On Two-user Fading Gaussian Broadcast Channels

with Perfect Channel State Information at the Receivers

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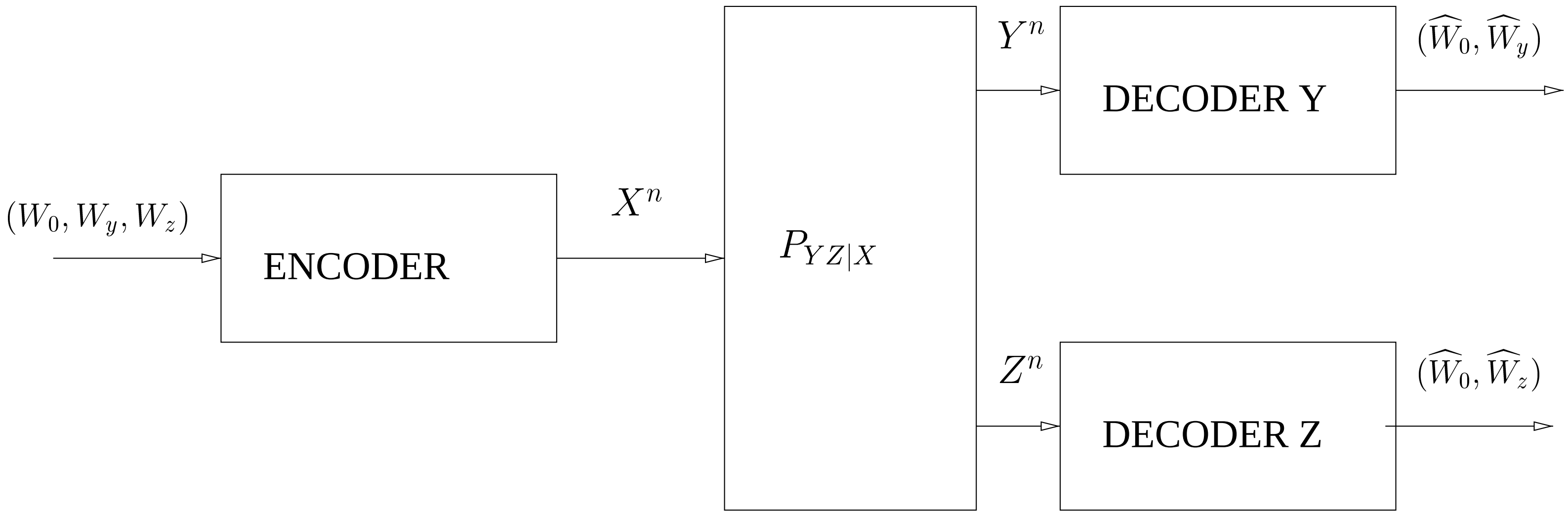
Motivation

- Fading Gaussian Broadcast channels (BS) model the down-link of wireless cellular systems with delay constraints much larger than the fading coherence time and no feedback available.
- The BC problem was formalized by Thomas Cover in 1972.
- Most of the available results on **general BCs** date back to the 70s. For the subsequent two decades few relevant works appeared.
- The BC problem regained attention recently in the unfaded multi-antenna *Gaussian* case (Caire-Shamai 2000).

The broadcast problem (Cover 1972)

- A general **discrete memoryless two-user Broadcast channel without feedback** consists of an input alphabet $\mathcal{X}$, two output alphabets $\mathcal{Y}$ and $\mathcal{Z}$, and a transition probability $P_{YZ|X}$;
- A $((2^{nR_0}, 2^{nR_y}, 2^{nR_z}), n, P_e^{(n)})$-code consists of
  - three message sets $\mathcal{W}_u = [2^{nR_u}]$ for $u = \{0, y, z\}$,
  - one encoder $\mathcal{W}_0 \times \mathcal{W}_y \times \mathcal{W}_z \to \mathcal{X}^n$;
  - two decoders $\mathcal{Y}^n \to \mathcal{W}_0 \times \mathcal{W}_y$ and $\mathcal{Z}^n \to \mathcal{W}_0 \times \mathcal{W}_z$,

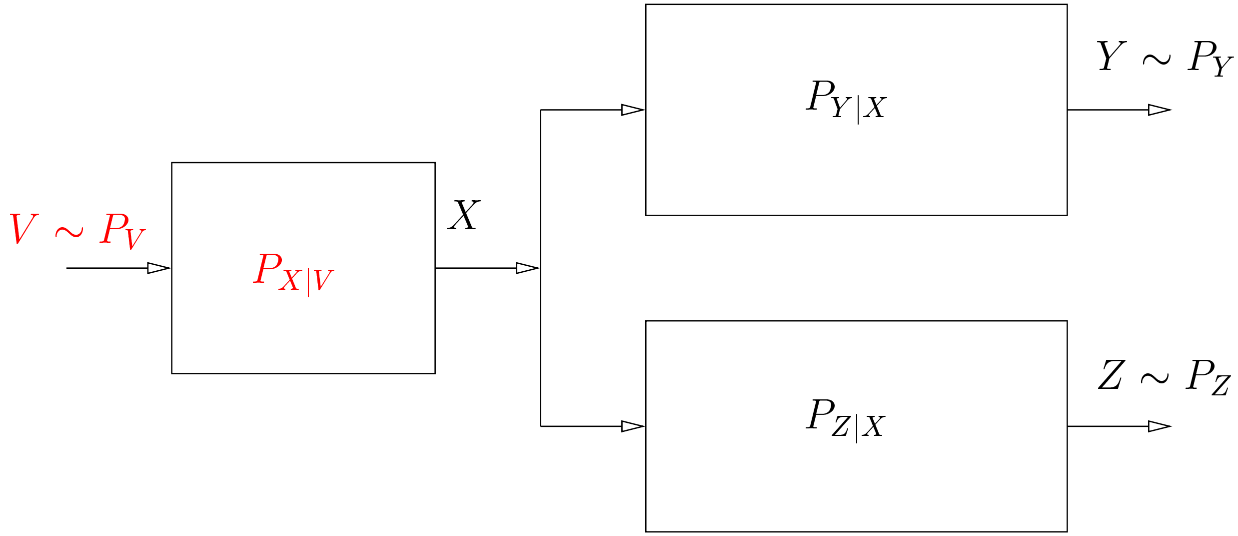
  and probability of error, with (w_0, w_y, w_z) uniformly distributed,
  $P_e^{(n)} = \Pr[\widehat{w}_0 \neq w_0 \text{ or } \widehat{w}_y \neq w_y \text{ or } \widehat{w}_z \neq w_z]$;
- The capacity region is the convex closure of rates $(R_0, R_y, R_z) \in \mathbb{R}_+^3$ for which there exists a sequence of $((2^{nR_0}, 2^{nR_y}, 2^{nR_z}), n, P_e^{(n)})$-codes with $P_e^{(n)} \to 0$.



- "$P_e^{(n)} \to 0$" depends only on the (conditional) marginals

$$P_{Y|X} = \int_z P_{YZ|X} \quad \text{and} \quad P_{Z|X} = \int_y P_{YZ|X}$$

- The capacity region of a **general BC** is NOT known.
- If channel $P_{Y|X}$ is "*stronger*" (classification by Korner-Marton 1975) than channel $P_{Z|X}$ then ...



The channel is said to be:

- **degraded**: $\exists\, \widetilde{P}_{Y|Z} : \sum_y \widetilde{P}_{Y|Z} P_{Y|X} = P_{Z|X}$,
- **less noisy**: $I(V;Z) \leq I(V;Y)$ for all P_{VX},
- **more capable**: $I(X;Z) \leq I(X;Y)$ for all P_X

Clearly {degraded} $\subset$ {less noisy} $\subset$ {more capable} $\subset$ {general BC}.
(deg.:Bergmans 1973, Gallager 1974), (l.n.: Korner-Marton 1975)
(m.c.: A.El Gamal 1979) (general:???).

Capacity region for more capable BCs (A.El Gamal 1979)

If $I(X;Z) \leq I(X;Y)$ for all P_X then

$$C^{(\mathrm{mc})} = \begin{cases} R_0 + R_y + R_z \leq & \min\{I(X;Y)\,,\, I(V;Z) + I(X;Y|V)\} \\ R_0 + R_z \leq & I(V;Z) \end{cases}$$

where $V \to X \to (Y, Z)$ and $|\mathcal{V}| \leq |\mathcal{X}| + 2$.

NB. The capacity is also known in the following cases:

- the channel has **one deterministic component** (Marton 1979, Gelanfd-Pinsker 1980) and
- $R_y = 0$ or $R_z = 0$, the so called **degraded message set** case, (Korner-Marton 1977).
- **sum and product of reversely degraded** BCs (A.El Gamal 1980).

The best inner bound (Marton 1979)

For a general two-user BC, the best inner bound is:

$$\mathcal{I} = \begin{cases} R_0 + R_y \leq & I(WU;Y) \\ R_0 + R_z \leq & I(WV;Z) \\ R_0 + R_y + R_z \leq & \min\{I(W;Y), I(W;Z)\} + \\ & + I(U;Y|W) + I(V;Z|W) - I(U;V|W) \end{cases}$$

where $(W, U, V) \to X \to (Y, Z)$.

No bounds on the cardinality of $\mathcal{W}, \mathcal{U}, \mathcal{V}$ ...

The best outer bound (Korner-Marton 1979)

For a general two-user BC, the best outer bound is

$$\mathcal{O} = \mathcal{O}_y \cap \mathcal{O}_z$$

where (exchange the role of the users to obtain $\mathcal{O}_z$)

$$\mathcal{O}_y = \begin{cases} R_y \leq & I(X;Y) \\ R_z \leq & I(V;Z) \\ R_y + R_z \leq & I(X;Y|V) + I(V;Z) \end{cases}$$

where $V \to X \to (Y,Z)$ and $|\mathcal{V}| \leq |\mathcal{X}| + 2$.

Remarks:

- If the channel is degraded, or less noisy, the capacity is

$$C^{(\mathrm{deg})} = \begin{cases} R_y \leq & I(X;Y|V) \\ R_0 + R_z \leq & I(V;Z) \end{cases}$$

  where $V \to X \to (Y,Z)$ and $|\mathcal{V}| \leq |\mathcal{X}|$ (Gallager 1974).

- In general $C^{(\mathrm{deg})}$ in neither an inner bound nor an outer bound!

- On the contrary, $C^{(\mathrm{mc})}$ is ALWAYS an inner bound, i.e.,

$$C^{(\mathrm{mc})} \equiv \mathcal{I}_y \triangleq \mathcal{I}|_{X=U,\, W=V}$$

  We shall refer to $\mathcal{I}_y$ as the "more-capable-like" inner bound.

AWGN BCs with RX-CSI only

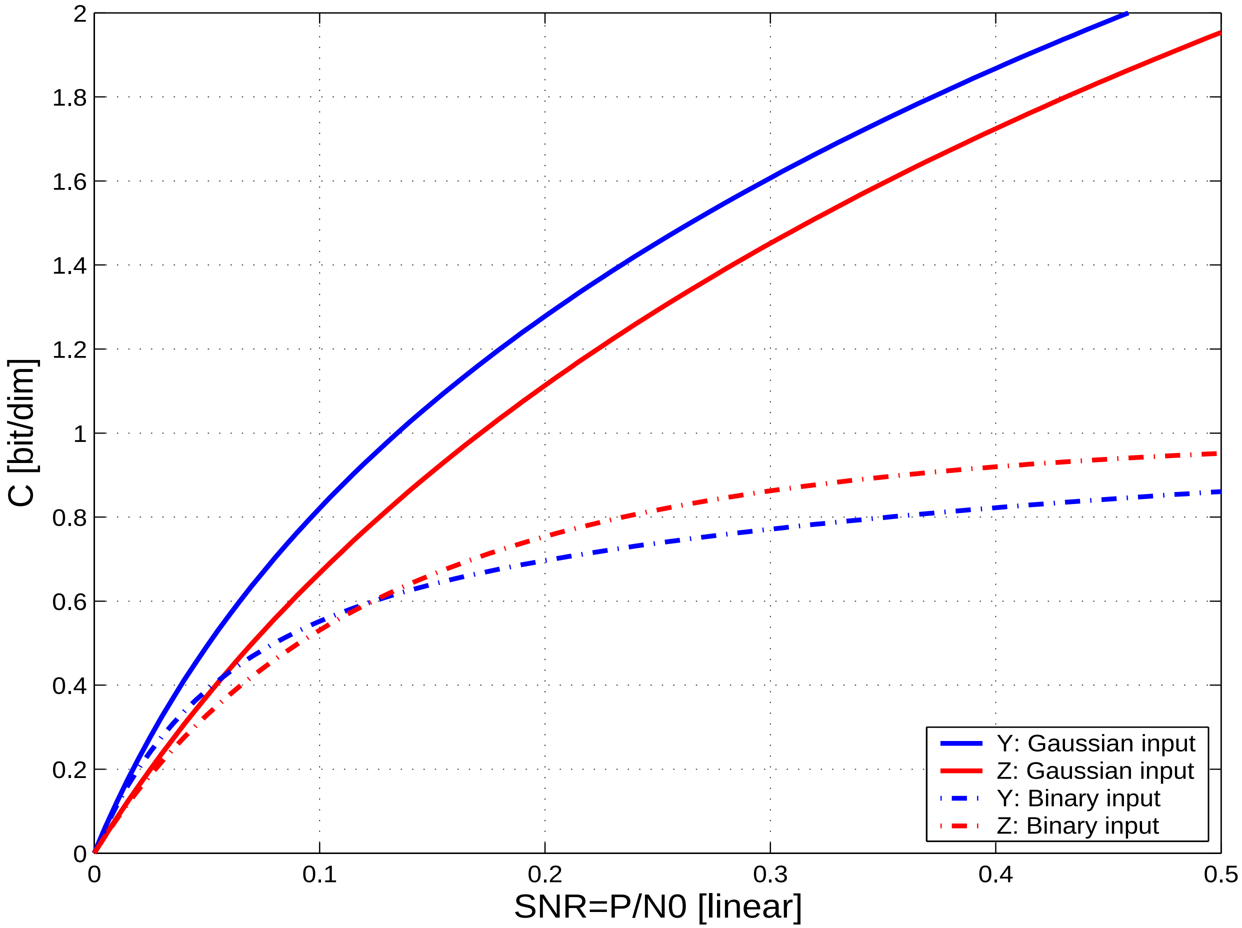
- We consider complex-valued fading *Gaussian* BCs

$$\begin{cases} Y & = & A\,X + N_y \\ Z & = & B\,X + N_z \end{cases}$$

  - A and B: ergodic processes known at the receivers only,
  - $N_u \sim \mathcal{N}_c(0, N)$: Gaussian noise at receiver $u \in \{y, z\}$ and
  - X: subject to the power constraint $\mathrm{E}[|X|^2] \leq P$.

- The output of channel-Y (resp. Z) is the pair (Y, A) (resp. (Z, B)).
- (W, U, V, X) are independent of A and B.
- In general, this channel is not *more capable*.

Example: $|A|^2 \in \{2, 16\}$ and $|B|^2 \in \{4, 8\}$ equiprobable



Remarks:

- Capacity for A and B constant (unfaded case) is achieved by

$$\begin{bmatrix} V \\ X \end{bmatrix} \sim \mathcal{N}_c \left(\mathbf{0}, P \begin{bmatrix} 1-\alpha & \sqrt{1-\alpha} \\ \sqrt{1-\alpha} & 1 \end{bmatrix} \right) \text{ for } \alpha \in [0,1]$$

- The "more-capable-like" inner bound region (with $R_0 = 0$) computed for the above jointly Gaussian distribution reduces to

$$\mathcal{I}_y = \bigcup_{\alpha \in [0,1]} \begin{cases} R_z & \leq C_z(P) - C_z(\alpha P) \\ R_y + R_z & \leq C_y(P) \\ R_y + R_z & \leq C_z(P) + C_y(\alpha P) - C_z(\alpha P) \end{cases}$$

where $C_y(P) = \mathrm{E}\left[\log(1 + |A|^2 P/N)\right]$ is the single user capacity which is achieved by $X \sim \mathcal{N}_c(0, P)$.

- Certain authors claim (GLOBECOM 2000) that if $C_y(P) \geq C_z(P)$ then the degraded region, computed for jointly Gaussian input, is an inner bound for the general fading case! Now, with Gaussian input, stripping at the receiver with largest single user capacity is possible if

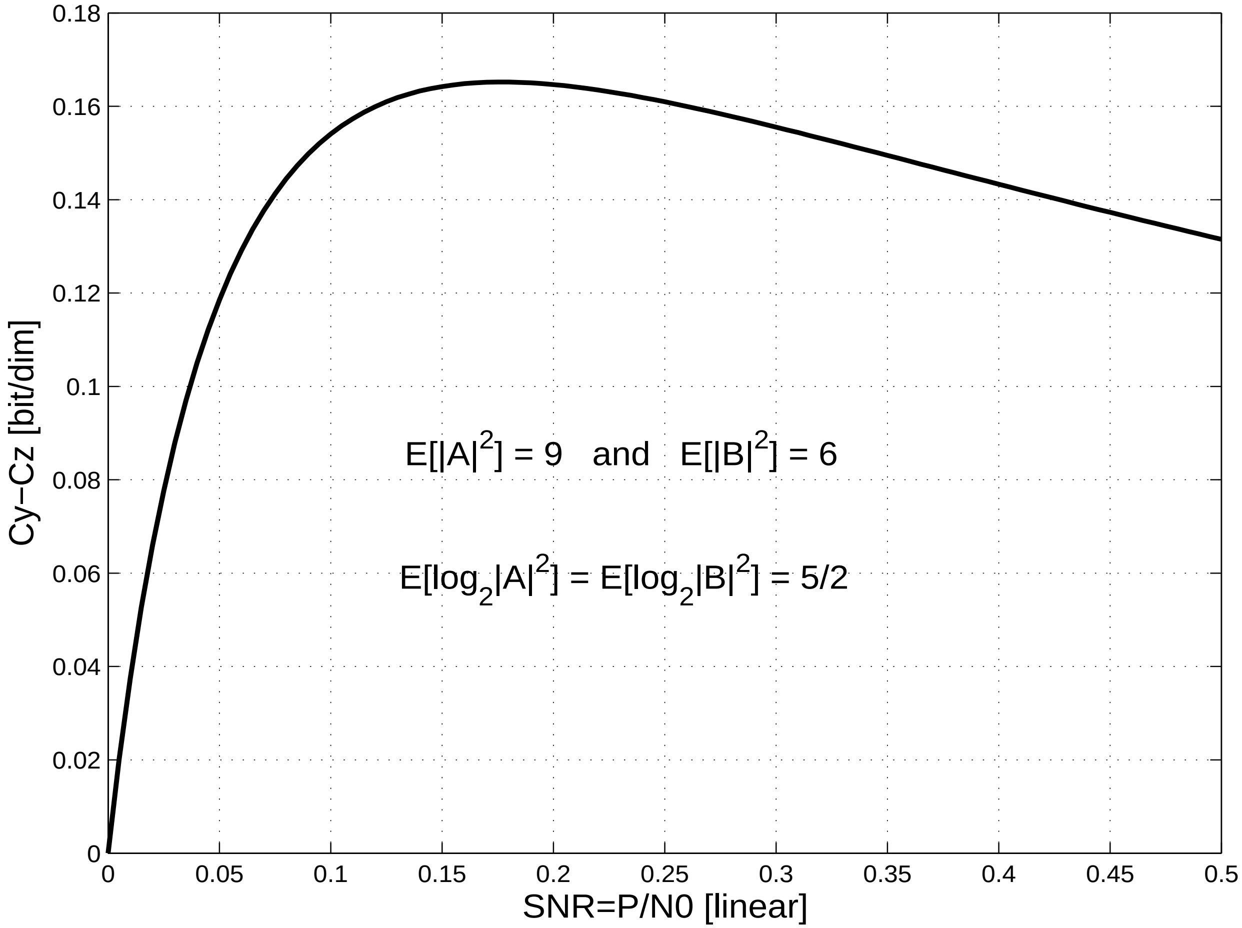
$$C_y(P) - C_z(P) \geq C_y(\alpha P) - C_z(\alpha P) \quad \forall \alpha \in [0, 1]$$

in other words, if

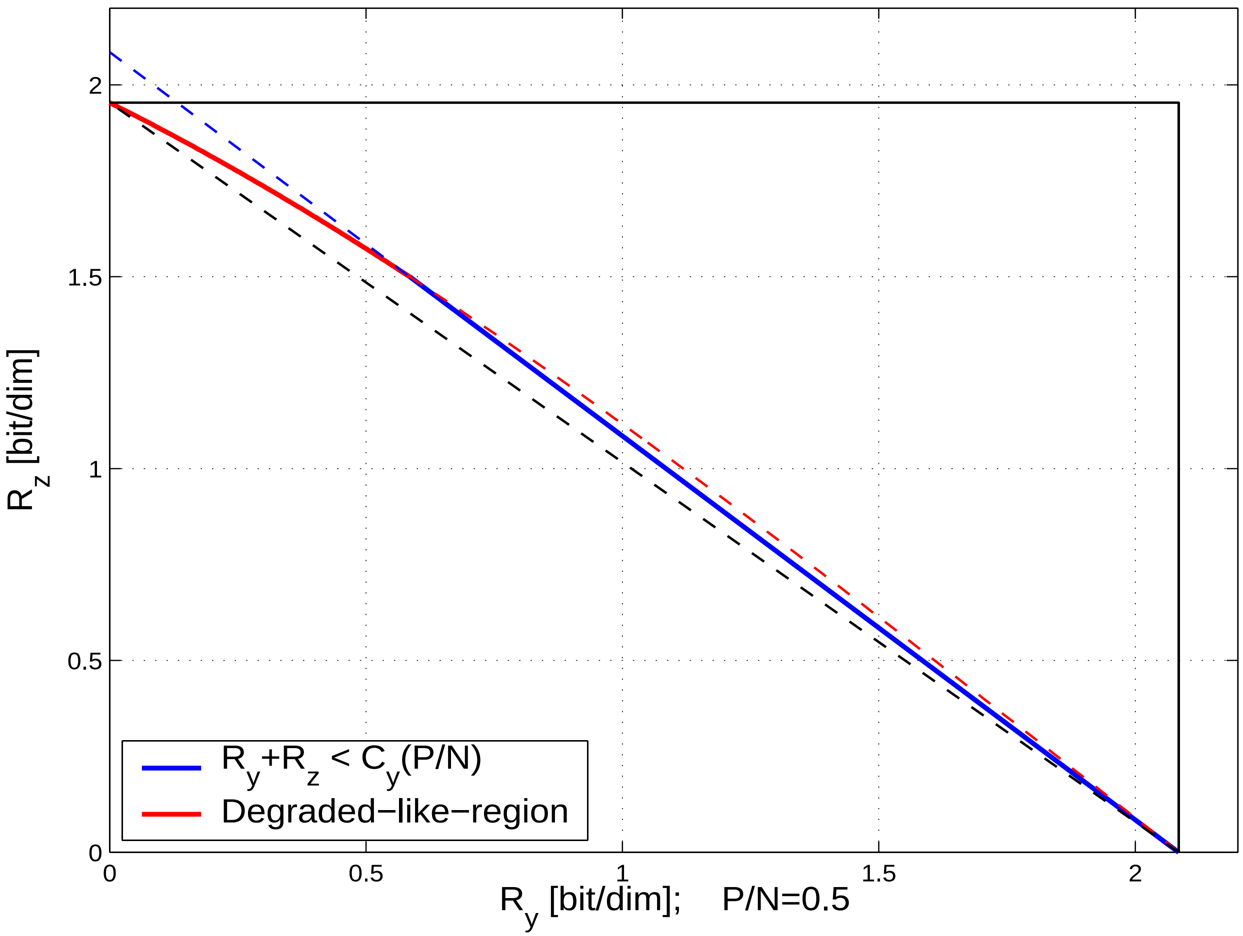
$$d(\alpha) \stackrel{\Delta}{=} C_y(\alpha P) - C_z(\alpha P)$$

as a function of $\alpha \in [0, 1]$ achieves its absolute maximum for $\alpha = 1$.

Example: $|A|^2 \in \{2, 16\}$ and $|B|^2 \in \{4, 8\}$ equiprobable



Example: $|A|^2 \in \{2, 16\}$ and $|B|^2 \in \{4, 8\}$ equiprobable



How far are we from capacity?

- The outer bound is of use if the union over **ALL** $P_{V\,X}$ is taken ...
- IDEA (our converse!):

$$\bigcup_{P_{V\,X}:\mathrm{E}[|X|^2]\leq P} = \bigcup_{h:h\leq g(P)} \bigcup_{P_{V\,X}:\mathrm{E}[|X|^2]\leq P\,,\,H(X|V)=h}$$

since

$$h = H(X|V) \leq H(X) \leq \log(\pi \mathrm{e} \mathrm{Var}[X]) \leq \log(\pi \mathrm{e} P) \stackrel{\Delta}{=} g(P)$$

A lower bound to the (conditional) entropy

Similarly to Bergman's proof (1974):

$$
\begin{aligned}
H(Z|VB) &\overset{\text{entropy power ineq.}}{\geq} \log\left(e^{H(BX|VB)} + e^{H(N_z|VB)}\right) \\
&= \log\left(e^{\mathrm{E}_B[H(b\cdot X|V,B=b)]} + e^{H(N_z)}\right) \\
&\overset{P_{VXB}=P_B P_V P_{X|V}}{=} g\left(\frac{e^{\mathrm{E}_B[\log(|B|^2)]H(X|V)}}{\pi e} + \frac{e^{g(N)}}{\pi e}\right) \\
&\overset{\text{Jensen ineq.}}{\geq} \mathrm{E}_B\left[g\left(|B|^2\frac{e^{H(X|V)}}{\pi e} + N\right)\right]
\end{aligned}
$$

An upper bound to the (conditional) entropy

$$
\begin{aligned}
H(Z|VB) &\overset{\mathcal{N}_c(\cdot,\cdot)\text{ max. H}}{\leq} \mathrm{E}_{B,V}\left[g\left(\sigma^2_{Z|VB}(v,b)\right)\right] \\
&\overset{P_{VXB}=P_B P_V P_{X|V}}{=} \mathrm{E}_{B,V}\left[g\left(N+|B|^2\sigma^2_{X|V}(v)\right)\right] \\
&\overset{\text{Jensen ineq.}}{\leq} \mathrm{E}_{B}\left[g\left(N+|B|^2\,\mathrm{E}[\sigma^2_{X|V}(v)]\right)\right]
\end{aligned}
$$

where $\sigma^2_{X|V}(v)$ is the variance of the random variable "$X|V=v$", i.e.,

$$
\sigma^2_{X|V}(v) = \int P_{X|V}(x|v)|x|^2 - \left|\int P_{X|V}(x|v)x\right|^2
$$

The above chain holds with = iif "$X|V=v$"$\sim \mathcal{N}_c(\cdot, g^{-1}(H(X|V)))$.

Everything can be expressed in terms of $H(X|V)$:

$$
\begin{aligned}
I(Z;V|B) &= H(Z|B) - H(Z|VB) \\
&\leq \mathrm{E}_B\left[g\left(N + |B|^2 P\right)\right] - \mathrm{E}_B\left[g\left(N + |B|^2 \frac{\mathrm{e}^{H(X|V)}}{\pi \mathrm{e}}\right)\right] \\
I(X;Y|VA) &= H(Y|VA) - H(Y|XA) \\
&\leq \mathrm{E}_A\left[g\left(N + |A|^2 \frac{\mathrm{e}^{H(X|V)}}{\pi \mathrm{e}}\right)\right] - g(N)
\end{aligned}
$$

Hence, for fading AWGN BCs with RX-CSI only

$$
\bigcup_{P_{VX}:\mathrm{E}[|X|^2]\leq P\,,\,H(X|V)=h} = \bigcup_{P_{VX}:X\sim\mathcal{N}_c(0,P)\,,\,X|V\sim\mathcal{N}_c(\cdot,g^{-1}(h))}
$$

$$
\bigcup_{h:h\leq g(P)} = \bigcup_{\alpha\in[0,1]:h=g(\alpha P)}
$$

We get

$$O_y = \bigcup_{\alpha\in[0,1]} \begin{cases} R_z & \le C_z(P) - C_z(\alpha P) \\ R_y & \le C_y(P) \\ R_y + R_z & \le C_z(P) + C_y(\alpha P) - C_z(\alpha P) \end{cases}$$

and recall that

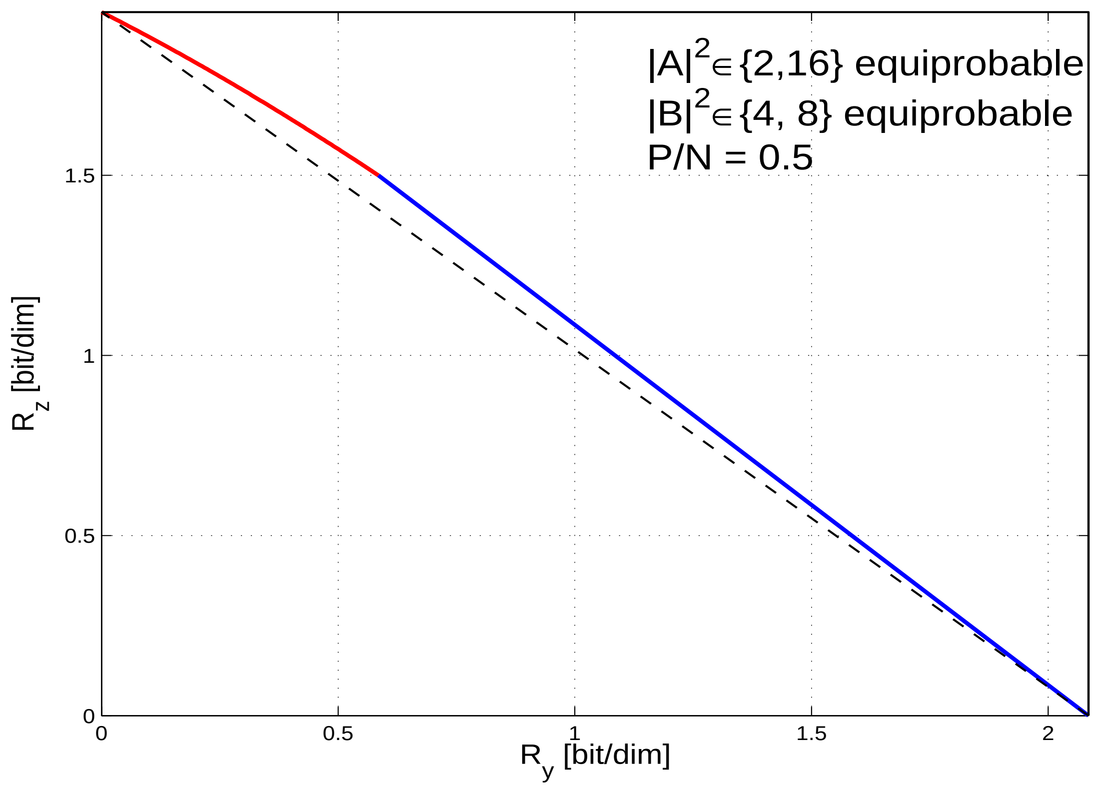
$$\mathcal{I}_y = \bigcup_{\alpha\in[0,1]} \begin{cases} R_z & \le C_z(P) - C_z(\alpha P) \\ R_y + R_z & \le C_y(P) \\ R_y + R_z & \le C_z(P) + C_y(\alpha P) - C_z(\alpha P) \end{cases}$$

it follows that ...

Main results

- **Theorem 1.** *The "degraded" region computed for Gaussian input is the capacity region for fading AWGN BCs with RX-CSI for which the condition for successive cancellation is verified, i.e.,* $d(\alpha) = C_y(\alpha\, P) - C_z(\alpha\, P)$ *has absolute maximum for* $\alpha = 1$.
- **Theorem 2.** *The "more-capable" region computed for Gaussian input is the capacity region for fading AWGN BCs with RX-CSI for which* $d(\alpha) = C_y(\alpha\, P) - C_z(\alpha\, P) \geq 0$ *for all* $\alpha \in [0, 1]$.
- **Theorem 3.** *If the fading AWGN BCs with RX-CSI BC is more-capable then Gaussian input is optimal.*

For fading AWGN BCs with RX-CSI, Theorem 2 comprises channels that are NOT more-capable, i.e., for which no coding theorem exists!



We are working the case "$d(\alpha)$ is not non-negative in $\alpha \in [0,1]$ and has not its absolute maximum for $\alpha = 1$" ($\notin \mathrm{Th.1} \cup \mathrm{Th.2}$)

Conclusions

We consider a general fading AWGN BC with RX-CSI:

1. We first derive an inner bound region which is valid for any fading distribution and has the flavor of the "more capable" capacity region.

2. From the inner bound with Gaussian inputs: the best receiver (largest single user capacity) must decode jointly the two messages and successive cancellation might incur inherent degradation.

3. Then, by using the entropy power inequality, we show that Gaussian inputs exhaust Korner-Marton's outer bound region.

4. We conclude by stating conditions under which inner and outer bound meet, yielding thus the capacity region.

5. Our capacity result holds for channels that neither degraded NOR more capable.