorder, we need only one function, $f : X \to \mathbb{R}$, together with a predetermined interval length r.

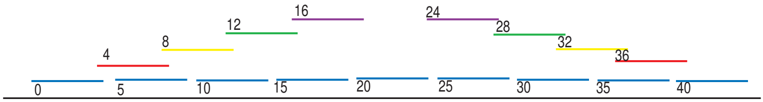
$x \prec y$ iff $f(x) + r < f(y)$.

A Semiorder

Making The Scale-Numerical Representations



We Can Do Better



How Good Can We Do?

Is there a minimal representation that preserves the semiorder relations?

What differentiation relations does this minimal representation preserve? Is everything ‘fixed’ on the scale?

Is the scale the same for all of these ‘minimal’ representations?

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Special Notation and Rules for Semiorders

Let (X, R) be a semiorder, and let $x, y \in X$.

- (Preference) We say that xPy if $x > y$ in (X, R)
- (Incomparability) We say that xIy if $x \parallel y$ in (X, R) (neither $x > y$ nor $y > x$).
- (Trace) We say that xTy if $f(x) > f(y)$

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Other Assumptions

- Any two elements must be represented with distinct intervals
- Every element is incomparable with its predecessor in the trace.
- Any two comparable elements are separated by at least one unit.

Inequalities

- If xPy then $f(x) \geq f(y) + r + 1$.
- If xIy then $|f(x) - f(y)| \leq r$
- $r > 0$ and $f(x_0) = 0$, where x_0 is our minimum element.

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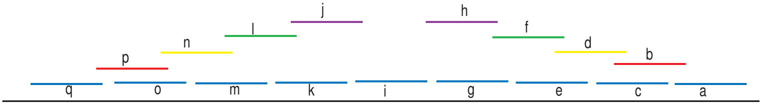
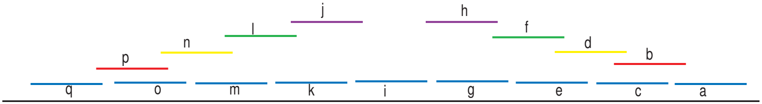


- $f(a) \geq f(c) + r + 1$
- $|f(d) - f(e)| \leq r$
- $f(g) \geq f(q) + r + 1$



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The Semiorder Polytope

The set of all representations forms a convex polytope in *n*-dimensional space, where *n* is the number of elements in our semiorder.

Note that we fix the minimum element at zero, and that we have one dimension for the length of intervals (r).

Too Many Inequalities

Our example on 17 elements gives us 158 inequalities to deal with....

Many of these are implied by the others, saving us some work.

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Saving Inequalites

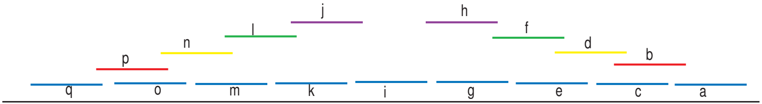
We actually need only need a subset of these inequalities to imply the others (implications by transitivity and order in the trace).

Noses and Hollows

- Noses: We say that xNy if xPy and any element z such that $xTzTy$ satisfies zIy and xIz.
- Hollows: We say that xHy if xIy, yTx and for all w and z such that $zTyTxTw$, zPw.

Noses and Hollows

- Noses: Two elements in a ‘nose’ relation are comparable, but only just barely so.
- Hollows: Two elements in a ‘hollow’ relation are incomparable, but only just barely so.



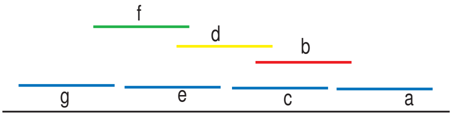
- aNc, cNe, eNg etc.
- bHa, dHb, fHd, etc.
- hNj iHh, jHi

Why Noses and Hollows?

We can list the preferences in a matrix.

The rows and columns correspond to the semiorder elements.

The matrix has only 0's and 1's, with a 1 indicating that the element corresponding to the row is preferred to the element in the column.



$$\begin{pmatrix} & a & b & c & d & e & f & g \\ a & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ b & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ c & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Step-Matrix

A semiorder always forms this type of step matrix. Our connectivity conditions keep this matrix away from the main diagonal and keeps differences between the rows and columns.

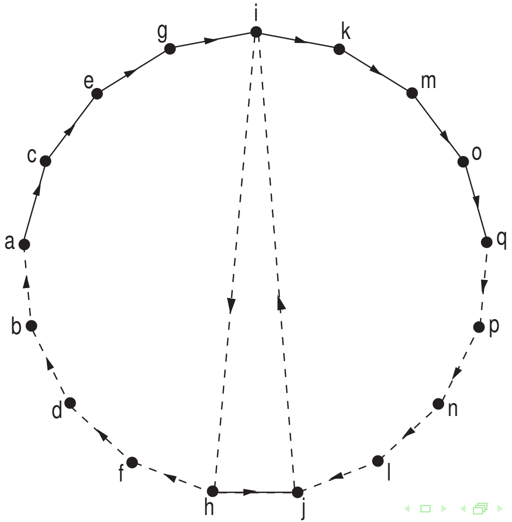
- The Noses are places where we could remove a 1 and still have a step-matrix.
- The Hollows are places where we could remove a 0 and still have a step-matrix.

Using Fewer Inequalities

The only relevant inequalities correspond to the noses and hollows of the preference matrix.

This reduces our large example of 158 inequalities to 19 inequalities.

The Super-Synthetic Graph: Noses and Hollows



Minimal Representation(s)

Pirlot (1990) proved the existence of a minimal representation of a given semiorder. This representation is minimal in the sense that

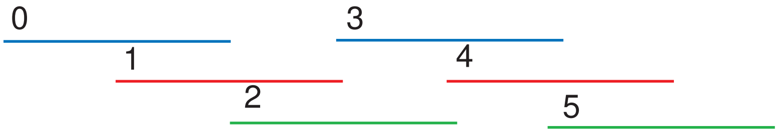
- The function values are as small as possible (the intervals are as far left as possible)
- The scale cannot be any smaller

Minimal Representation(s)

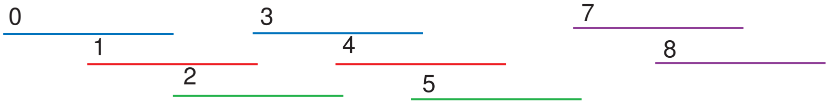
There exist other ‘minimal’ representations in the sense that the scale is the shortest possible and the function values are the smallest possible to satisfy a maximal number of noses and hollows.

These minimal representations are exactly the vertices of the semiorder polytope.

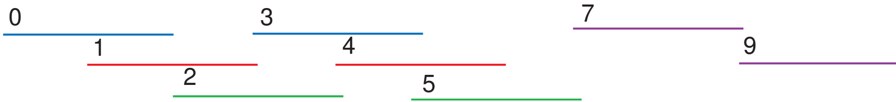
One Vertex



Two Vertices



OR



Finding the vertices of the semiorder polytope corresponds to finding cycles in the *SSG*.

Those inequalities on the cycles found are satisfied with equality.

A minimal representation will satify a maximal number of noses and hollows with equality.

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Computational Analysis

We used *PORTA* to compute the solution space to the linear system $Ax \leq b$, where A is derived from the incidence matrix of the SSG.

Formally, we must augment A by a column vector corresponding to r, the cycle length.

PORTA returns the vertices and extremal rays of each system.

Our natural (and seemingly innocuous) question was ‘Do all minimal scales have the same interval length?’

In other words, do all vertices have the same value for r? Do they all lie in the $r = k$ plane?

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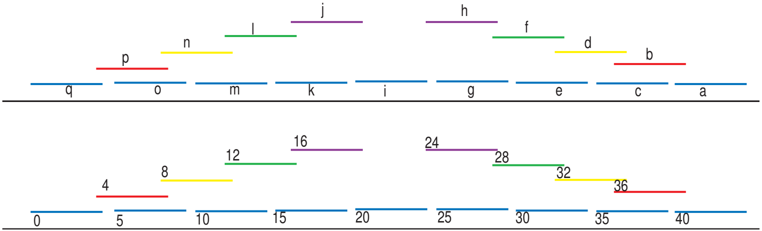
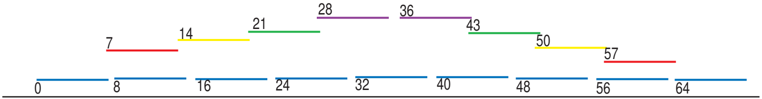
One Semiorder, Two Scales



$r = 4$



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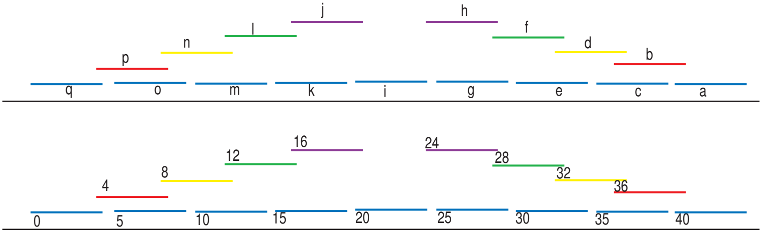

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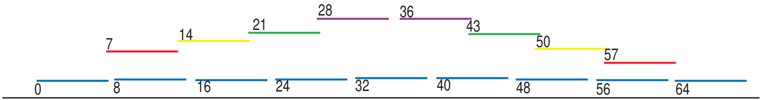

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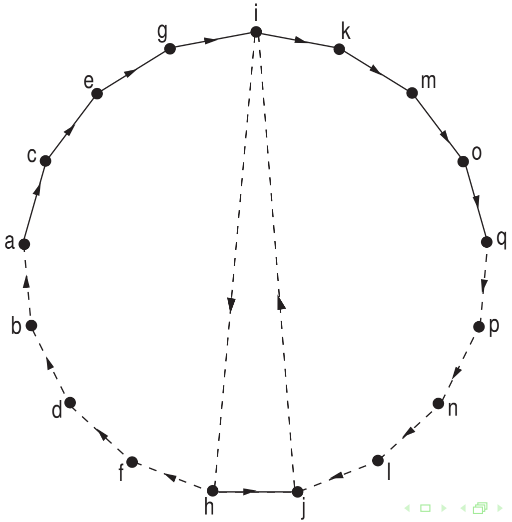
A semiorder polytope has all of its vertices on the same $r = k$ plane if all cycles of the corresponding SSG ‘behave’.

Two Cycle Types



More Pathological Examples

This example is part of a larger family of examples which have exactly two vertices, one at $r = k$ and one at $r = 2k + 1$.

We have constructed more examples with 3, 4, and more different r values for the vertices.

Some Results

- Any vertex of the semiorder polytope must satisfy at least one cycle of the SSG with equality.
- The minimal interval length r^* is equal to the length of the longest conformal cycle in the SSG. Any cycles of this length will be satisfied in any vertex in the plane $r = r^*$.
- The extremal rays of the semiorder polytope correspond exactly to 'cycle-breaking' subsets of the edges.

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Other Questions

Can we classify the minimal representations of a semiorder by examining the *SSG* and the Nose and Hollow Inequalities?

Can we determine which semiorders have certain desirable properties? (eg: all vertices with the same r value, only one vertex, a specified number of vertices, etc.)

Must the value of r always be an integer?

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THANK YOU