Lipschitz Quotients

[S. Bates], W.B.J., J. Lindenstrauss,

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Background

Benyamini-Lindenstrauss, *Geometric nonlinear functional analysis,* **AMS Colloquium Publications** (1999).

A mapping $f : X \to Y$, is a **co-Lipschitz** map provided there is a constant C so that for all x in X and all r,

$$f[B_r(x)] \supset B_{r/C}(f(x)).$$

co-Lip(f) denotes the smallest such C.

A co-Lipschitz map is open in a Lipschitz sense. A function is a **Lipschitz quotient map** if it is both Lipschitz and co-Lipschitz. Thus a one-to-one map is a Lipschitz quotient mapping iff it is bi-Lipschitz.

If there is a Lipschitz quotient map f from X onto Y, we say Y is a Lipschitz quotient of X (λ-Lipschitz quotient if Lip(f)$\cdot$co-Lip(f) $\leq \lambda$).

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Related concept [David-Semmes]

$T : X \to Y$ is ball non collapsing provided $\exists \ \omega > 0$ s.t. $\forall x \in X \ \ \exists y \in Y$ s.t.

$$TB_r(x) \supset B_{\omega r}(y).$$

Example of a ball non collapsing Lipschitz map which is NOT a Lipschitz quotient: fold a sheet of paper.

Examples of Lipschitz quotients maps in $\mathbb{R}^n$

From $\mathbb{R}$ to $\mathbb{R}$ they must be bi-Lipschitz.

From $\mathbb{R}^2$ to $\mathbb{R}$, they carry considerable structure. For example, the number of components of $f^{-1}(t)$ is bounded and each component of $f^{-1}(t)$ separates the plane.

Define f on $\mathbb{R}^2$ to be the homogenous extension to $\mathbb{R}^2$ of the mapping $z \mapsto z^n$ on the unit circle. This is a Lipschitz quotient mapping which is "typical"– EVERY Lipschitz quotient map on $\mathbb{R}^2$ can be written as $P \circ h$ where P is a (complex) polynomial and h is a homeomorphism of $\mathbb{R}^2$.

From $\mathbb{R}^3$ to $\mathbb{R}^2$, $f^{-1}(t)$ can contain a plane but cannot be a plane. [Csornyei]

References for non linear quotients in $\mathbb{R}^n$: [JLPS], [Csornyei], [Heinrich], [Randriantoanina], [Maleva].

A mapping $f : X \to Y$, is a **co-Lipschitz** map provided there is a constant C so that for all x in X and all r,

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Let $f : X \to Y$ be a surjective Lipschitz map. Then co-Lip$(f) < \lambda$ iff for all finite weighted trees T, $t_0 \in T$, $g : T \to Y$ with Lip$(g) \leq 1$, and $x_0 \in X$ with $f(x_0) = g(t_0)$, there exists a lifting $\tilde{g} : T \to X$ so that $\tilde{g}(t_0) = x_0$, Lip$(\tilde{g}) \leq \lambda$, and $g = f \circ \tilde{g}$.

For Banach spaces, the fundamental question is:

If Y is a Lipschitz quotient of X, [when] must Y be a linear quotient of X?

In every case where we know "Y is a Lipschitz quotient of $X \implies Y$ is a linear quotient of X" we also know that the existence of a ball non collapsing Lipschitz map from X to Y implies that Y is a linear quotient of X.

We do not know whether in general the existence of a ball non collapsing Lipschitz map from X to Y implies that Y is a Lipschitz quotient of X.

f admits **affine localization** if for every $\varepsilon > 0$ and every ball $B \subset X$ there is a ball $B_r \subset B$ and an affine function $L : X \to Y$ so that

$$\|f(x) - Lx\| \leq \varepsilon r, \qquad x \in B_r.$$

The couple (X, Y) has the **approximation by affine property (AAP)** if every Lipschitz map from X into Y admits affine localization.

AAP is enough to ensure that if f is a Lipschitz quotient map from X to Y then (for ε small enough) the linear approximant is a linear quotient map; and if f is a λ-Lipschitz quotient, (i.e., Lip(f)·co-Lip(f) $\leq \lambda$) the linear approximant is a $\lambda + \epsilon$ linear quotient.

f admits δ-**affine localization** if for every $\varepsilon > 0$ and every ball $B \subset X$ there is a ball $B_r \subset B$ and an affine function $L : X \to Y$ so that

$$\|f(x) - Lx\| \le \varepsilon r, \qquad x \in B_r$$

and $r \ge \delta(\varepsilon)\,\text{radius}(B) \quad (\delta(\varepsilon) > 0 \;\; \forall \varepsilon > 0)$.

The couple (X, Y) has the **uniform approximation by affine property (UAAP)** if there is a function $\delta(\varepsilon) > 0$ so that every Lipschitz map with constant one from X into Y admits δ-affine localization.

This notion (not the terminology) was introduced by [David-Semmes]. They proved that (X, Y) has the UAAP if both spaces are finite dimensional.

Theorem. *The couple (X, Y) has the UAAP iff one of the spaces is super-reflexive and the other is finite dimensional.*

A Banach space is super-reflexive iff it is isomorphic to a uniformly convex space iff it is isomorphic to a uniformly smooth space.

Repeat:

(1) If (X, Y) has the AAP and Y is a λ-Lipschitz quotient of X then Y is a $(\lambda+\epsilon)$-isomorphic to a linear quotient of X.

(2) If X is super-reflexive and Y is finite dimensional, then (X, Y) has the AAP.

Therefore:

(3) If X is super-reflexive and Z is a λ-Lipschitz quotient of X, then every finite dimensional quotient of Z is $(\lambda+\epsilon)$-isomorphic to a linear quotient of X ($\iff$ every finite dimensional subspace of Z^* is $(\lambda+\epsilon)$-isomorphic to a subspace of X^*).

(4) If Z is a λ-Lipschitz quotient of a Hilbert space, then Z is λ-isomorphic to a Hilbert space.

(5) If Z is a λ-Lipschitz quotient of L_p, $1 \leq p < \infty$, then Z is λ-isomorphic to a quotient of L_p.

The classification of Lipschitz quotients of ℓ_p, $1 < p \neq 2 < \infty$ is open. A Lipschitz quotient of ℓ_p is a Lipschitz quotient of L_p. For $2 \leq r < p < \infty$, the space ℓ_r is linear quotient of L_p but is not a Lipschitz quotient of ℓ_p.

There are known to exist non separable Banach spaces X and Y which are bi-Lipschitz equivalent but not isomorphic [Aharoni-Lindenstrauss]. It turns out that Y is not even a isomorphic to a linear quotient of X.

It may be that separable Banach spaces that are bi-Lipschitz equivalent must be isomorphic. The results on quotients suggest that if X is separable and Y is a Lipschitz quotient of X, then Y is isomorphic to a linear quotient of X (at least if X is one of the classical examples of Banach spaces). However,....

Metric trees and Lipschitz Quotients of spaces containing ℓ_1 [JLPS]

A metric space X is a *metric tree* provided it is complete, metrically convex, and there is a unique arc (which then by metric convexity must be a geodesic arc) joining each pair of points in X. There is an equivalent constructive definition of a separable metric tree, which we term an SMT because the equivalence to separable metric tree is not needed. Using the constructive definition, it is more-or-less clear that every metric tree is obtained by starting with a (possibly infinite) weighted tree and filling in each edge with an interval whose length is the distance between the vertices of the edge.

The ℓ_1 union of two metric spaces

If $X \cap Y = \{p\}$, the ℓ_1 union is $(X \cup Y, d)$, where the metric d agrees with d_X on X, d agrees with d_Y on Y, and if $x \in X$, $y \in Y$, then $d(x, y)$ is defined to be $d_X(x, p) + d_Y(p, y)$.

Construction of an SMT

Let I_1 be a closed interval or a closed ray and define $T_1 := I_1$. The metric space T_1 is the first approximation to our SMT. Having defined T_n, let I_{n+1} be a closed interval or a closed ray whose intersection with T_n is an end point, p_n, of I_{n+1}, and define $T_{n+1} := T_n \cup_1 I_{n+1}$. The completion, T, of $\bigcup_{n=1}^{\infty} T_n$ is an SMT. If each I_n is a ray with end point p_{n-1} for $n > 1$ and the set $\{p_n\}_{n=1}^{\infty}$ of nodal points is dense in T, then we call T an 'ℓ_1 tree' and say that $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ describe an allowed construction of T.

Proposition. *Let T be an ℓ_1 tree. Then every separable, complete, metrically convex metric space is a 1-Lipschitz quotient of T.*

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Let Y be a separable, complete, metrically convex metric space. Build the desired Lipschitz quotient map by defining it on T_n by induction (where $\{I_n\}_{n=1}^{\infty}$, $\{T_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ describe an allowed construction of T).

Suppose you have a 1-Lipschitz map $f : T_n \to Y$, and y is taken from some countable dense subset Y_0 of Y. Extend f to T_{n+1} by mapping I_{n+1} to a geodesic arc $[f(p_n), y]$ which joins $f(p_n)$ to y; f is an isometry on $\{z \in I_{n+1} : d(p_n, z) \leq d(f(p_n), y)\}$ and f maps points on I_{n+1} whose distance to p_n is larger than $d(f(p_n), y)$ to y. This makes f act like a Lipschitz quotient at p_n relative to $[f(p_n), y]$. Since the nodal points are dense in T, a judicious selection of the points from Y_0 will produce a 1-Lipschitz quotient mapping.

Lemma. *Assume that X and Y are 1-absolute Lipschitz retracts which intersect in a single point, p. Then $X \cup_1 Y$ is also a 1-absolute Lipschitz retract.*

A metric space X is a 1-absolute Lipschitz retract if and only if X is metrically convex and every collection of mutually intersecting closed balls in X have a common point.

Corollary. *Let T be an SMT. Then T is a 1-absolute Lipschitz retract.*

Proposition. *Every SMT is a* 1*-Lipschitz quotient of* $C(\Delta)$*, where* Δ *is the Cantor set* $\{-1,1\}^{\mathbb{N}}$*.*

Let r_n be the nth coordinate projection on Δ. In the space $C(\Delta)$, the sequence $\{r_n\}_{n=1}^{\infty}$ is isometrically equivalent to the unit vector basis of ℓ_1. For $n = 1, 2, \ldots$, let E_n be the functions in $C(\Delta)$ which depend only on the first n coordinates. Notice that if x is in E_n and $m > n$ then for all real t, $\|x + tr_m\| = \|x\| + |t|$. In other words, if I is a ray in the direction of r_m emanating from a point p in E_n, then, in $C(\Delta)$, the set $E_n \cup I$ is an ℓ_1 union of E_n and I. That $\{r_n\}_{n=1}^{\infty}$ acts like the ℓ_1 basis *over* $C(\Delta)$ is the key to proving the above proposition. The lemma is used to extend a 1-Lipschitz mapping from $E_n \cup I$ into the SMT to a 1-Lipschitz mapping from E_m into the SMT.

Corollary. *If Y is a separable, complete, metrically convex metric space, then Y is a 1-Lipschitz quotient of $C(\Delta)$.*

In particular, every separable Banach space is a 1-Lipschitz quotient of $C(\Delta)$, but it is well known that e.g. ℓ_1 is NOT isomorphic to a linear quotient of $C(\Delta)$.

From known results in the linear theory it then follows:

Theorem. *Let X be a separable Banach space which contains a subspace isomorphic to ℓ_1 and let $\varepsilon > 0$. Then every separable, complete, metrically convex metric space is a $(1+\varepsilon)$-Lipschitz quotient of X. (Moreover, the Gateaux derivative of the Lipschitz quotient mapping has rank at most one wherever it exists.)*

$f : X \to \mathbb{R}^n$ is *measure non collapsing* provided $\mu f(B_r(x)) \geq \delta r^n$; f is *ball non collapsing* if $f(B_r(x)) \supset B_{\delta r}(y)$ ($\mu =$ Lebesgue measure).

[David-Semmes] if $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz and measure non collapsing then it is ball non collapsing. $\mathbb{R}^m$ can be replaced by any super-reflexive space [BJLPS].

If X is a separable Banach space containing an isomorph of ℓ_1 then $\exists$ $f : X \to \mathbb{R}^2$ Lipschitz, measure non collapsing, but $f(X)$ is closed and has empty interior (hence f is NOT ball non collapsing).

Problems and concluding remarks

(1) Classify the metric spaces which are Lipschitz quotients of a Hilbert space. In particular, must each such space bi-Lipschitz embed into a Hilbert space?

(2) Classify the metric spaces which are Lipschitz quotients of a subset of a Hilbert space.

We know only:

(2.1) There are metric spaces which are not Lipschitz quotients of any subsets of a Hilbert space.

(2.2) There are metric spaces which are Lipschitz quotients of subsets of a Hilbert space but which do not bi-Lipschitz embed into a Hilbert space.

Quantitative versions of problem 2 might be interesting.

(3) Estimate, in terms of λ and N, the largest Euclidean distortion of an N-point metric space which is a λ-Lipschitz quotient of a subset of a Hilbert space.

Recall the definition of δ-affine localization:

f admits δ-**affine localization** if for every $\varepsilon > 0$ and every ball $B \subset X$ there is a ball $B_r \subset B$ and an affine function $L : X \to Y$ so that

$$\|f(x) - Lx\| \leq \varepsilon r, \qquad x \in B_r$$

and $r \geq \delta(\varepsilon)\,\text{radius}(B)$.

(4) Is there an analogue of δ-affine localization for Lipschitz mappings between other classes of metric spaces? Maybe metric groups? Are there conditions which will guarantee that a pair (X, Y) has the analogue of UAAP?