Improved Lower Bounds for Coded Caching

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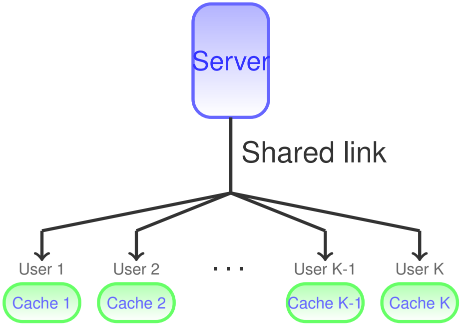
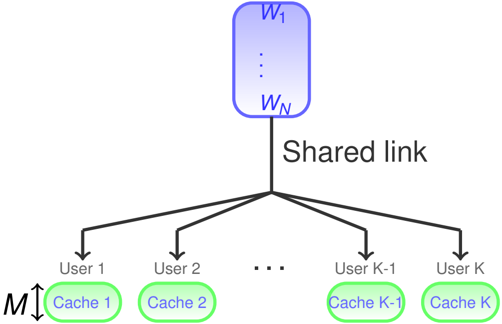
Conventional Content Delivery with Caching



- Mechanism for reducing transmission rates from server to clients.
  - Conventional approach: clients cache portions of popular content.
- Coding in the cache and coded transmission from server are typically not considered.

Coded Caching Formulation *[Maddah-Ali & Niesen '13]*

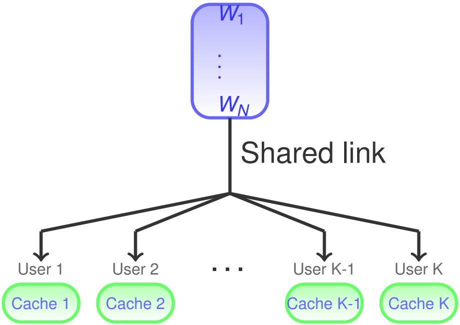
- Server contains N files each of size F bits.
- K users each with a cache of size MF bits.
- The i-th user requests file $d_i \in \{1, \ldots, N\}$.



Coded Caching Formulation *[Maddah-Ali & Niesen '13]*

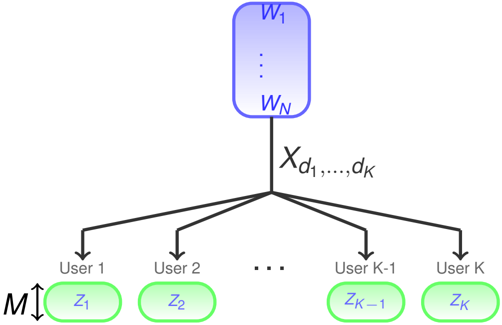
1. *Placement phase*: The content of the caches are populated, does not depend on users actual requests.

2. *Delivery phase*: the server transmits a signal of rate RF bits over the shared link so that each user's request is satisfied.



Coded Caching Formulation *[Maddah-Ali & Niesen '13]*

- N files $\{W_n\}_{n=1}^N$,
- i-th user requests the file W_{d_i},
- Cache content: Z_i,
- Delivery phase signal: $X_{d_1,d_2,\ldots,d_K}$,
- Decoding file for i-th user: $\hat{W}_{d_1,\ldots,d_K;i}$,
- Probability of error: $\max_{d_1,\ldots,d_K} \max_i P(\hat{W}_{d_1,\ldots,d_K;i} \neq W_{d_i})$.



Achievable Pair (M, R):

The pair is said to be achievable if for any $\epsilon > 0$ there exist a file size F large enough and a (M, R) caching scheme with probability of at most ϵ.

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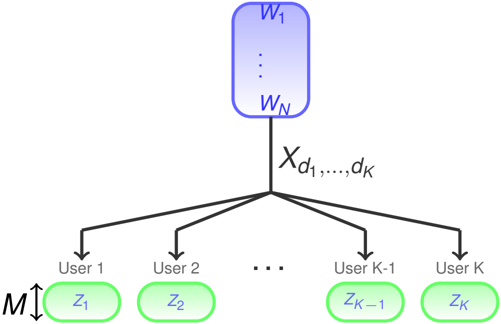
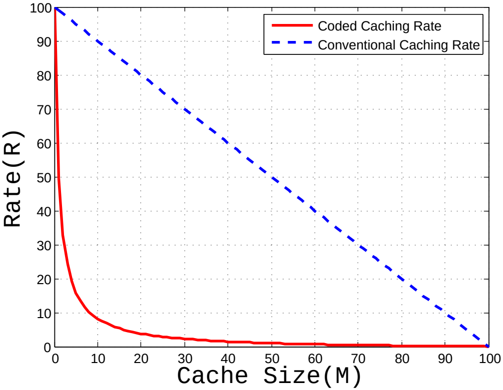


Memory-rate tradeoff

$$R^\star(M) = \inf\{R : \ (M, R) \ \text{ is achievable}\}.$$

Achievable rates $N = 1000, K = 100$



$$R_C(M) = K\left(1 - \frac{M}{N}\right) \cdot \min\left\{\frac{1}{1 + KM/N}, \frac{N}{K}\right\},$$

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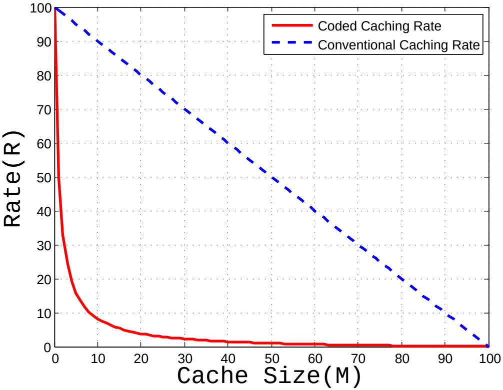

$$R_C(M) = K\left(1 - \frac{M}{N}\right) \cdot \min\left\{\frac{1}{1 + KM/N}, \frac{N}{K}\right\},$$

- However, tight lower bounds on $R_C(M)$ are not known at this point.

Related Work

- Cutset bound [Maddah-Ali & Niesen '13]. Show that $R_C(M)/R^{star}(M) \leq 12$ (multiplicative gap).
- Parallel works
  - Improved bounds using Han's inequality [Sengupta, Tandon, Clancy '15]. Show a multiplicative gap of 8.
  - Another approach (can be considered a special case of our work) by [Ajaykrishnan et al. 15].
  - Computational approach of [Tian '15] (Arxiv preprint) for the specific case of $N = K = 3$.

An Example: $N = K = 3$ and $M = 1$.

$$2R^\star F + 2MF \geq H(Z_1) + H(X_{1,2,3}) + H(Z_2) + H(X_{3,1,2})$$
$$\geq H(Z_1, X_{1,2,3}) + H(Z_2, X_{3,1,2})$$
$$\geq I(W_1; Z_1, X_{1,2,3}) + H(Z_1, X_{1,2,3}|W_1) + I(W_1; Z_2, X_{3,1,2}) + H(Z_2, X_{3,1,2}|W_1)$$

$$I(W_1; Z_1, X_{1,2,3}) = H(W_1) - H(W_1|Z_1, X_{1,2,3})$$
$$\geq F(1-\epsilon)$$

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Writing mutual information another way ...

$$\begin{aligned} I(W_1; Z_1, X_{1,2,3}) &= H(W_1) - H(W_1 | Z_1, X_{1,2,3}) \\ &\geq F(1 - \epsilon) \end{aligned}$$

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Since W_1 can be recovered from Z_1 and $X_{1,2,3}$ with ϵ-error.

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$$\geq I(W_1; Z_1, X_{1,2,3}) + H(Z_1, X_{1,2,3}|W_1) + I(W_1; Z_2, X_{3,1,2})$$
$$+ H(Z_2, X_{3,1,2}|W_1),$$
$$= 2F(1-\epsilon) + H(Z_1, X_{1,2,3}|W_1) + H(Z_2, X_{3,1,2}|W_1)$$
$$\geq 2F(1-\epsilon) + H(Z_1, Z_2, X_{1,2,3}, X_{3,1,2}|W_1)$$
$$= 2F(1-\epsilon) + I(W_2, W_3; Z_1, Z_2, X_{1,2,3}, X_{3,1,2}|W_1)$$
$$+ H(Z_1, Z_2, X_{1,2,3}, X_{3,1,2}|W_1, W_2, W_3)$$
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&= 2F(1-\epsilon) + H(Z_1, X_{1,2,3}|W_1) + H(Z_2, X_{3,1,2}|W_1) \\
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Final Result

- $2R^\star + 2M \geq 4$
- $\implies R^\star \geq 1$. (Known to be achievable).
- Non-cutset based bound. Generalizes a strategy that appeared in *[Maddah-Ali & Niesen '13]*

Equivalent description on directed tree

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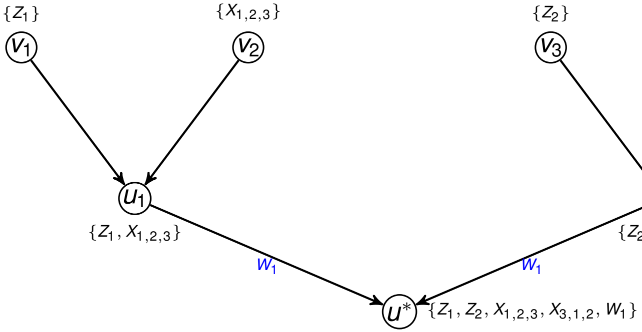
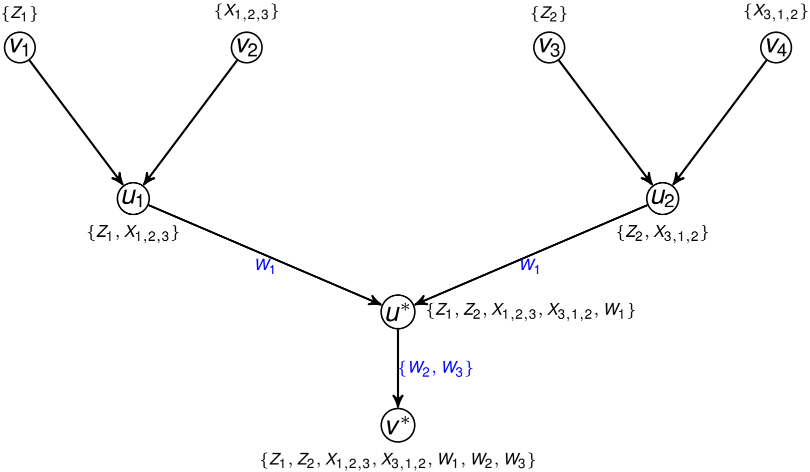


The pairs $Z_1, X_{1,2,3}$ and $Z_2, X_{3,1,2}$ each recover a new source W_1.

Equivalent description on directed tree



The set of cache and delivery phase signals $\{Z_1, Z_2, X_{1,2,3}, X_{3,1,2}\}$ recovers the sources W_1, W_2, W_3. W_1 has already been recovered earlier. The new sources are thus W_2, W_3.

Problem Instance: $P(\mathcal{T}, \alpha, \beta, L, N, K)$

- Problem Input.
  - Number of files N and users K.
  - Tree $\mathcal{T}$ with α leaves labeled with delivery phase signals and β leaves labeled with cache signals.
- Algorithm returns lower bound $\alpha R + \beta M \geq L$.

Natural question

- For a given *N*, *K* and α and β.
  - Determine the optimal tree $\mathcal{T}^\star$ and its labeling so that the lower bound *L* is maximized.
  - Refer to this as the optimal problem instance.
- Solution to this would yield the best possible lower bound using *this* technique.

Sketch of ideas

Observation

For problem instance $P(\mathcal{T}, \alpha, \beta, L, N, K)$, the lower bound $L \leq \alpha \min(\beta, K)$. For N large enough, we can always find an instance where $L = \alpha \min(\beta, K)$.

Sketch of ideas

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Example

Let $\alpha = 2, \beta = 3$ and $N = \alpha\beta = 6$ and $K = 3$.

- Choose cache signals: Z_1, Z_2, and Z_3.
- Choose delivery phase signals, such that each cache recovers a different file: $X_{1,2,3}$ and $X_{4,5,6}$.

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&\geq H(Z_1, Z_2, Z_3, X_{1,2,3}, X_{4,5,6}) \\
&= H(W_1, W_2, \ldots, W_6) \\
&= 6F.
\end{aligned}
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&= H(W_1, W_2, \ldots, W_6) \\
&= 6F.
\end{aligned}
$$

Observation

We don't really need six files to get a lower bound of $6F$.

Formal definition of saturation number

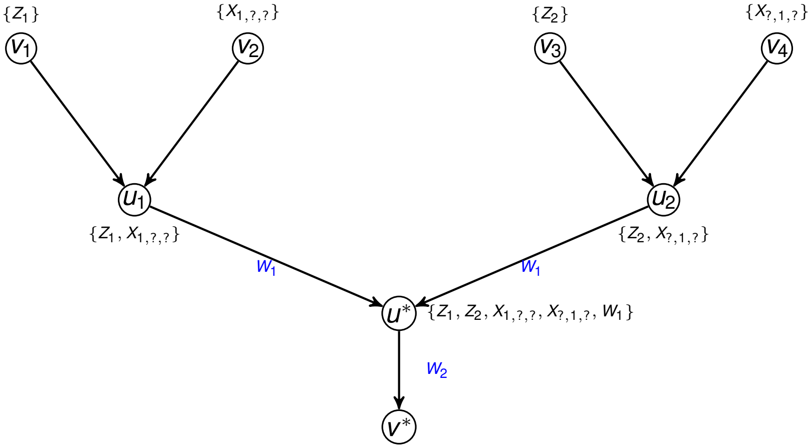
Definition

Saturation number. Consider an instance $P^*(\mathcal{T}^*, \alpha, \beta, L^*, N^*, K)$, where $L^* = \alpha \min(\beta, K)$, such that for all problem instances of the form $P(\mathcal{T}, \alpha, \beta, L^*, N, K)$, we have $N^* \leq N$. We call N^* the saturation number of instances with parameters (α, β, K) and denote it by $N_{sat}(\alpha, \beta, K)$.

- Saturated instances use the files most efficiently in obtaining the lower bound.
- If $N = \alpha\beta$, it is easy to demonstrate an instance where $L = \alpha\beta$ (precisely, the idea of the cutset bound!).

Intuition about saturation number

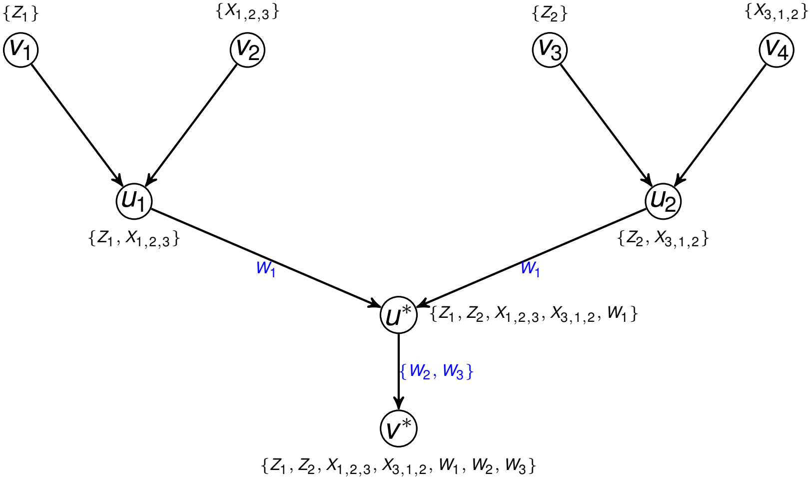
Suppose that $\alpha = \beta = 2, \mathbf{N} = \mathbf{2}, K = 3$.



- Regardless of the value of ?'s in the delivery phase signals, the lower bound can be at most 3.
- Cannot reach $\alpha\beta = 4$ under any possible labeling.

Intuition about saturation number

Suppose that $\alpha = \beta = 2, \mathbf{N} = \mathbf{3}, K = 3.$



- With $N = 3$, we can obtain an instance where $L = \alpha\beta = 4$.

Key Lemma

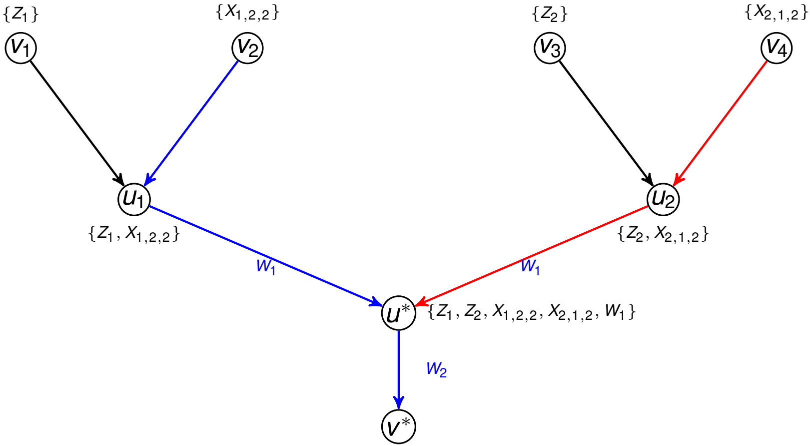
Lemma

Let $P = P(\mathcal{T}, \alpha, \beta, L, K, N)$ be an instance where $L < \alpha \min(\beta, K)$. Then, we can construct a new instance $P' = P(\mathcal{T}', \alpha, \beta, L', K, N + 1)$, where $L' = L + 1$.

- Simple argument that changes the label of one delivery phase signal to exploit the new file.

Example: $X_{2,1,2}$ is inefficient

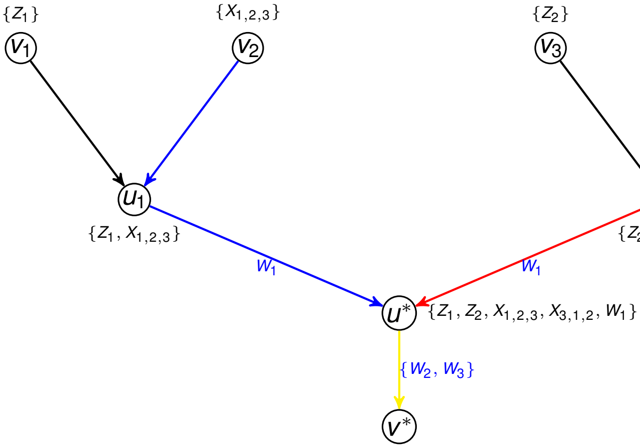
Suppose that $\alpha = \beta = 2, \mathbf{N} = \mathbf{2}, K = 3$.



- Identified the inefficiency of $X_{2,1,2}$.

Example: Fixing the inefficiency of $X_{2,1,2}$

Suppose that $\alpha = \beta = 2, \mathbf{N} = \mathbf{3}, K = 3.$



- Changed $X_{2,1,2}$ to $X_{3,1,2}$. Can be done systematically in general.

Main theorem

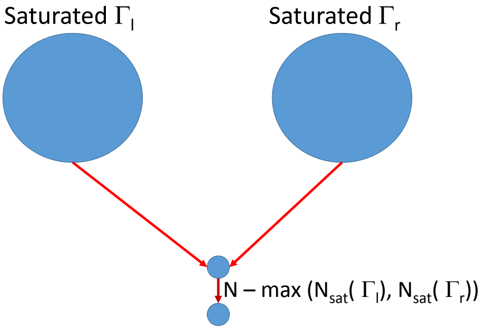
Theorem

Suppose that there exists an optimal and atomic problem instance $P_o(\mathcal{T} = (V, A), \alpha, \beta, L_o, N, K)$. Then, there exists optimal and atomic problem instance $P^*(\mathcal{T}^* = (V^*, A^*), \alpha, \beta, L^*, N, K)$ where $L^* = L_o$ with the following properties. Let us denote the last edge in P^* with (u^*, v^*). Let $P_l^* = P(\mathcal{T}^*_{u^*(l)}, \alpha_l, \beta_l, L_l^*, N_l, K)$ and $P_r^* = P(\mathcal{T}^*_{u^*(r)}, \alpha_r, \beta_r, L_r^*, N_r, K)$. Then, we have

$$
\begin{aligned}
L_l^* &= \alpha_l \min(\beta_l, K), \\
L_r^* &= \alpha_r \min(\beta_r, K), \ \textit{and} \\
L^* &= \min\left(\alpha \min(\beta, K), L_l^* + L_r^* + N - N_0\right),
\end{aligned}
$$

where $N_0 = \max(N_{sat}(\alpha_l, \beta_l, K), N_{sat}(\alpha_r, \beta_r, K))$. Furthermore, at least one of β_l or β_r is strictly smaller than K.

Implication: Optimal problem instances



- Upper bounds on N_{sat} allow us to obtain valid lower bounds as well.

Discussion

- Cutset bound

$$\underbrace{\lfloor N/s \rfloor}_{\alpha} R^\star + \underbrace{s}_{\beta} M \geq s \lfloor N/s \rfloor \qquad s = 1, \dots, \min(N, K)$$

- Special case of our bound. Simply choose $Z_1, \dots, Z_s$ as cache nodes, and $\lfloor N/s \rfloor$ delivery phase signals with disjoint file requests.

Discussion: Cutset bound on $\alpha R + \beta M$

$N \geq \alpha\beta$.

Example

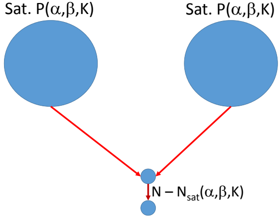
$N = 64$, $K = 12, M = 16/3$

$$9R^\star + 7M \geq 63$$
$$\implies R^\star \geq 2.852. \text{ (best lower bound using cutsets)}$$

Discussion: Bound $2\alpha R + 2\beta M$ instead

- Suppose $\alpha\beta < N$ but $4\alpha\beta > N$. Then,

$$2\alpha R + 2\beta M \geq 2\alpha\beta + N - N_{sat}(\alpha, \beta, K)$$
$$\implies \alpha R + \beta M \geq \alpha\beta + \frac{N - N_{sat}(\alpha, \beta, K)}{2}.$$



Example

$$18R^{\star} + 14M \geq 126 + 64 - N_{sat}(9, 7, 12)$$
$$\geq 126 + 21$$
$$\implies R^{\star} \geq 4.018. \text{ (improvement)}$$

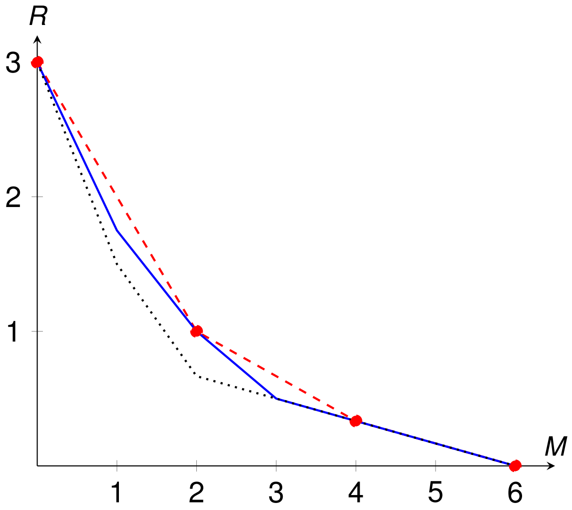
Optimizing over choices for α and β

Example

$N = 64$, $K = 12, M = 16/3$.

$$
\begin{aligned}
12R^\star + 8M &\geq \min(12 \times 8, 6 \times 4 + 6 \times 4 + 64 - N_{sat}(6, 4, 12)) \\
&\geq \min(96, 112 - \hat{N}_{sat}(6, 4, 12)) = \min(96, 112 - 17) = 95 \\
\implies R^\star &\geq \frac{157}{36} = 4.361 \\
R_c &= 5.5 \text{ (achievable rate)}
\end{aligned}
$$

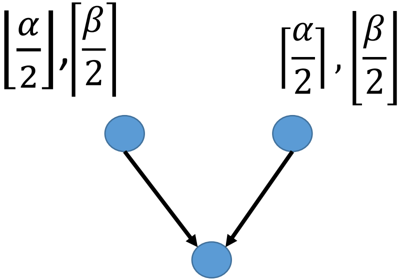
Plot for $N = 6, K = 3$



$N = 6$, $K = 3$, Blue: Proposed bound, Dotted Black: Cut-set bound, Dashed Red: Achievable rate

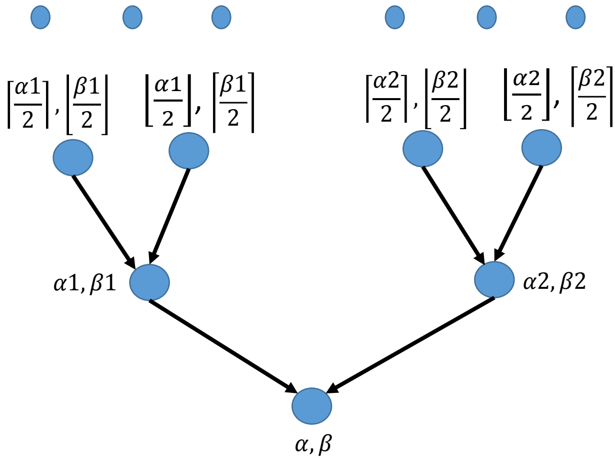
Upper bound on saturation number $N_{sat}(\alpha, \beta, K)$

- For given α, β and K, consider "roughly" balanced splits.



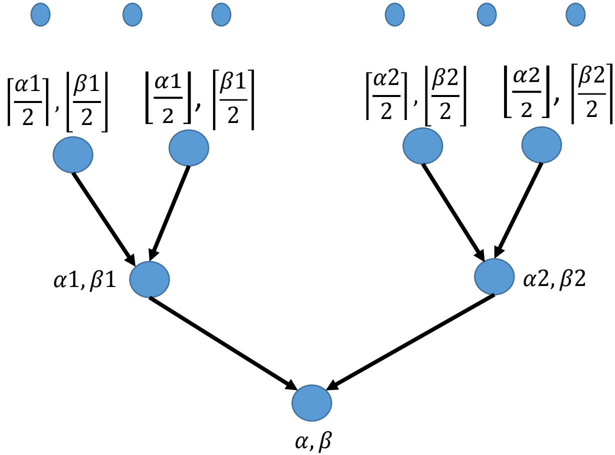
Upper bound on saturation number $N_{sat}(\alpha, \beta, K)$

- Continue recursively, at all levels, maintaining roughly balanced splits, until leaves are reached.



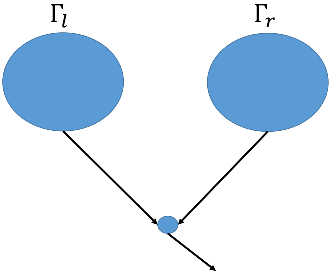
Upper bound on saturation number $N_{sat}(\alpha, \beta, K)$

- Use $N = \alpha\beta$ files to obtain an instance with lower bound $L = \alpha\beta$.



Upper bound on saturation number $N_{sat}(\alpha, \beta, K)$

- Structural properties of saturated instances.
- Let Γ_l be the file indices used in the left branch of some node (likewise Γ_r).
- Then, either $\Gamma_l \subseteq \Gamma_r$ or $\Gamma_r \subseteq \Gamma_l$.
- Procedure to (iteratively) modify the instance so that this condition is met at all nodes; number of files is guaranteed to decrease at each step.



Upper bound on saturation number $N_{sat}(\alpha, \beta, K)$

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N_{sat}(\alpha, \beta, K) &\leq \frac{2\alpha\beta + \alpha + \beta}{3} \\
&< \alpha\beta \text{ (for large enough values of } \alpha \text{ and } \beta)
\end{aligned}
$$

Multiplicative gap results

- Nontrivial upper bound on $N_{sat}(\alpha, \beta, K)$ when $\beta \leq K$.

$$\begin{aligned} N_{sat}(\alpha, \beta, K) &\leq \frac{2\alpha\beta + \alpha + \beta}{3} \\ &< \alpha\beta \text{ (for large enough values of } \alpha \text{ and } \beta) \end{aligned}$$

- With some work, this yields a multiplicative gap of at most 4 between our lower bound and the achievability scheme.

$$\frac{R_C(M)}{R^*(M)} \leq 4.$$

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  - Our bound is superior for reasonably large α and β

$$\frac{1}{\alpha} + \frac{1}{\beta} \leq \frac{1}{2}$$

  - For small values of $M \leq 1$, their bound is better, especially when $N \leq K$.
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  - For small values of $M \leq 1$, their bound is better, especially when $N \leq K$.
  - We have a better multiplicative gap.
- Approach of [Tian '15] for the case of $N = K = 3$ has one inequality that is strictly better than us. However, it is unclear whether this approach is practical for arbitrary N and K.

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