On Testing Properties in Directed Graphs

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Dealing with "BigData" in Graphs

- We want to process graphs quickly
  - Detect basic properties
  - Analyze their structure

- For large graphs, by "quickly" we often would mean: in time *constant* or *sublinear* in the size of the graph

Dealing with "BigData" in Graphs

One approach:

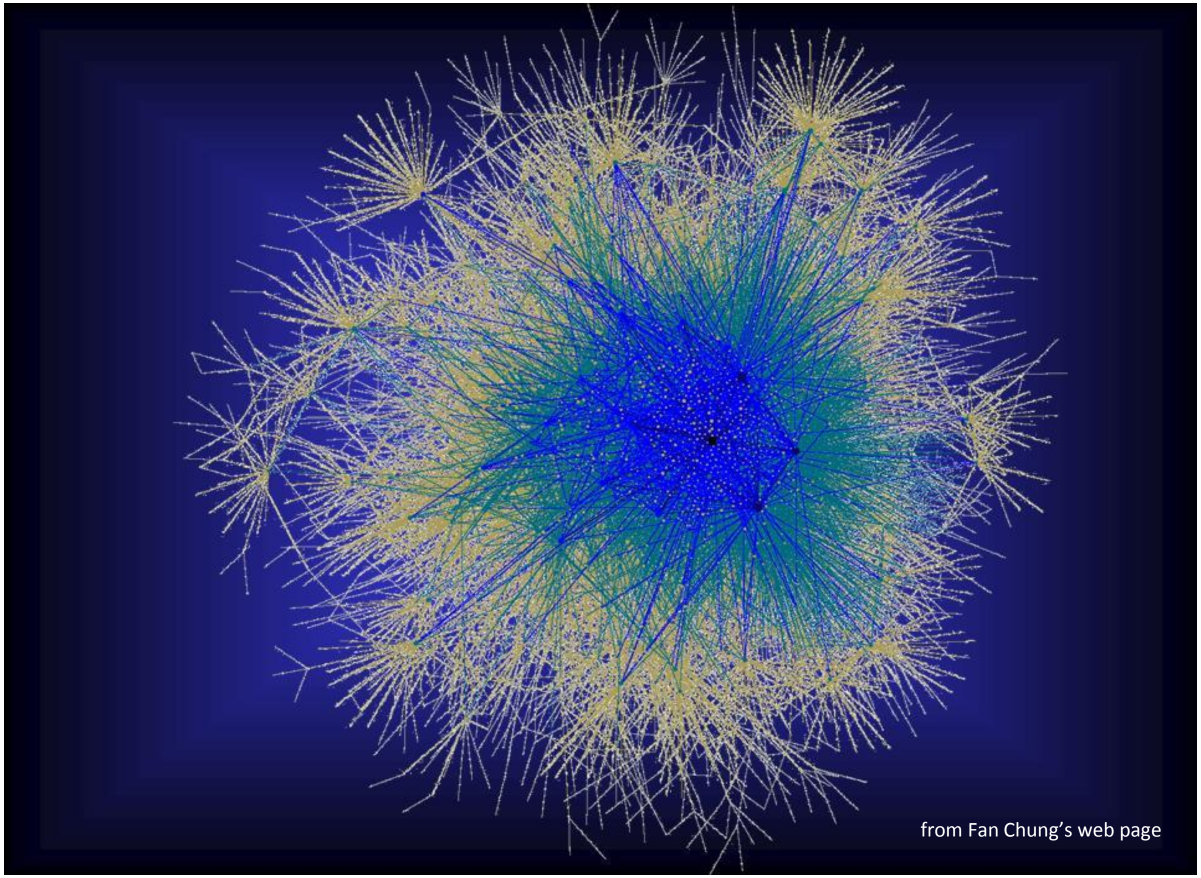
- How to test basic properties of graphs
  in the framework of **property testing**

Framework of property testing

- We cannot quickly give 100% precise answer
- We need to approximate

- Distinguish graphs that have specific property from those that are far from having the property

Fast Testing of Graph Properties



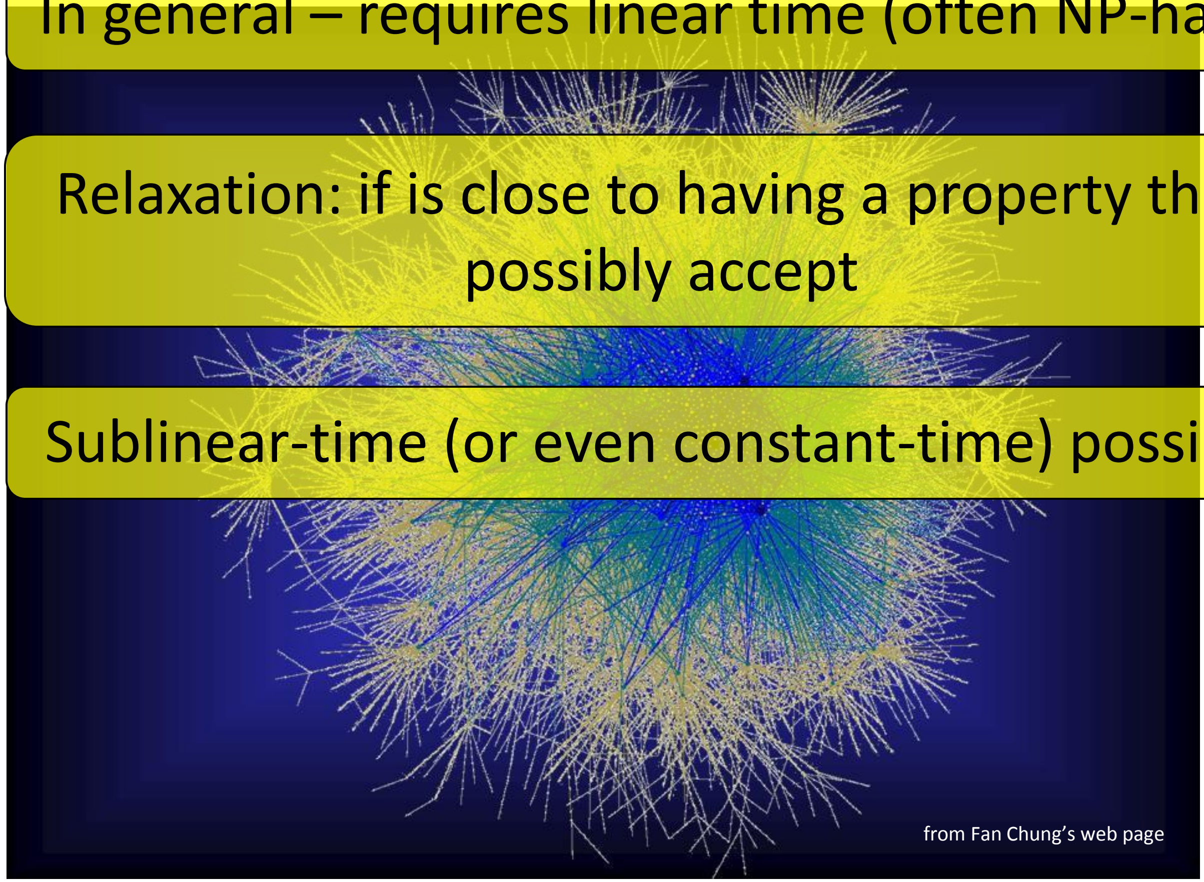
- Does this graph have a clique of size 11?
- Does it have a given H as its subgraph?
- Is this graph planar?
- Is it bipartite?
- Is it k-colorable?
- Does it have good expansion?
- Does it have good clustering?

Fast Testing of Graph Properties

In general – requires linear time (often NP-hard)

Relaxation: if is close to having a property then possibly accept

Sublinear-time (or even constant-time) possible



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Testing properties of graphs

Input:

- graph property P;
- proximity parameter ε;
- input graph $G = (V, E)$ of maximum degree d.

Output:

- if G satisfies property P then ACCEPT
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Testing properties of graphs

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G is ε–far from satisfying P if one has to modify $\leq d|V|$ edges of G to obtain a graph satisfying P

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- if we can err only for REJECTION then one-sided error
- if we can also err for ACCEPTs then two-sided error

Fast Testing of Graph Properties

- Started with Rubinfeld-Sudan (1996) and Goldreich-Goldwasser-Ron (1998)
- Now we know a lot
  - If G is dense, given as an oracle to adjacency matrix, then every hereditary property can be tested in constant time
  - If G is sparse, given as an oracle to adjacency list, then many properties can be tested in constant time, many can be tested in sublinear time
  - If G is directed then … essentially nothing is known
    - unless there is a trivial reduction to undirected graphs

Fast Testing of Digraph Properties

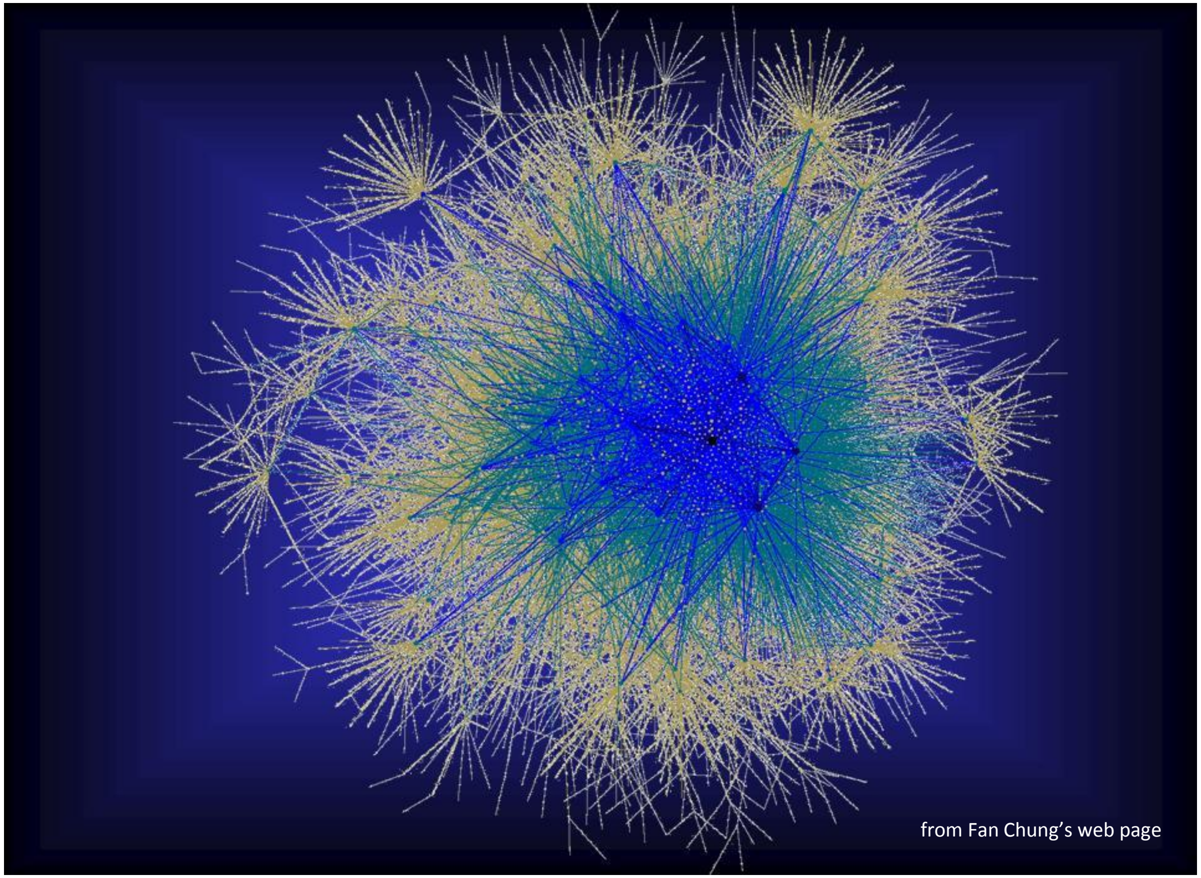
Models introduced by Bender-Ron (2002):

- Digraphs with bounded maximum in- and out-degrees
- Oracle with access to adjacency list
- Two main models:
  - Bidirectional: outgoing and incoming edges
    - shares properties of undirected graphs;
    - not suitable in many scenarios/applications
  - One-directional: access to outgoing edges only
    - major difference wrt undirected graphs
    - more natural in many scenarios/applications

Sometimes very fast

More challenging

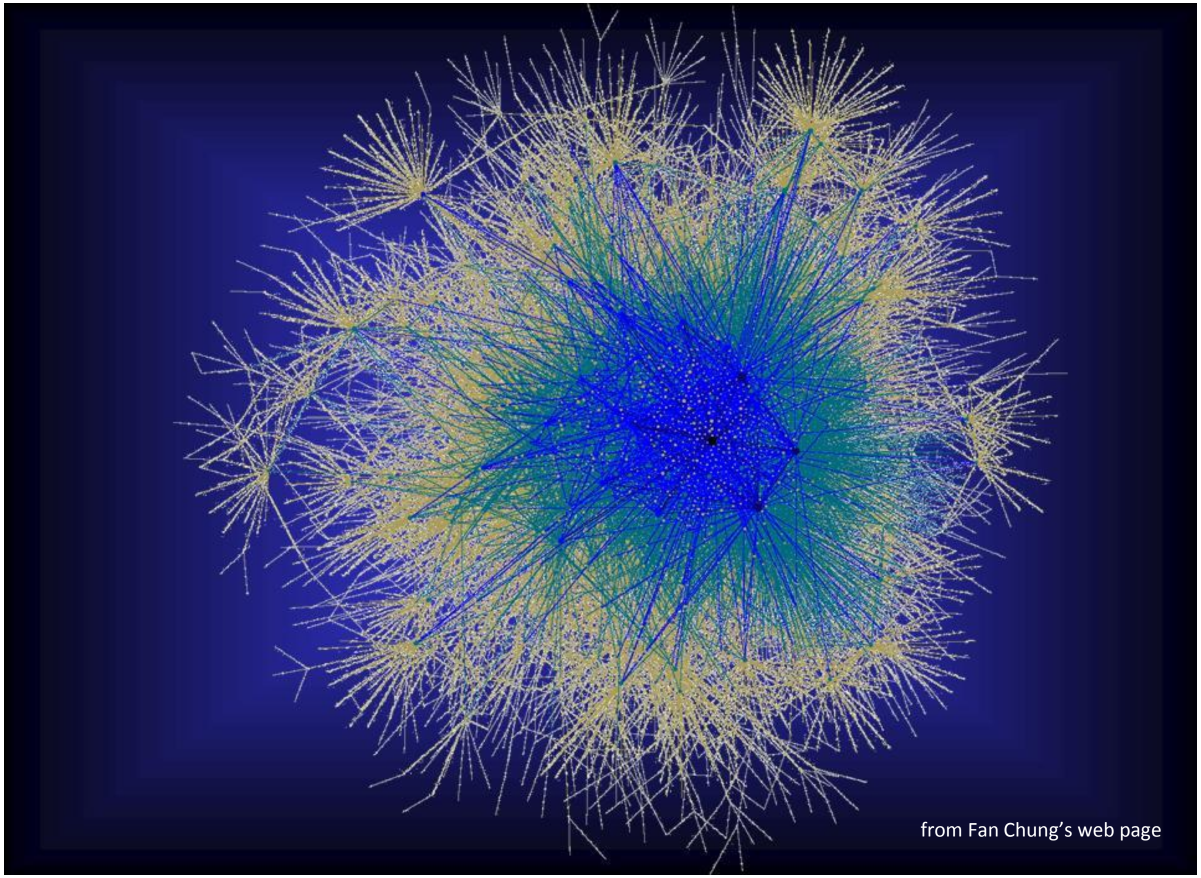
Big networks



- Is it weakly connected?
  (or close to it)
- Is it planar?
  (or close to it)

If we have access to both directional edges then this reduces to a problem in undirected graphs (which we understand well)

Big networks



- Is it strongly connected?
  (or close to it)
- Is it acyclic?
  (or close to it)
- Is it C_{33}-free?
  (or close to it)

Highly non-trivial if we have no access to incoming edges
For example: we cannot easily check if a node has in-degree 0

OBJECTIVE: Study the dependency between the models

There is a tester for property P with constant query time in bidirectional model

⬇

We can test P in one-directional model with sublinear $n^{1-\Omega_{\varepsilon,d}(1)}$ query time (in two-sided error model)

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Application:
Every hyperfinite property can be tested with sublinear complexity in one-directional model

What is known for digraphs

Not much

What is known for digraphs

Strong connectivity

- Constant complexity in bidirectional model (Bender-Ron'02)
- One-directional queries:
  - requires $\Omega(\sqrt{n})$ complexity (Bender-Ron'02)
  - can be done with $n^{1-\Omega_{\varepsilon,d}(1)}$ complexity (Goldreich'11, Hellweg-Sohler'12)
  - requires $\Omega(n)$ complexity in one-sided-error model (Goldreich'11, Hellweg-Sohler'12)

What is known for digraphs

Bidirectional model:

- testing Eulerianity (Orenstein-Ron'11)
- testing k-edge-connectivity (Orenstein-Ron'11 ,Yoshida-Ito'10)
- testing k-vertex connectivity (Orenstein-Ron'11)
- acyclicity requires $\Omega(n^{1/3})$ queries (Bender-Ron'02)

- Testing H-freeness
  - constant complexity in bidirectional model (folklore)
  - $O(n^{1-1/k})$ complexity, where k is # of connected components of H with no incoming edge from another part of H (Hellweg-Sohler'12)
- 3-star-freeness:
  - requires $\Omega(n^{2/3})$ complexity (Hellweg-Sohler'12)

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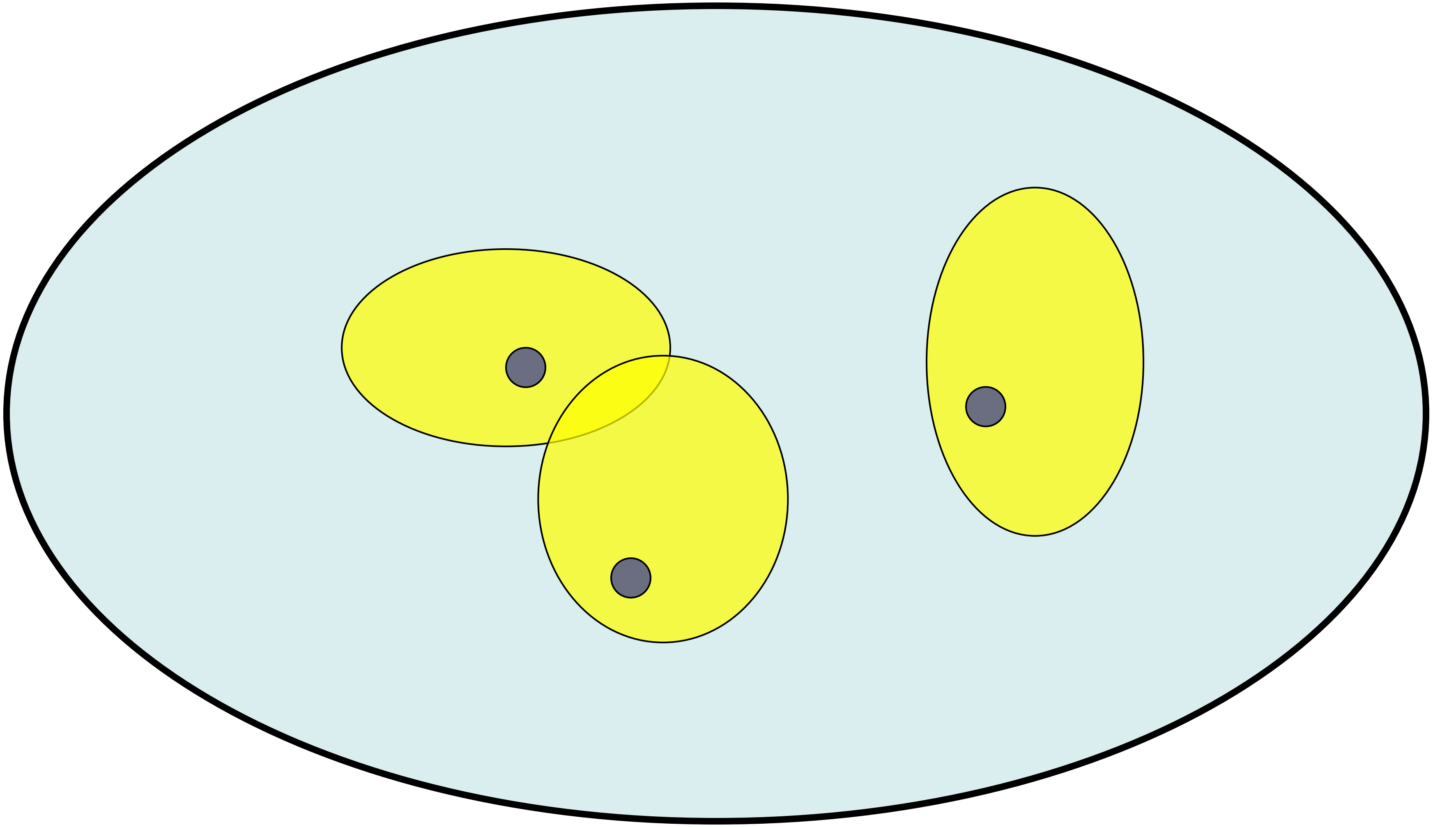
This cannot be improved much:

- two-sided error is required (cf. strong connectivity)
- $\Omega(n^{2/3})$ "simulation" slowdown is required (cf. 3-star-freeness)

Conjecture: bound is tight

Key ideas

What a constant-complexity tester in bidirectional model can do?



What a constant-complexity tester in bidirectional model can do?

Tester of complexity $q = q(\varepsilon, d, n)$

Cannot do more than

- Randomly sample q vertices
- Explore q neighborhood of the sampled vertices
  - neighborhood = using edges of either direction
- Accept or reject on the basis of the explored digraph

Key ideas

- We can characterize properties testable with constant number of queries ➔ canonical testers
- Canonical tester will do the following:
  - Samples a constant number of random vertices
  - Explores bounded-radius discs rooted at sampled vertices
  - Decides whether to accept or reject on the basis of a check if the explored digraph is isomorphic to any digraph from a forbidden collection of rooted discs

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Further property:
* If G satisfies P then bounded-radius discs at randomly sampled vertices will be isomorphic to any element from the forbidden collection with prob $\leq 1/3$
* If G is ε–far, then the discs will be isomorphic with prob $\geq 2/3$

Key ideas

- We can characterize properties testable with constant number of queries ➔ canonical testers
- Goal of one-directional tester
  - Simulate canonical bidirectional testers
  - We want to "estimate" the structure of random q discs of (bidirectional) radius q

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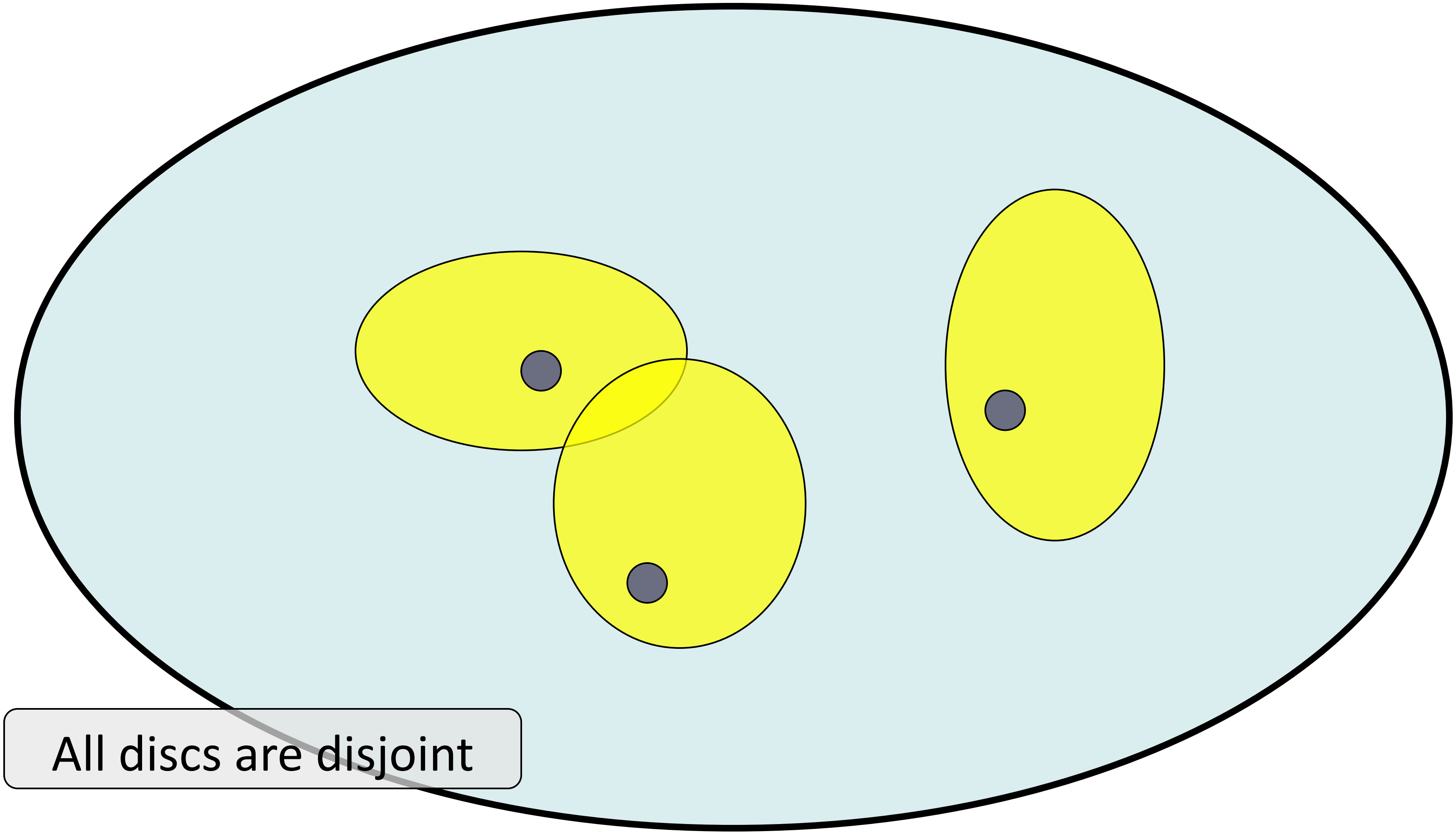
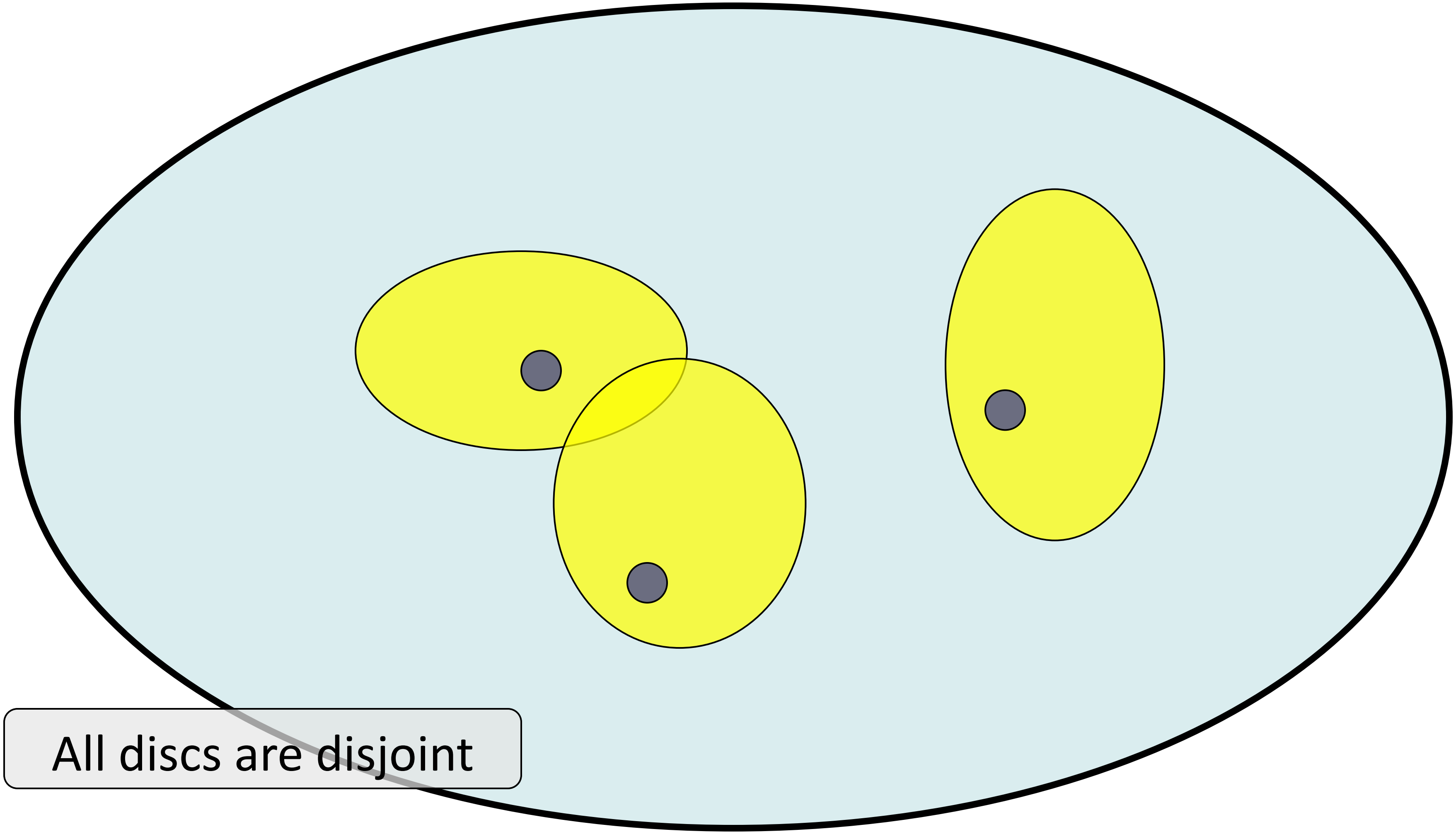


All discs are disjoint

What a ~~constant-complexity~~ tester in ~~bidirectional~~ one-directional model can do?



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  - Let $H_{q,d}$ be the set of q rooted digraphs of (bidirectional) radius q of maximum in-/out-degree d
    - Note: $|H_{q,d}| = f(q, d, \varepsilon)$, and $q = q(\varepsilon, d)$ ➔ $|H_{q,d}| = O_{\varepsilon,d}(1)$
  - We can approximate the number of copies of any $H \in H_{q,d}$ in the input digraph G

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  - We want to “estimate” the structure of random q discs of (bidirectional) radius q
  - By randomly sampling $n^{1-\Omega_{\varepsilon,d}(1)}$ edges, we can approximate well the number of occurrences of any $H \in H_{q,d}$ in the input digraph G

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  ➔ We can simulate canonical bidirectional tester

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Application:
Every hyperfinite property can be tested with sublinear complexity in one-directional model

Hyperfinite graphs and properties

- Graph is hyperfinite if we can remove small fraction of edges to split it into small connected components
  - E.g. bounded degree planar graphs, bounded degree graphs defined by a finite collection of forbidden minors

- Property is hyperfinite if it contains only hyperfinite graphs
  - E.g. planarity

Hyperfinite graphs and properties

Newman-Sohler (2013) proved that every (undirected) graph property of a hyperfinite graph is testable with constant complexity. Also: every hyperfinite property is testable with constant query complexity.

We can extend this to digraphs (in bidirectional model)

This extends the claims to one-directional model, giving two-sided error testers with query complexity $n^{1-\Omega_{\varepsilon,d}(1)}$

Conclusions

While testing of undirected graphs is rather well understood, we know little about directed graphs

In this talk: progress towards our understanding of testing digraph properties in one-directional model