We can add cooperation, coordination, and get a smoother topic.
Correlation
- Correlation
- Communication
- Correlation
- Communication
- Complexity
Correlation

Communication

Complexity

Competition
We can add
We can add

**Cooperation, Coordination, Concealed Correlation,**...,...
We can add

Cooperation, Coordination, Concealed Correlation,...,

and get a smoother topic:
We can add

Cooperation, Coordination, Concealed Correlation,...,...

and get a smoother topic: $C^\infty$
The classical paradigm of game theory assumes full rationality of the interactive agents.
The classical paradigm of game theory assumes full rationality of the interactive agents.

In particular, it often assumes unlimited computational power.
General Introduction

The classical paradigm of game theory assumes full rationality of the interactive agents.

In particular, it often assumes unlimited computational power.

However, there are many decision problems and games for which it is impossible to assume that the agents (players) can either *compute* or *implement* an optimal (or best response or approximate optimal) strategy.
Design and Implementation

It is often argued that evolutionary self selection leaves us with agents that act optimally. Therefore, the complexity of finding an optimal (or approximate optimal) strategy is conceptually less disturbing. However, the computational feasibility and the computational cost of implementing various strategies should be taken into account.
Design and Implementation

It is often argued that evolutionary self selection leaves us with agents that act optimally.
It is often argued that evolutionary self selection leaves us with agents that act optimally.

Therefore, the complexity of finding an optimal (or approximate optimal) strategy is conceptually less disturbing.
It is often argued that evolutionary self selection leaves us with agents that act optimally.

Therefore, the complexity of finding an optimal (or approximate optimal) strategy is conceptually less disturbing.

However, the computational feasibility and the computational cost of implementing various strategies should be taken into account.
Design and Implementation

One can imagine scenarios where the design and choice of strategies is by rational agents with (essentially) unlimited computation power and the selected strategies need be implemented by players with restricted computational resources.
Design and Implementation

One can imagine scenarios where the design and choice of strategies is by rational agents with (essentially) unlimited computation power and the selected strategies need be implemented by players with restricted computational resources.

- A corporation
- The USA Navy
- A soccer team
- A chess player
- A computer network
In theory, mixed and behavioral strategies are equivalent (in games of perfect recall). In practice, mixed and behavioral strategies are not equivalent. Recall that a mixed strategy reflects uncertainty regarding the chosen pure strategy, and a behavioral strategies randomizes actions at the decision nodes.
In theory, mixed and behavioral strategies are equivalent (in games of perfect recall)
Pure Mixed and Behavioral

- In theory, mixed and behavioral strategies are equivalent (in games of perfect recall).
- In practice, mixed and behavioral strategies are not equivalent.
Pure Mixed and Behavioral

- In theory, mixed and behavioral strategies are equivalent (in games of perfect recall).
- In practice, mixed and behavioral strategies are not equivalent.

Recall that

- A mixed strategy reflects uncertainty regarding the chosen pure strategy, and
- A behavioral strategies randomizes actions at the decision nodes.
Strategies in the Repeated Game

- The number of pure strategies of the repeated game grows at a double exponential rate in the number of repetitions.
- Many of the strategies are not implementable by reasonable sized computing agents.
General Objective

The impact on

strategic interactions
the value and equilibrium payoffs

of variations of the game where players are restricted to employ

Simple Strategies
Simple Strategies

Computable Strategies
Simple Strategies

Finite Automata
Sample of References: F.A.

- Ben-Porath (1993) *J. of Econ. Theory*
  Repeated Games with Finite Automata

- Neyman (1985) *Economics Letters*
  Bounded Complexity Justifies Cooperation in the Finitely Repeated Prisoner’s Dilemma

- Neyman (1997) *in Cooperation: Game-Theoretic Approaches, Hart and Mas Colell, (eds.)*
  Cooperation, Repetition, and Automata

  Finitely Repeated Games with Finite Automata
References: Finite Automata

  Two-person R. Games with Finite Automata

  Stochastic Games with Automata (Hebrew)

- Aumann (1981) in *Essays in Game Theory and Mathematical Economics in Honor of O. Morgenstern*
  Survey of Repeated Games
References: F. A.

References: Finite Automata

- Kalai (1990) *in Game Theory and Applications, Ichiishi, Neyman and Tauman (eds.)*
  Bounded Rat. and Strat. Complexity in R. G.

- Piccione (1989) *Journal of Economic Theory*
  Finite Automata Eq. with Discounting and Unessential Modifications of the Stage Game

- Rubinstein (1986) *Journal of Economic Theory*
  Finite Automata Play the R. P.’s Dilemma

- Zemel (1989) *Journal of Economic Theory*
  Small Talk and Cooperation: A Note on Bounded Rationality
Simple Strategies Recall

Bounded Recall
References: Bounded Recall

- Lehrer (1988) *Journal of Economic Theory*  
  R.G.s with Bounded Recall Strategies

- Lehrer (1994) *Games and Economic Behavior*  
  Many Players with Bounded Recall in Infinite Repeated Games
References: Bounded Recall

- Neyman (1997) in *Cooperation: Game-Theoretic Approaches*, *Hart and Mas Colell* (eds.)
  Cooperation, Repetition, and Automata

- Aumann and Sorin (1990) *GEB*
  Cooperation and Bounded Recall

- Bavly and Neyman (forthcoming)
  Concealed Correlation by Boundedly Rational Players
Simple Strategies

Bounded Strategic Entropy
References: Bounded Entropy

Neyman and Okada

- Strategic Entropy and Complexity in Repeated Games
  Games and Economic Behavior (1999)

- Repeated Games with Bounded Entropy
  Games and Economic Behavior (2000)
Simple Strategies

Kolmogorov’s Complexity
References/Origin

- Solomonov (1964) A formal theory of inductive inference, Information and Control
- Kolmogorov (1965) Three approaches to the quantitative definition of information, Problems in Information Transmission
- Chaitin
- Neyman (forthcoming) Finitely Repeated Games with Bounded Kolmogorov’s Strategic Complexity
Simple Strategies

- Computable Strategies
- Finite Automata
- Bounded Recall
- Bounded Strategic Entropy
- Kolmogorov’s Complexity
Notation-Finite Automata
Notation-Finite Automata

\[ M := \max_{a \in A} \min_{b \in B} g(a, b) \]
\[ V := \min_{y \in \Delta(B)} \max_{a \in A} g(a, y) \]
\[ = \max_{x \in \Delta(A)} \min_{b \in B} g(x, b) \]

\[ mm(k_1, k_2) := \min_{\tau \in \Sigma_2(k_2)} \max_{\sigma \in \Sigma_1(k_1)} G(\sigma, \tau) \]
\[ := \min \max (k_1, k_2) \geq \]

\[ Mn(k_1, k_2) := \min_{\tau \in \Delta(\Sigma_2(k_2))} \max_{\sigma \in \Sigma_1(k_1)} G(\sigma, \tau) \]
\[ := Min \max (k_1, k_2) \]
2-P 0-sum FA: The Questions
2-P 0-sum FA: The Questions

Assume $k_2 \geq k_1 \rightarrow \infty$
2-P 0-sum FA: The Questions

Assume \( k_2 \geq k_1 \rightarrow \infty \)

What are the asymptotic relations between the size of \( k_1 \) and \( k_2 \) of the automata of P1 and P2 so that
2-P 0-sum FA: The Questions

Assume $k_2 \geq k_1 \to \infty$

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 so that

\[ Mm(k_1, k_2) = V \]
2-P 0-sum FA: The Questions

Assume $k_2 \geq k_1 \to \infty$
What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 so that

- $M_m(k_1, k_2) = V$
- $M_m(k_1, k_2) = M$
Assume \( k_2 \geq k_1 \to \infty \)
What are the asymptotic relations between the size of \( k_1 \) and \( k_2 \) of the automata of P1 and P2 so that

- \( Mm(k_1, k_2) = V \)
- \( Mm(k_1, k_2) = M \)
- \( Mm(k_1, k_2) = x \quad \text{where} \quad M < x < V \)
Assume $k_2 \geq k_1 \to \infty$

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 so that

- $Mm(k_1, k_2) = V$
- $Mm(k_1, k_2) = M$
- $Mm(k_1, k_2) = x$ where $M < x < V$
- $mm(k_1, k_2) = V$
- $mm(k_1, k_2) = M$
### Table-Finite Automata

<table>
<thead>
<tr>
<th>k_1</th>
<th>k_2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ben-Porath (86, 93)

Neyman (97)
<table>
<thead>
<tr>
<th>$k_2 \geq k_1 \rightarrow \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
### Table-Finite Automata

<table>
<thead>
<tr>
<th>$k_2 \geq k_1 \rightarrow \infty$</th>
<th>$mm(k_1, k_2)$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table-Finite Automata

<table>
<thead>
<tr>
<th>$k_2 \geq k_1 \rightarrow \infty$</th>
<th>$mm(k_1, k_2)$</th>
<th>$\geq$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ben-Porath (86, 93)

Neyman (97)

$k_2 \gg k_1 \log k_1 = o(k_1)$
## Table-Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>$mm(k_1, k_2)$</th>
<th>$\geq$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td>$mm(k_1, k_2)$</td>
<td>$\geq$</td>
<td>$Mm(k_1, k_2)$</td>
</tr>
<tr>
<td>---------------------------------</td>
<td>----------------</td>
<td>-------</td>
<td>----------------</td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td></td>
<td>$V$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ben-Porath (86, 93)</td>
</tr>
</tbody>
</table>
### Table: Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>$mm(k_1, k_2)$</th>
<th>$\geq$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \to \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td></td>
<td>Ben-Porath (86, 93)</td>
</tr>
<tr>
<td>$k_2 \geq k_1^C k_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table: Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>$mm(k_1, k_2)$</th>
<th>$\geq$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td>$Mm(k_1, k_2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C$ s.t. $M$</td>
<td>$\Rightarrow$</td>
<td>$M$</td>
</tr>
</tbody>
</table>

Ben-Porath (86, 93)

Neyman (97)

$V > 0$ fixed $k_2 = 2k_1$
<table>
<thead>
<tr>
<th>$k_2 \geq k_1 \to \infty$</th>
<th>$mm(k_1, k_2)$</th>
<th>$\geq$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td>$V$</td>
<td>Ben-Porath (86, 93)</td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C$ s.t. $M$</td>
<td>$\implies$</td>
<td>$M$</td>
</tr>
<tr>
<td>$k_2 \geq 2^{Ck_1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4C: Correlation, Communication, Complexity, and Competition – p. 25/81
### Table: Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \to \infty$</td>
<td>$mm(k_1, k_2)$</td>
<td>$\geq Mm(k_1, k_2)$</td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td>$V$</td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>$\implies M$</td>
</tr>
<tr>
<td>$k_2 \geq 2^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>Neyman (97)</td>
</tr>
</tbody>
</table>

Ben-Porath (86, 93)
### Table: Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>mm($k_1, k_2$)</th>
<th>$\geq$</th>
<th>Mm($k_1, k_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \to \infty$</td>
<td>$mm(k_1, k_2)$</td>
<td>$\geq$</td>
<td>$Mm(k_1, k_2)$</td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>$\implies$</td>
<td>$M$</td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>$\implies$</td>
<td>$M$</td>
</tr>
<tr>
<td>$k_2 \geq 2^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>$\implies$</td>
<td>$M$</td>
</tr>
<tr>
<td>$k_2 \gg k_1 \log k_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ben-Porath (86, 93)

Neyman (97)
<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td>$mm(k_1, k_2) \geq Mm(k_1, k_2)$</td>
<td></td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td>$V$</td>
<td>Ben-Porath (86, 93)</td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td></td>
</tr>
<tr>
<td>$k_2 \geq 2^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>Neyman (97)</td>
</tr>
<tr>
<td>$k_2 \gg k_1 \log k_1$</td>
<td>$\leq V$</td>
<td>Neyman (97)</td>
</tr>
</tbody>
</table>
## Table-Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td>$mm(k_1, k_2)$</td>
<td>$\geq Mm(k_1, k_2)$</td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td>$V$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ben-Porath (86, 93)</td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td>$\implies M$</td>
</tr>
<tr>
<td></td>
<td>Ben-Porath (86, 93)</td>
<td></td>
</tr>
<tr>
<td>$k_2 \geq 2^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Neyman (97)</td>
<td></td>
</tr>
<tr>
<td>$k_2 \gg k_1 \log k_1$</td>
<td>$\leq V$</td>
<td>$\implies \leq V$</td>
</tr>
<tr>
<td></td>
<td>Neyman (97)</td>
<td></td>
</tr>
</tbody>
</table>
Table: Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>$mm(k_1, k_2)$</th>
<th>$\geq$</th>
<th>$Mm(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k_2 \geq k_1^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td></td>
<td>$V$ Ben-Porath (86, 93)</td>
</tr>
<tr>
<td>$k_2 \geq 2^{Ck_1}$</td>
<td>$\exists C \text{ s.t. } M$</td>
<td></td>
<td>Neyman (97)</td>
</tr>
<tr>
<td>$k_2 \gg k_1 \log k_1$</td>
<td>$\leq V$</td>
<td></td>
<td>$\leq V$</td>
</tr>
<tr>
<td>$\theta &gt; 0 \text{ fixed } k_2 = 2^{\theta k_1}$</td>
<td></td>
<td></td>
<td>$f(\theta)$</td>
</tr>
</tbody>
</table>

Ben-Porath (86, 93)
## Table-Finite Automata

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
<th>( mm(k_1, k_2) )</th>
<th>( \geq )</th>
<th>( Mm(k_1, k_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_2 \geq k_1 \rightarrow \infty )</td>
<td></td>
<td></td>
<td>( \geq )</td>
<td>( Mm(k_1, k_2) )</td>
</tr>
<tr>
<td>( \log k_2 = o(k_1) )</td>
<td>?</td>
<td>( V )</td>
<td>Ben-Porath (86, 93)</td>
<td></td>
</tr>
<tr>
<td>( k_2 \geq k_1^{Ck_1} )</td>
<td>( \exists C ) s.t. ( M )</td>
<td>( \implies ) ( M )</td>
<td>Ben-Porath (86, 93)</td>
<td></td>
</tr>
<tr>
<td>( k_2 \geq 2^{Ck_1} )</td>
<td>( \exists C ) s.t. ( M )</td>
<td>( \implies ) ( \leq V )</td>
<td>Neyman (97)</td>
<td></td>
</tr>
<tr>
<td>( k_2 &gt;&gt; k_1 \log k_1 )</td>
<td>( \leq V )</td>
<td>( \implies ) ( \leq V )</td>
<td>Neyman (97)</td>
<td></td>
</tr>
<tr>
<td>( \theta &gt; 0 ) fixed ( k_2 = 2^{\theta k_1} )</td>
<td></td>
<td></td>
<td>( f(\theta) )</td>
<td></td>
</tr>
</tbody>
</table>
Let $Mm(T; k_1, k_2)$ be the minmax the $T$-stage game when P2 minimizes over all mixtures of automata of size $k_2$ and P1 maximizes over all automata of size $k_1$. Similarly $mm(T; k_1, k_2)$
The Questions

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 and the number of repetitions $T$ so that
The Questions
What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 and the number of repetitions $T$ so that

$$Mm(T; k_1, k_2) = V$$
The Questions

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 and the number of repetitions $T$ so that

- $Mm(T; k_1, k_2) = V$
- $Mm(T; k_1, k_2) = M$
The Questions

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 and the number of repetitions $T$ so that

- $Mm(T; k_1, k_2) = V$
- $Mm(T; k_1, k_2) = M$
- $Mm(T; k_1, k_2) = x$ where $M < x < V$
2-P 0-sum Finitely Repeated FA

The Questions

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 and the number of repetitions $T$ so that

- $M m(T; k_1, k_2) = V$
- $M m(T; k_1, k_2) = M$
- $M m(T; k_1, k_2) = x$ where $M < x < V$
The Questions

What are the asymptotic relations between the size of $k_1$ and $k_2$ of the automata of P1 and P2 and the number of repetitions $T$ so that

- $Mm(T; k_1, k_2) = V$
- $Mm(T; k_1, k_2) = M$
- $Mm(T; k_1, k_2) = x$ \hspace{1cm} \text{where } M < x < V$
- $mm(T; k_1, k_2) = V$
- $mm(T; k_1, k_2) = M$
Let $G(T; k_1; k_2)$ be the $T$-stage game when $P_2$ uses machines of size $k_2$ and $P_1$ uses machines of size $k_1$.

The question: What are the asymptotic relations between the sizes $k_1$ and $k_2$ and the number of repetitions $T$ so that the set of equilibrium payoffs of $G(T; k_1; k_2)$ converge to the equilibrium payoffs of the infinitely repeated game $G$. 
Let $G(T; k_1, k_2)$ be the $T$-stage game when P2 uses machines of size $k_2$ and P1 uses machines of size $k_1$.

The Questions
Let $G(T; k_1, k_2)$ be the $T$-stage game when P2 uses machines of size $k_2$ and P1 uses machines of size $k_1$.

**The Questions**

What are the asymptotic relations between the sizes $k_1$ and $k_2$ and the number of repetitions $T$ so that...
Let $G(T; k_1, k_2)$ be the $T$-stage game when P2 uses machines of size $k_2$ and P1 uses machines of size $k_1$.

The Questions

What are the asymptotic relations between the sizes $k_1$ and $k_2$ and the number of repetitions $T$ so that

- The set of equilibrium payoffs of $G(T; k_1, k_2)$ converge to the equilibrium payoffs of the infinitely repeated game $G^*$. 
The objective is the study of the equilibrium of $G(k_1; \ldots; k_n)$ and of $G(T; k_1; \ldots; k_n)$.

It requires the analysis of the individual rational payoff of say player 1, namely of $\text{Min Max } G(\cdot; 1)$ where the min is over all strategy profiles $\cdot = (\ldots)$ where $j$ is a mixture of automata of $P_j$ of size $k_j$ and the max is over all automata of $P_1$ of size $k_1$. 
The objective is the study of the equilibrium of
\[ G(k_1, \ldots, k_n) \]
and of
\[ G(T; k_1, \ldots, k_n). \]
The objective is the study of the equilibrium of

\[ G(k_1, \ldots, k_n) \]

and of

\[ G(T; k_1, \ldots, k_n). \]

It requires the analysis of the individual rational payoff of say player 1, namely of

\[ \text{Min Max } G(\sigma^{-1}, \sigma^1) \]

where the min is over all strategy profiles \( \sigma^{-1} = (\sigma^j)_{j \neq 1} \)
where \( \sigma^j \) is a mixture of automata of \( \text{Pj} \) of size \( k_j \) and the max is over all automata of \( \text{P1} \) of size \( k_1 \).
Notation-Bounded Recall
Notation-Bounded Recall

\[ M = \max_{a \in A} \min_{b \in B} g(a, b) \]

\[ V = \min_{y \in \Delta(B)} \max_{a \in A} g(a, y) \]

\[ = \max_{x \in \Delta(A)} \min_{b \in B} g(x, b) \]

\[ mm(k_1, k_2) = \min \max (k_1, k_2) \]

\[ = \min_{\tau \in BR_2(k_2)} \max_{\sigma \in BR_1(k_1)} G(\sigma, \tau) \]

\[ Mn(k_1, k_2) = \Min \max (k_1, k_2) \]

\[ = \min_{\tau \in \Delta(BR_2(k_2))} \max_{\sigma \in BR_1(k_1)} G(\sigma, \tau) \]
Table-Bounded Recall
Table-Bounded Recall

<table>
<thead>
<tr>
<th>$k_2 \geq k_1 \rightarrow \infty$</th>
<th>$mm(k_1, k_2)$</th>
<th>$Mn(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Table-Bounded Recall

<table>
<thead>
<tr>
<th>Condition</th>
<th>( mm(k_1, k_2) )</th>
<th>( Mn(k_1, k_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_2 \geq k_1 \rightarrow \infty )</td>
<td>( \log k_2 = o(k_1) )</td>
<td>?</td>
</tr>
<tr>
<td>( \log k_2 = o(k_1) )</td>
<td>\text{V Lehrer}</td>
<td></td>
</tr>
</tbody>
</table>

4C: Correlation, Communication, Complexity, and Competition – p. 30/81
# Table-Bounded Recall

<table>
<thead>
<tr>
<th>Condition</th>
<th>$mm(k_1, k_2)$</th>
<th>$Mn(k_1, k_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_2 \geq k_1 \rightarrow \infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\log k_2 = o(k_1)$</td>
<td>?</td>
<td>$V$ Lehrer</td>
</tr>
<tr>
<td>$k_2 &gt;&gt;</td>
<td>A \times B</td>
<td>^{k_1}$</td>
</tr>
</tbody>
</table>

4C: Correlation, Communication, Complexity, and Competition – p. 30/81
## Table-Bounded Recall

<table>
<thead>
<tr>
<th>Condition</th>
<th>( mm(k_1, k_2) )</th>
<th>( Mn(k_1, k_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_2 \geq k_1 \to \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \log k_2 = o(k_1) )</td>
<td>( ? )</td>
<td>( V )</td>
</tr>
<tr>
<td>( k_2 \gg</td>
<td>A \times B</td>
<td>^{k_1} )</td>
</tr>
<tr>
<td>( k_2 &gt; Ck_1 )</td>
<td>( \exists C ) such that ( \leq V )</td>
<td>( \leq V )</td>
</tr>
</tbody>
</table>

Lehrer

Neyman and Okada
Complexity and Competition
Complexity and Competition

Ben-Porath 85
Complexity and Competition

- Ben-Porath 85
- Lehrer 88
Complexity and Competition

- Ben-Porath 85
- Lehrer 88
- Neyman 97
Complexity and Competition

- Ben-Porath 85
- Lehrer 88
- Neyman 97
- Stearns 97
Complexity and Competition

- Ben-Porath 85
- Lehrer 88
- Neyman 97
- Stearns 97
- Neyman and Okada
Complexity and Cooperation

2-person finitely repeated games
Complexity and Cooperation

2-person finitely repeated games

Meggido and Wigderson 86
Complexity and Cooperation

2-person finitely repeated games

- Meggido and Wigderson 86
- Neyman 85,98
Complexity and Cooperation

2-person finitely repeated games

- Meggido and Wigderson 86
- Neyman 85,98
- Papadimitriou and Yanakakis 94
Complexity and Cooperation

2-person finitely repeated games

- Meggido and Wigderson 86
- Neyman 85,98
- Papadimitriou and Yanakakis 94
- Zemel 89
Complexity and Cooperation

2-person finitely repeated games

- Meggido and Wigderson 86
- Neyman 85, 98
- Papadimitriou and Yanakakis 94
- Zemel 89
- ...
Complexity and Cooperation

2-person finitely repeated games

- Meggido and Wigderson 86
- Neyman 85, 98
- Papadimitriou and Yanakakis 94
- Zemel 89
- ...

2-person infinitely repeated games
Complexity and Cooperation

n-person games \((n > 2)\)
Complexity and Cooperation

\( n \)-person games \((n > 2)\)

- Ben-Porath 92
Complexity and Cooperation

\[ n \text{-} \text{person games } (n > 2) \]

- Ben-Porath 92
- Lehrer 94
Complexity and Cooperation

\( n \)-person games \((n > 2)\)

- Ben-Porath 92
- Lehrer 94
- Neyman 97
Complexity and Cooperation

\( n \)-person games \((n > 2)\)

- Ben-Porath 92
- Lehrer 94
- Neyman 97...
Complexity and Cooperation

\(n\)-person games \((n > 2)\)

- Ben-Porath 92
- Lehrer 94
- Neyman 97...
- Gossner Hernandez and Neyman
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)

Lehrer 93 (Bounded Recall)
Neyman 97 (Bounded Recall and Finite Automata)

1 weak and 1 or many strong conceal correlation from a median one

Neyman and Bavly 03 (Bounded Recall and FA)

n weak and 1 strong conceal correlation from a median one
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)
  - 2 weak players conceal correlation from a stronger one

- Lehrer 93 (Bounded Recall)
- Neyman 97 (Bounded Recall and Finite Automata)
- Neyman and Bavly 03 (Bounded Recall and FA)
  - $n \geq 2$ weak and 1 strong conceal correlation from a median one
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)
  - 2 weak players conceal correlation from a stronger one)
- Lehrer 93 (Bounded Recall)
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)
  - 2 weak players conceal correlation from a stronger one)
- Lehrer 93 (Bounded Recall)
- Neyman 97 (Bounded Recall and Finite Automata)
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)
  - 2 weak players conceal correlation from a stronger one)

- Lehrer 93 (Bounded Recall)

- Neyman 97 (Bounded Recall and Finite Automata)
  - 1 weak and 1 or many strong conceal correlation from a median one
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)
  - 2 weak players conceal correlation from a stronger one
- Lehrer 93 (Bounded Recall)
- Neyman 97 (Bounded Recall and Finite Automata)
  - 1 weak and 1 or many strong conceal correlation from a median one
- Neyman and Bavly 03 (Bounded Recall and FA)
Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)
  - 2 weak players conceal correlation from a stronger one
- Lehrer 93 (Bounded Recall)
- Neyman 97 (Bounded Recall and Finite Automata)
  - 1 weak and 1 or many strong conceal correlation from a median one
- Neyman and Bavly 03 (Bounded Recall and FA)
  - $n \geq 2$ weak and 1 strong conceal correlation from a median one
Concealed Correlation
Concealed Correlation

Gossner and Tomala
Concealed Correlation

- Gossner and Tomala
- Gossner Tomala and Laraki
Online Concealed Correlation
by Boundedly Rational Players

Gilad Bavly and Abraham Neyman
Online Concealed Correlation
by Boundedly Rational Players

Gilad Bavly and Abraham Neyman
Distributions on Cartesian Products
Distributions on Cartesian Products

Consider a stochastic process with values in $A^\infty$
Distributions on Cartesian Products

Consider a stochastic process with values in $A^\infty$ where $A$ is a product set, e.g., $A = A^1 \times A^2 \times A^3$. 
Consider a stochastic process with values in $A^\infty$ where $A$ is a product set, e.g., $A = A^1 \times A^2 \times A^3$

i.e., a probability distribution $P$ over streams $a_1, a_2, \ldots, a_t, \ldots$

with

$$a_t = (a^1_t, a^2_t, a^3_t) \in A = A^1 \times A^2 \times A^3$$
Distributions on Cartesian Products

Consider a stochastic process with values in $A^\infty$ where $A$ is a product set, e.g., $A = A^1 \times A^2 \times A^3$

i.e., a probability distribution $P$ over streams $a_1, a_2, \ldots, a_t, \ldots$ with

$$a_t = (a^1_t, a^2_t, a^3_t) \in A = A^1 \times A^2 \times A^3$$

The law $P$ of the process is governed by a list of independent rules, $\sigma^1$, $\sigma^2$, and $\sigma^3$, each governing its own factor $A^1$, $A^2$, and $A^3$, respectively.
The independent rules = strategies
The independent rules = strategies

The rule \( \sigma^i \) specifies, for each \( t \), the coordinate \( a_t^i \) as a function of \( a_1, \ldots, a_{t-1} \).
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a^i_t$ as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A^i$
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a_t^i$ as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A_i$
- A behavioral rule: $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over $A_i$
- A mixed rule is a mixture of deterministic rules
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a^i_t$ as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A^i$
- A behavioral rule: $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over $A^i$
- A mixed rule is a mixture of deterministic rules
- A mixed behavioral rule is a mixture of behavioral rules
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a^i_t$ as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A^i$
- A behavioral rule: $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over $A^i$
- A mixed rule is a mixture of deterministic rules
- A mixed behavioral rule is a mixture of behavioral rules
  $k$-recall rules
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a^i_t$ as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A^i$
- A behavioral rule: $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over $A^i$
- A mixed rule is a mixture of deterministic rules
- A mixed behavioral rule is a mixture of behavioral rules

$k$-recall rules

A deterministic $k$-recall rule $\sigma^i$ specifies $a^i_t$ as a function of the last $k$ stages, i.e as a function of $a^i_{t-k}, \ldots, a^i_{t-1}$. 
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a^i_t$ as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A^i$
- A behavioral rule: $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over $A^i$
- A mixed rule is a mixture of deterministic rules
- A mixed behavioral rule is a mixture of behavioral rules
  
  $k$-recall rules

- A deterministic $k$-recall rule $\sigma^i$ specifies $a^i_t$ as a function of the last $k$ stages, i.e as a function of $a^i_{t-k}, \ldots, a^i_{t-1}$.
- A behavioral $k$-recall rule
The independent rules = strategies

The rule $\sigma^i$ specifies, for each $t$, the coordinate $a^i_t$ as a function of $a_1, \ldots, a_{t-1}$.

- **A deterministic rule:** $\sigma^i(a_1, \ldots, a_{t-1})$ an element of $A^i$
- **A behavioral rule:** $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over $A^i$
- **A mixed rule** is a mixture of deterministic rules
- **A mixed behavioral rule** is a mixture of behavioral rules

$k$-recall rules

- **A deterministic $k$-recall rule** $\sigma^i$ specifies $a^i_t$ as a function of the last $k$ stages, i.e as a function of $a^i_{t-k}, \ldots, a^i_{t-1}$.
- **A behavioral $k$-recall rule**
- **A mixed $k$-recall rule**
Product marginals

In what follows we assume that the mixtures $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent.
Product marginals

Kuhn 1953: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent, then
Product marginals

Kuhn 1953: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent, then the distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_1, \ldots, a_{t-1}$ is a product distribution.
Product marginals

Kuhn 1953: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent, then the distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_1, \ldots, a_{t-1}$ is a product distribution.

Early 1990s: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent mixtures of $k_i$-recall strategies, and $k_1, k_2 \leq m$, then
Kuhn 1953: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent, then the distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_1, \ldots, a_{t-1}$ is a product distribution.

Early 1990s: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent mixtures of $k_i$-recall strategies, and $k_1, k_2 \leq m$, then the distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_{t-m}, \ldots, a_{t-1}$ is essentially a product distribution.
Kuhn 1953: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent, then the distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_1, \ldots, a_{t-1}$ is a product distribution.

Early 1990s: If $\sigma^1$, $\sigma^2$, and $\sigma^3$ are independent mixtures of $k_i$-recall strategies, and $k_1, k_2 \leq m$, then the distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_{t-m}, \ldots, a_{t-1}$ is essentially a product distribution when $m \to \infty$ ($k_i = k_i(m)$).
The distribution of $\alpha_t = (\alpha_t^1, \alpha_t^2, \alpha_t^3)$

given $\alpha_{t-m}, \ldots, \alpha_{t-1}$,
The distribution of $a_t = (a^1_t, a^2_t, a^3_t)$

given $a_{t-m}, \ldots, a_{t-1}$

If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, then for every $(b_1, \ldots, b_m, b_{m+1})$ we compute

$$P_\sigma((a_{t-m}, \ldots, a_{t-1}, a_t) = (b_1, \ldots, b_m, b_{m+1}))$$
The distribution of \( a_t = (a_t^1, a_t^2, a_t^3) \)

given \( a_{t-m}, \ldots, a_{t-1} \)

If \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \), has \((k_1, k_2, k_3)\)-recall,
The distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_{t-m}, \ldots, a_{t-1}$

If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, has $(k_1, k_2, k_3)$-recall, then for every
$(b_1, \ldots, b_m, b_{m+1})$
The distribution of $a_t = (a^1_t, a^2_t, a^3_t)$

given $a_{t-m}, \ldots, a_{t-1}$

If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, has $(k_1, k_2, k_3)$-recall, then for every $(b_1, \ldots, b_m, b_{m+1})$ the empirical probability
The distribution of $a_t = (a^1_t, a^2_t, a^3_t)$
given $a_{t-m}, \ldots, a_{t-1}$

If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, has $(k_1, k_2, k_3)$-recall, then for every $(b_1, \ldots, b_m, b_{m+1})$ the empirical probability

$$\frac{1}{n} \sum_{t=m+1}^{n} P_\sigma((a_{t-m}, \ldots, a_{t-1}, a_t) = (b_1, \ldots, b_m, b_{m+1}))$$
The distribution of \( \alpha_t = (\alpha^1_t, \alpha^2_t, \alpha^3_t) \) given \( \alpha_{t-m}, \ldots, \alpha_{t-1} \)

If \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \), has \((k_1, k_2, k_3)\)-recall, then for every \((b_1, \ldots, b_m, b_{m+1})\) the empirical probability

\[
\frac{1}{n} \sum_{t=m+1}^{n} P_\sigma((\alpha_{t-m}, \ldots, \alpha_{t-1}, \alpha_t) = (b_1, \ldots, b_m, b_{m+1}))
\]

converges as \( n \to \infty \)
The distribution of $a_t = (a^1_t, a^2_t, a^3_t)$ given $a_{t-m}, \ldots, a_{t-1}$

If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, has $(k_1, k_2, k_3)$-recall, then for every $(b_1, \ldots, b_m, b_{m+1})$ the empirical probability

$$\frac{1}{n} \sum_{t=m+1}^{n} P_\sigma((a_{t-m}, \ldots, a_{t-1}, a_t) = (b_1, \ldots, b_m, b_{m+1}))$$

converges as $n \to \infty$

Thus inducing a probability $P_\sigma$ on $B^{m+1}$ where $B = A$. 

4C: Correlation, Communication, Complexity, and Competition – p. 40/81
The distribution of $\alpha_t = (\alpha_t^1, \alpha_t^2, \alpha_t^3)$

given $\alpha_{t-m}, \ldots, \alpha_{t-1}$

If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, has $(k_1, k_2, k_3)$-recall, then for every $(b_1, \ldots, b_m, b_{m+1})$ the empirical probability

$$\frac{1}{n} \sum_{t=m+1}^{n} P_\sigma((\alpha_{t-m}, \ldots, \alpha_{t-1}, \alpha_t) = (b_1, \ldots, b_m, b_{m+1}))$$

converges as $n \to \infty$

Thus inducing a probability $P_\sigma$ on $B^{m+1}$ where $B = A$.

We study the distribution of $b_{m+1}$ conditional on $b_1, \ldots, b_m$
The Questions
The Questions

- What are the asymptotic relation between $m$ and $k_1, k_2, k_3$, such that...
The Questions

- What are the asymptotic relation between $m$ and $k_1, k_2, k_3$, such that any distributions $Q$ on $A$ can be “realized” as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ w.r.t. some $P_\sigma$ where $\sigma$ has $(k_1, k_2, k_3)$-recall
What are the asymptotic relation between $m$ and $k_1, k_2, k_3$, such that

- any distributions $Q$ on $A$ can be “realized” as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ w.r.t. some $P_\sigma$ where $\sigma$ has $(k_1, k_2, k_3)$-recall
- the marginal on $A^1 \times A^2$ of the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is a product distribution w.r.t. any $P_\sigma$ with $\sigma$ having $(k_1, k_2, k_3)$-recall.
The Questions

- What are the asymptotic relation between $m$ and $k_1, k_2, k_3$, such that
  - any distributions $Q$ on $A$ can be “realized” as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ w.r.t. some $P_\sigma$ where $\sigma$ has $(k_1, k_2, k_3)$-recall
  - the marginal on $A^1 \times A^2$ of the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is a product distribution w.r.t. any $P_\sigma$ with $\sigma$ having $(k_1, k_2, k_3)$-recall.

- For a given asymptotic relation between $m$ and $k_1, k_2, k_3$, what are the distributions $Q$ on $A$ that can be “realized” as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ w.r.t. some $P_\sigma$ where $\sigma$ has $(k_1, k_2, k_3)$-recall
If $m$ is subexponential in $k_1$ (i.e., $\log m = o(k_1)$) and $m \leq k_2$, then any distributions $Q$ on $A$ can be "realized" as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$.

(Bavley-N) If $m$ is superexponential in $k_1$ and $k_2$ (s.t. $m \leq Ck_1 + Ck_2$) then the marginal on $A_1 A_2$ of the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is a product distribution.

(Early 90s) If $m \leq k_1, k_2$ then the marginal on $A_1 A_2$ of the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is a product distribution.
Answers A

Assume $k_1 \leq k_2 \leq k_3$. 
If $m$ is subexponential in $k_1$ (i.e., $\log m = o(k_1)$) and $m \ll k_2, k_3$ then any distributions $Q$ on $A$ can be “realized” as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$. 
If $m$ is subexponential in $k_1$ (i.e., $\log m = o(k_1)$) and $m \ll k_2, k_3$ then any distributions $Q$ on $A$ can be “realized” as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$.

(Bavley-N) If $m$ is superexponential in $k_1$ and $k_2$ ($\exists C$ s.t. $m \geq e^{Ck_1+Ck_2}$) then the marginal on $A^1 \times A^2$ of the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is a product distribution.
If \( m \) is subexponential in \( k_1 \) (i.e., \( \log m = o(k_1) \)) and \( m \ll k_2, k_3 \) then any distributions \( Q \) on \( A \) can be “realized” as the distribution of \( b_{m+1} \) given \( b_1, \ldots, b_m \).

(Bavley-N) If \( m \) is superexponential in \( k_1 \) and \( k_2 \) (\( \exists C \) s.t. \( m \geq e^{Ck_1+Ck_2} \)) then the marginal on \( A^1 \times A^2 \) of the distribution of \( b_{m+1} \) given \( b_1, \ldots, b_m \) is a product distribution.

(Early 90s) If \( m \geq k_1, k_2 \) then the marginal on \( A^1 \times A^2 \) of the distribution of \( b_{m+1} \) given \( b_1, \ldots, b_m \) is a product distribution.
(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \leq k_3$ then there is a distribution $Q$ on $A$ such that the marginal of $Q$ on $A_1 \times A_2$ is not a product distribution and the distribution of $b_{m+1}$ given $b_1; \ldots; b_m$ is $Q$.

$H_Q(a_1; a_2; a_3) = H_Q(a_1) + H_Q(a_2)$.
Assume $k_1 \leq k_2 \leq k_3$. 
(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then there is a distribution $Q$ on $A$ such that the marginal of $Q$ on $A^1 \times A^2$ is not a product distribution and the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$. 
(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then there is a distribution $Q$ on $A$ such that the marginal of $Q$ on $A^1 \times A^2$ is not a product distribution and the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$.

(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then any distribution $Q$ on $A$ such that

$$H_Q(a^1, a^2, a^3) \geq H_Q(a^1) + H_Q(a^2)$$

can be realized as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$. 
(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then there is a distribution $Q$ on $A$ such that the marginal of $Q$ on $A^1 \times A^2$ is not a product distribution and the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$.

(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then any distribution $Q$ on $A$ such that

$$H_Q(a^1, a^2, a^3) \geq H_Q(a^1) + H_Q(a^2)$$

can be realized as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$. 
(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then there is a distribution $Q$ on $A$ such that the marginal of $Q$ on $A^1 \times A^2$ is not a product distribution and the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$.

(Bavly-N) If $m$ is subexponential in $k_1$ and $k_2$ and $m \ll k_3$ then any distribution $Q$ on $A$ such that

$$H_Q(a^1, a^2, a^3) \geq H_Q(a^1) + H_Q(a^2)$$

can be realized as the distribution of $b_{m+1}$ given $b_1, \ldots, b_m$ is $Q$. 
(Bavly-N) If \( m \) is subexponential in \( k_1 \) and \( k_2 \) and \( m \ll k_3 \) then there is a distribution \( Q \) on \( A \) such that the marginal of \( Q \) on \( A^1 \times A^2 \) is not a product distribution and the distribution of \( b_{m+1} \) given \( b_1, \ldots, b_m \) is \( Q \).

(Bavly-N) If \( m \) is subexponential in \( k_1 \) and \( k_2 \) and \( m \ll k_3 \) then any distribution \( Q \) on \( A \) such that

\[
H_Q(a^1, a^2, a^3) \geq H_Q(a^1) + H_Q(a^2)
\]

can be realized as the distribution of \( b_{m+1} \) given \( b_1, \ldots, b_m \) is \( Q \).
Gossner and Hernandez

Part of the talk will focus on a joint project of Gossner, Hernandez, and Neyman

Online Matching Pennies

Optimal Use of Communication Resources

More to come
Gossner and Hernandez

Part of the talk will focus on a joint project of Gossner, Hernandez, and Neyman
Part of the talk will focus on a joint project of Gossner, Hernandez, and Neyman

- Online Matching Pennies
Part of the talk will focus on a joint project of Gossner, Hernandez, and Neyman

- Online Matching Pennies
- Optimal Use of Communication Resources
Gossner and Hernandez

Part of the talk will focus on a joint project of Gossner, Hernandez, and Neyman

- Online Matching Pennies
- Optimal Use of Communication Resources
- More to come
The $n$-stage game
The $n$-stage game

Sequence of temporal states of nature

$$x = (x_1, \ldots, x_n) \in I^n$$
The $n$-stage game

- Sequence of temporal states of nature
  \[ x = (x_1, \ldots, x_n) \in I^n \]

- Pure strategies of player 2:
  either \[ y = (y_1, \ldots, y_n) \] where \[ y_t : I^n \rightarrow J \]
The $n$-stage game

- Sequence of temporal states of nature
  
  \[ x = (x_1, \ldots, x_n) \in I^n \]

- Pure strategies of player 2:
  
  either \[ y = (y_1, \ldots, y_n) \text{ where } y_t : I^n \to J \]

  or \[ y = (y_1, \ldots, y_n) \text{ where } y_t : I^n \times K^{t-1} \to J \]
The $n$-stage game

- Sequence of temporal states of nature
  \[ x = (x_1, \ldots, x_n) \in I^n \]

- Pure strategies of player 2:
  either
  \[ y = (y_1, \ldots, y_n) \text{ where } y_t : I^n \to J \]
  or
  \[ y = (y_1, \ldots, y_n) \text{ where } y_t : I^n \times K^{t-1} \to J \]

- Pure strategies of player 3:
  \[ z = (z_1, \ldots, z_n) \]
  \[ z_t : I^{t-1} \times J^{t-1} \to K \]
Payoffs

Players 2 and 3 form a team, against Player 1.

Stage payoff function to the team:

\[ g(i; j; k) \]

n-stage payoff to the team:

\[ G(x; y; z) = 1 \]

\[ \prod_{t=1}^{n} g(x_t; y_t; z_t) \]
Players 2 and 3 form a team, against Player 1.
Players 2 and 3 form a team, against Player 1.

Stage payoff function to the team:

\[ g(i, j, k) \]
Payoffs

Players 2 and 3 form a team, against Player 1.

Stage payoff function to the team:

\[ g(i, j, k) \]

\( n \)-stage payoff to the team:

\[ G(x, y, z) = \frac{1}{n} \sum_{t=1}^{n} g(x_t, y_t, z_t) \]
Example

\[ I = J = K = \{0, 1\} \text{ and} \]

\[ g(i, j, k) = \begin{cases} 
1 & \text{if } i = j = k \\
0 & \text{otherwise}
\end{cases} \]

\[
\begin{array}{cc}
1 & 0 \\
0 & 0
\end{array}
\quad
\begin{array}{cc}
0 & 0 \\
0 & 1
\end{array}
\]
The team problem

What are good strategies for the team?

The forecaster can play the sequence \( y = x \) and the follower can play a sequence of \((1, 2), (1, 2), \ldots\) i.i.d.: securing a payoff of \(1, 2\) against all sequences.

Can they do better?
The team problem

What are *good* strategies for the team?
The team problem

What are good strategies for the team?

The forecaster can play the sequence $y = x$ and the follower can play a sequence of $(\frac{1}{2}, \frac{1}{2})$ i.i.d.
The team problem

What are good strategies for the team?

The forecaster can play the sequence $y = x$ and the follower can play a sequence of $(\frac{1}{2}, \frac{1}{2})$ i.i.d.:

securing a payoff of $\frac{1}{2}$ against all sequences.
The team problem

What are good strategies for the team?

The forecaster can play the sequence $y = x$ and the follower can play a sequence of $(\frac{1}{2}, \frac{1}{2})$ i.i.d.:

securing a payoff of $\frac{1}{2}$ against all sequences.

Can they do better?
In pure strategies
In pure strategies

The forecaster can play on odd stages the next action of Player 1 and on even stages the follower and the forecaster play the previous action of the forecaster.
In pure strategies

The forecaster can play on odd stages the next action of Player 1 and on even stages the follower and the forecaster play the previous action of the forecaster. The follower plays an arbitrary sequence of actions on the odd stages.
In pure strategies

The forecaster can play on odd stages the next action of Player 1 and on even stages the follower and the forecaster play the previous action of the forecaster. The follower plays an arbitrary sequence of actions on the odd stages.

Resulting sequences of actions:

\[ x = (x_1, x_2, x_3, x_4, \ldots, x_{80}) \]

\[ y = (x_2, x_2, x_4, x_4, \ldots, x_{80}) \]

\[ z = (z_1, x_2, z_3, x_4, \ldots, x_{80}) \]
Payoffs for these strategies

Against a sequence distributed (1/2, 1/2) i.i.d.:
- Payoff of 1 at even stages.
- Expected payoff of \( \frac{1}{4} \) at odd stages.
- Average expected payoff of 0.625.

Against the worst possible case:
- Payoff of 1 at even stages.
- Payoff of zero at odd stages.
- Average payoff of 0.5.
Payoffs for these strategies

Against a sequence distributed \((1/2,1/2)\) i.i.d.:

- Payoff of 1 at even stages.
- Expected payoff of \(\frac{1}{4}\) at odd stages.
- Average expected payoff of 0.625.

Against the worst possible case:

- Payoff of 1 at even stages.
- Payoff of zero at odd stages.
- Average payoff of 0.5.
Payoffs for these strategies

- Against a sequence distributed (1/2,1/2) i.i.d.:
  - Payoff of 1 at even stages.
Payoffs for these strategies

- Against a sequence distributed \((1/2,1/2)\) i.i.d.:
  - Payoff of 1 at even stages.
  - Expected payoff of \(\frac{1}{4}\) at odd stages.
Payoffs for these strategies

- Against a sequence distributed \((1/2,1/2)\) i.i.d.:
  - Payoff of 1 at even stages.
  - Expected payoff of \(\frac{1}{4}\) at odd stages.
  - Average expected payoff of 0.625.
Payoffs for these strategies

- Against a sequence distributed (1/2, 1/2) i.i.d.:
  - Payoff of 1 at even stages.
  - Expected payoff of $\frac{1}{4}$ at odd stages.
  - Average expected payoff of 0.625.

- Against the worst possible case:
Payoffs for these strategies

- Against a sequence distributed \((1/2, 1/2)\) i.i.d.:
  - Payoff of 1 at even stages.
  - Expected payoff of \(\frac{1}{4}\) at odd stages.
  - Average expected payoff of 0.625.

- Against the worst possible case:
  - Payoff of 1 at even stages.
Payoffs for these strategies

- Against a sequence distributed \( (1/2,1/2) \) i.i.d.:
  - Payoff of 1 at even stages.
  - Expected payoff of \( \frac{1}{4} \) at odd stages.
  - Average expected payoff of 0.625.

- Against the worst possible case:
  - Payoff of 1 at even stages.
  - Payoff of zero at odd stages.
Payoffs for these strategies

- Against a sequence distributed (1/2, 1/2) i.i.d.:
  - Payoff of 1 at even stages.
  - Expected payoff of $\frac{1}{4}$ at odd stages.
  - Average expected payoff of 0.625.

- Against the worst possible case:
  - Payoff of 1 at even stages.
  - Payoff of zero at odd stages.
  - Average payoff of 0.5.
Question

How much can the team get?

In expected payoffs?

In the worst case?

Can mixed strategies do better for the latter?
Question

How much can the team get?
Question

How much can the team get?

- In expected payoffs?
Question

How much can the team get?

- In expected payoffs?
- In the worst case?
Question

How much can the team get?

- In expected payoffs?
- In the worst case?
- Can mixed strategies do better for the latter?
What is your answer?
There exists $809 < v < 810$ such that: there exist pure strategies for the team that guarantee $v$ against all sequences. Against an i.d.d. sequence $(1,2;1,2)$, no strategy of the team can obtain more than $v$. $v$ is defined by $H(v) + (1 - v) \log 3 = 1$ where $H$ is the entropy function.
There exists \(0.809 < \nu^* < 0.81\) such that:
Answer

There exists $0.809 < v^* < 0.81$ such that:

- There exist pure strategies for the team that guarantee $v^* - o(1)$ against all sequences.
There exists \( 0.809 < v^* < 0.81 \) such that:

- There exist *pure* strategies for the team that guarantee \( v^* - o(1) \) against all sequences.
- Against an i.d.d. sequence \((\frac{1}{2}, \frac{1}{2})\), no strategy of the team can obtain more than \( v^* \).
There exists $0.809 < v^* < 0.81$ such that:

- There exist pure strategies for the team that guarantee $v^* - o(1)$ against all sequences.
- Against an i.d.d. sequence $(\frac{1}{2}, \frac{1}{2})$, no strategy of the team can obtain more than $v^*$.
- $v^*$ is defined by

\[ H(v^*) + (1 - v^*) \log 3 = 1 \]

where $H$ is the entropy function.
For general games: iid sequences
For general games: iid sequences

\[ \forall \mu \in \Delta(I) \ \exists v^*(\mu) \text{ s.t.:} \]
For general games: iid sequences

\[ \forall \mu \in \Delta(I) \exists v^*(\mu) \text{ s.t.:} \]

- If the sequence of states of nature is i.i.d. according to \( \mu \), then \( \forall \) strategies of the forecaster and the follower, their payoff in the \( n \)-stage version of the game does not exceed \( v^*(\mu) \).
For general games: iid sequences

∀μ ∈ Δ(I) ∃v*(μ) s.t.:

- If the sequence of states of nature is i.i.d. according to μ, then ∀ strategies of the forecaster and the follower, their payoff in the n-stage version of the game does not exceed v*(μ).

- ∀ n, ∃ pure strategies for the team in the n-stage version that achieves a payoff of at least v*(μ) − o(1) against a μ iid sequence.
For general games: iid sequences

\[ \forall \mu \in \Delta(I) \exists v^*(\mu) \text{ s.t.:} \]

- If the sequence of states of nature is i.i.d. according to \( \mu \), then \( \forall \) strategies of the forecaster and the follower, their payoff in the \( n \)-stage version of the game does not exceed \( v^*(\mu) \).

- \( \forall \ n, \exists \ pure \) strategies for the team in the \( n \)-stage version that achieves a payoff of at least \( v^*(\mu) - o(1) \) against a \( \mu \) iid sequence.

- \( \exists \ pure \) strategies for the team in the \( \infty \)-stage game with expected average payoff in the \( n \)-stages converging as \( n \to \infty \) to \( v^*(\mu) \) against a \( \mu \) iid sequence.
General games: worst case
General games: worst case

Set $v^* = \min_{\mu \in \Delta(I)} v^*(\mu)$:
Set $v^* = \min_{\mu \in \Delta(I)} v^*(\mu)$:

- \(\forall n, \exists\) pure strategies for the team in the \(n\)-stage game that achieves a payoff of at least \(v^* - o(1)\) against all sequences of actions of player 1.
General games: worst case

Set $v^* = \min_{\mu \in \Delta(I)} v^*(\mu)$:

- $\forall n$, $\exists$ pure strategies for the team in the $n$-stage game that achieves a payoff of at least $v^* - o(1)$ against all sequences of actions of player 1.

- $\exists$ a sequence $v^*_n = v^* - o(1)$ and pure strategies for the team in the $\infty$-stage game that achieve an average payoff in the $n$-stages $\geq v^*_n = v^* - o(1)$ against any sequence.
General games: worst case

Set \( v^* = \min_{\mu \in \Delta(I)} v^*(\mu) \):

- \( \forall n, \exists \text{ pure strategies for the team in the } n\text{-stage game that achieves a payoff of at least } v^* - o(1) \text{ against all sequences of actions of player 1.} \)

- \( \exists \text{ a sequence } v_n^* = v^* - o(1) \text{ and pure strategies for the team in the } \infty\text{-stage game that achieve an average payoff in the } n\text{-stages } \geq v_n^* = v^* - o(1) \text{ against any sequence.} \)

- \( \exists \mu \in \Delta(I) \text{ s.t. when player 1’s sequence of actions is i.i.d. according to } \mu, \forall \text{ strategies of the forecaster and the follower, their payoff in the } n\text{-stage version of the game does not exceed } v^*. \)
Remarks

an $\varepsilon$-optimal strategy for player one is given by an i.i.d. sequence according to some distribution $\mu$ independent of $n$. 
Remarks

- an $\varepsilon$-optimal strategy for player one is given by an i.i.d. sequence according to some distribution $\mu$ independent of $n$.

- the existence of $\varepsilon$-optimal pure strategies for the team.
Characterization of $\nu^*(\mu)$
Characterization of $\nu^*(\mu)$

For $\mu \in \Delta(I)$, let $Q(\mu)$ be the class of distributions $Q$ on $I \times J \times K$ such that:

The marginal of $Q$ on $I$ is $\mu$, and

$$H(i \mid k) + H(j \mid i, k) = H(i)$$
Characterization of $\nu^*(\mu)$

For $\mu \in \Delta(I)$, let $Q(\mu)$ be the class of distributions $Q$ on $I \times J \times K$ such that:

- The marginal of $Q$ on $I$ is $\mu$, and

\[ H(i \mid k) + H(j \mid i, k) = H(i) \]

Then

\[ \nu^*(\mu) = \max_{Q \in Q(\mu)} \mathbb{E}_Q(g(i, j, k)) \]
Characterization of $v^*(\mu)$

For $\mu \in \Delta(I)$, let $Q(\mu)$ be the class of distributions $Q$ on $I \times J \times K$ such that:

The marginal of $Q$ on $I$ is $\mu$, and

$$H(i \mid k) + H(j \mid i, k) = H(i)$$

Then

$$v^*(\mu) = \max_{Q \in Q(\mu)} E_Q(g(i, j, k))$$

and

$$v^* = \min_{\mu} v^*(\mu) = \min_{\mu} \max_{Q \in Q(\mu)} E_Q(g(i, j, k))$$
More forecasters and/or followers?

Existence of $\varepsilon$-optimal pure strategies for the team enables the extension of the result to $1 + s + f = n$ - person games where there are $s$ forecasters and $f$ followers. Replace the set of $s$ forecasters by a single forecaster with an action set equal to the cartesian product of the action sets of the forecasters, and the $f$ followers by a single follower with an action set equal to the product of the action sets of the followers.
Near term plan

Proof in the special case of the example.
Near term plan

Proof in the special case of the example.
1. Reminder on entropy.
Near term plan

Proof in the special case of the example.

1. Reminder on entropy.

2. Prove that no strategy of the team can achieve more than $v^*(\mu)$. 
   - Use of additivity of entropies.

3. Prove there exists strategies for the team that achieve $v$ against a iid sequence:
   - Use of coding theory.

4. Prove there exists strategies for the team that achieve $v$ against all sequences:
   - Use of coding theory.
Near term plan

Proof in the special case of the example.

1. Reminder on entropy.

2. Prove that no strategy of the team can achieve more than $v^*(\mu)$.
   - Use of additivity of entropies.

3. Prove there exists strategies for the team that achieve $v^*(\mu)$ against a $\mu$ iid sequence:
   - Use of coding theory.
Near term plan

Proof in the special case of the example.

1. Reminder on entropy.

2. Prove that no strategy of the team can achieve more than $v^*(\mu)$.
   - Use of additivity of entropies.

3. Prove there exists strategies for the team that achieve $v^*(\mu)$ against a $\mu$ iid sequence:
   - Use of coding theory.

4. Prove there exists strategies for the team that achieve $v^*$ against all sequences:
   - Use of coding theory.
Reminder on entropy

- $X, Y$ pair of random variables.
Reminder on entropy

- $X, Y$ pair of random variables.
- $H(X) = - \sum_x P(x) \log P(x)$,
  with $\log = \log_2$ and $0 \log 0 = 0$. 

Additivity of entropies:

$H(X; Y) = H(X|Y) + H(Y)$. 

4C: Correlation, Communication, Complexity, and Competition – p. 60/81
Reminder on entropy

- $X, Y$ pair of random variables.
- $H(X) = - \sum_x P(x) \log P(x)$, with $\log = \log_2$ and $0 \log 0 = 0$.
- $h(X \mid y) = - \sum_x P(x \mid y) \log P(x \mid y)$. 

Additivity of entropies:

$H(X; Y) = H(X \mid Y) + H(Y)$. 

4C: Correlation, Communication, Complexity, and Competition – p. 60/81
Reminder on entropy

- $X, Y$ pair of random variables.
- $H(X) = - \sum_x P(x) \log P(x)$, with $\log = \log_2$ and $0 \log 0 = 0$.
- $h(X \mid y) = - \sum_x P(x \mid y) \log P(x \mid y)$.
- $H(X \mid Y) = - \sum_y P(y) h(X \mid y)$. 

Additivity of entropies:

$$H(X; Y) = H(X \mid Y) + H(Y)$$
Reminder on entropy

- \( X, Y \) pair of random variables.

\[
H(X) = - \sum_x P(x) \log P(x),
\]
with \( \log = \log_2 \) and \( 0 \log 0 = 0 \).

\[
h(X \mid y) = - \sum_x P(x \mid y) \log P(x \mid y).
\]

\[
H(X \mid Y) = - \sum_y P(y) h(X \mid y).
\]

Additivity of entropies: \( H(X, Y) = H(X \mid Y) + H(Y) \).
First part
Assume that the distribution of $X = (X_1, \ldots, X_n)$ has entropy $nh$ ($0 \leq h \leq 1$).
Assume that the distribution of $X = (X_1, \ldots, X_n)$ has entropy $nh$ ($0 \leq h \leq 1$).
Let $Y$ and $Z$ be pure strategies of P2 and P3.
First part

Assume that the distribution of $X = (X_1, \ldots, X_n)$ has entropy $nh$ ($0 \leq h \leq 1$).
Let $Y$ and $Z$ be pure strategies of P2 and P3.

$$H(X_1, Y_1, \ldots, X_n, Y_n) = H(X_1, \ldots, X_n) = nh$$
Assume that the distribution of $X = (X_1, \ldots, X_n)$ has entropy $nh$ ($0 \leq h \leq 1$). Let $Y$ and $Z$ be pure strategies of P2 and P3.

$$H(X_1, Y_1, \ldots, X_n, Y_n) = H(X_1, \ldots, X_n) = nh$$

Let $\mathcal{F}_t$ be the algebra of events spanned by the random variables $X_1, Y_1, \ldots, X_t, Y_t$. 
Assume that the distribution of $X = (X_1, \ldots, X_n)$ has entropy $nh$ ($0 \leq h \leq 1$).

Let $Y$ and $Z$ be pure strategies of P2 and P3.

$$H(X_1, Y_1, \ldots, X_n, Y_n) = H(X_1, \ldots, X_n) = nh$$

Let $\mathcal{F}_t$ be the algebra of events spanned by the random variables $X_1, Y_1, \ldots, X_t, Y_t$.

$$g_t = \mathbb{E}_\mu (\mathbb{I}(X_t = Z_t = Y_t) \mid \mathcal{F}_{t-1})$$

is $\mathcal{F}_{t-1}$-measurable.
Where does the $\log 3$ come from?

Conditional on $\mathcal{F}_{t-1}$ (and also to $Z_t$):

\[
X_t = Y_t = Z_t^1 \cdot g_t
\]
Where does the $\log 3$ come from?

Conditional on $\mathcal{F}_{t-1}$ (and also to $Z_t$):

$$1 - g_t \quad g_t$$

$$X_t = Y_t = Z_t$$
Where does the $\log 3$ come from?

Conditional on $\mathcal{F}_{t-1}$ (and also to $Z_t$):

$$X_t = Y_t = Z_t$$

1. $1 - g_t$
2. $g_t$

- both wrong
- follower wrong only
- forecaster wrong only
Where does the $\log 3$ come from?

Conditional on $\mathcal{F}_{t-1}$ (and also to $Z_t$):

$$X_t = Y_t = Z_t$$

$h(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3$
Adding entropies up

Therefore,

\[ h(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3 \]
Adding entropies up

Therefore,

\[ h(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3 \]
Adding entropies up

Therefore,

\[ h(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3 \]

\[ \mathbb{E} \left( H(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \right) \leq \mathbb{E}_\mu (H(g_t) + (1 - g_t) \log 3) \]
Adding entropies up

Therefore,

\[ h(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3 \]

\[ \mathbb{E} H(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq \mathbb{E}_\mu(H(g_t) + (1 - g_t) \log 3) \]

Sum over \( t \)
Adding entropies up

Therefore,

\[ h(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3 \]

\[ E \left( H(X_t, Y_t \mid X_1 \ldots Y_{t-1}) \right) \leq E_{\mu} \left( H(g_t) + (1 - g_t) \log 3 \right) \]

Sum over \( t \)

\[ nh \leq \sum_{1}^{n} E_{\mu} \left( H(g_t) + (1 - g_t) \log 3 \right) \]
Conclusion of the first part

With $g = \mathbb{E}_\mu \left( \frac{1}{n} \sum_{t=1}^{n} g_t \right)$, $(g, h)$ is in the convex hull of $V = \{(x, y) \leq (x, H(x) + (1 - x) \log 3)\}$
Conclusion of the first part

With $g = \mathbf{E}_\mu \left( \frac{1}{n} \sum_{t=1}^{n} g_t \right)$, $(g, h)$ is in the convex hull of

$V = \{ (x, y) \leq (x, H(x) + (1 - x) \log 3) \}$
Conclusion of the first part

With \( g = \mathbb{E}_\mu \left( \frac{1}{n} \sum_{t=1}^{n} g_t \right) \), \((g, h)\) is in the convex hull of
\[
V = \{(x, y) \leq (x, H(x) + (1 - x) \log 3)\}
\]

\[
y = H(x) + (1 - x) \log 3
\]
Conclusion of the first part

With \( g = \mathbb{E}_\mu \left( \frac{1}{n} \sum_{t=1}^{n} g_t \right) \), \( (g, h) \) is in the convex hull of
\[
V = \{(x, y) \leq (x, H(x) + (1 - x) \log 3)\}
\]

\[
y = H(x) + (1 - x) \log 3
\]
Conclusion of the first part

With \( g = E_\mu \left( \frac{1}{n} \sum_{t=1}^{n} g_t \right) \), \((g, h)\) is in the convex hull of
\( V = \{(x, y) : (x, H(x) + (1 - x) \log 3) \} \)

\[ y = H(x) + (1 - x) \log 3 \]
Conclusion of the first part

With \( g = E_\mu \left( \frac{1}{n} \sum_{t=1}^{n} g_t \right) \), \( (g, h) \) is in the convex hull of
\[
V = \{(x, y) \leq (x, H(x) + (1 - x) \log 3)\}
\]

\[
y = H(x) + (1 - x) \log 3
\]
Second part: idea

- Strategies are defined over blocks of length $n$. 

- Strategies are defined over blocks of length $n$. 

- Strategies are defined over blocks of length $n$. 

Second part: idea

- Strategies are defined over blocks of length $n$.
- In a block, the forecaster tells the follower what to play in the next block.
Second part: idea

- Strategies are defined over blocks of length $n$.
- In a block, the forecaster tells the follower what to play in the next block.
- Two possibilities for transmitting information:
  - Sending information to the follower when the follower makes a mistake. (1 bit)
  - Make a mistake when the follower is “right”.
  - Is the second a good idea?
Second part: idea

- Strategies are defined over blocks of length $n$.
- In a block, the forecaster tells the follower what to play in the next block.
- Two possibilities for transmitting information:
  - Sending information to the follower when the follower makes a mistake. (1 bit)
  - Make a mistake when the follower is “right”.
  - Is the second a good idea?
- We look for an “optimal” codification scheme.
Second part: idea

- Strategies are defined over blocks of length \( n \).
- In a block, the forecaster tells the follower what to play in the next block.
- Two possibilities for transmitting information:
  - Sending information to the follower when the follower makes a mistake. (1 bit)
  - Make a mistake when the follower is “right”.
  - Is the second a good idea?
- We look for an “optimal” codification scheme.
**Second part: idea**

- Strategies are defined over blocks of length $n$.
- In a block, the forecaster tells the follower what to play in the next block.
- Two possibilities for transmitting information:
  - Sending information to the follower when the follower makes a mistake. (1 bit)
  - Make a mistake when the follower is “right”.
  - Is the second a good idea?
- We look for an “optimal” codification scheme.
Second part: idea

- Strategies are defined over blocks of length $n$.
- In a block, the forecaster tells the follower what to play in the next block.
- Two possibilities for transmitting information:
  - Sending information to the follower when the follower makes a mistake. (1 bit)
  - Make a mistake when the follower is “right”.
  - Is the second a good idea?
- We look for an “optimal” codification scheme.
Search for best codification

- Remember the $\log 3$?
Search for best codification

Remember the $\log 3$?

In order to have a “tight” inequality, conditional on the fact that one of the team members is wrong, all three possibilities should have equal probabilities:

- Both are wrong.
- Only the follower is wrong.
- Only the forecaster is wrong.
Search for best codification

- Remember the $\log 3$?
- In order to have a “tight” inequality, conditional on the fact that one of the team members is wrong, all three possibilities should have equal probabilities:
  - Both are wrong.
  - Only the follower is wrong.
  - Only the forecaster is wrong.
Search for best codification

- Remember the $\log_3$?
- In order to have a “tight” inequality, conditional on the fact that one of the team members is wrong, all three possibilities should have equal probabilities:
  - Both are wrong.
  - Only the follower is wrong.
  - Only the forecaster is wrong.
Search for best codification

- Remember the $\log 3$?

- In order to have a “tight” inequality, conditional on the fact that one of the team members is wrong, all three possibilities should have equal probabilities:
  - Both are wrong.
  - Only the follower is wrong.
  - Only the forecaster is wrong.
Let $0 < x < 1$ s.t. $H(x) + (1 - x) \log 3 = 1$.
Define $q = \frac{2}{3}(1 - x)$ and $p = 1 - \frac{x}{q}$. 
Tuning

Let $0 < x < 1$ s.t. $H(x) + (1 - x) \log 3 = 1$. Define $q = \frac{2}{3}(1 - x)$ and $p = 1 - x/q$.

- $x$: % of stages during which both are right.
Let $0 < x < 1$ s.t. $H(x) + (1 - x) \log 3 = 1$.

Define $q = \frac{2}{3} (1 - x)$ and $p = 1 - x/q$.

- $x$: % of stages during which both are right.
- $q$: % of stages at which the follower is wrong.
- $p$: the % of stages at which the forecaster is wrong, conditional on the follower right.
Tuning

Let $0 < x < 1$ s.t. $H(x) + (1 - x) \log 3 = 1$.
Define $q = \frac{2}{3}(1 - x)$ and $p = 1 - x/q$.

- $x$: % of stages during which both are right.
- $q$: % of stages at which the follower is wrong.
- $p$ is the % of stages at which the forecaster is wrong, conditional on the follower right.
How many messages?
How many messages?

The follower is wrong for $nq$ stages
How many messages?

The follower is wrong for $nq$ stages

$\implies 2^{nq}$ messages.
How many messages?

The follower is wrong for \( nq \) stages
\[ \implies 2^{nq} \text{ messages.} \]

When the follower is right, the forecaster makes a mistake a proportion \( p \) of the time
How many messages?

The follower is wrong for $nq$ stages
\[ \implies 2^{nq} \text{ messages}. \]

When the follower is right, the forecaster makes a mistake a proportion $p$ of the time
\[ \implies \binom{n(1-q)}{n(1-q)p} \sim 2^{n(1-q)H(p)} \text{ messages}. \]
The follower is wrong for $nq$ stages

$\implies 2^{nq}$ messages.

When the follower is right, the forecaster makes a mistake a proportion $p$ of the time

$\implies \binom{n(1-q)}{n(1-q)p} \sim 2^{n(1-q)H(p)}$ messages.

$2^{n(q+(1-q)H(p))}$ messages can be sent.
Both trees are equivalent:
Both trees are equivalent:

\[ H(q) + q + (1 - q)H(p) = H(x) + (1 - x) \log 3 = 1. \]

Therefore, if both messages can be sent, then

\[ 2^n(q + (1 - q)H(p)) = 2^n(1 - x). \]
Both trees are equivalent:

\[
\begin{align*}
q &\quad 1 - q \\
\frac{1}{2} &\quad \frac{1}{2} & p &\quad 1 - p \\
&\quad & &\quad \\
\text{both wrong} &\quad \text{follower only} & \text{forecaster only} & \text{both right}
\end{align*}
\]

\[
\begin{align*}
1 - x &\quad x \\
\frac{1}{3} &\quad \frac{1}{3} & \frac{1}{3} &\quad \frac{1}{3} \\
&\quad & &\quad \\
\text{both wrong} &\quad \text{follower only} & \text{forecaster only} & \text{both right}
\end{align*}
\]

Therefore

\[
\begin{align*}
q + (1 - q) \log 3 &= 1 \\
\text{and thus} \\
2^n (q + (1 - q)) \log 3 &= 2^n (1) \\
\end{align*}
\]

messages can be sent.
Both trees are equivalent:

\[ H(q) + q + (1 - q)H(p) \]

Both right
Both trees are equivalent:

\[ H(q) + q + (1 - q)H(p) = H(x) + (1 - x) \log 3 \]
Both trees are equivalent:

\[
H(q) + q + (1 - q)H(p) = H(x) + (1 - x) \log 3 = 1.
\]
Both trees are equivalent:

\[
H(q) + q + (1 - q)H(p) = H(x) + (1 - x) \log 3 = 1.
\]

Therefore \( q + (1 - q)H(p) = 1 - H(q) \) and thus

\[
2^n(q + (1 - q)H(p)) = 2^n(1 - H(q))
\]

messages can be sent.
Question

Does there exist a set $A \subset 2^n$ such that

$$|A| = 2^{(1-H(q)+o(1))n}$$

and s.t.: $\forall x \in 2^n \ \exists y \in A$ s.t.

$$d_H(x, y) = (1 - q)n.$$

where $d_H$ is the Hamming distance?
Existence of $A$

Probabilistic proof:

Take a set $A = \{a_i\}$ of $2^{(1-H(q))n}$ points taken randomly i.i.d. uniformly in $2^n$.

For every fixed $x \in 2^n$ the probability that there is no $z \in 2^n$ so that $d_H(x, y) = [qn]$ is

$$
\leq (1 - \left( \frac{n}{qn} \right)/2^n)^{2^{(1-H(q))n}} \leq \exp \left( -2^n(H(q)+1-H(q)) \right)
$$

We prove that the probablity that $A$ feeds our needs is positive.

Hence, such $A$ exists.
Example 1

Consider, for instance, $I = f_E; W$; $J = f_T; B$; and $K = f_L; R$. The correlated distribution $Q$ on $I \cap J \cap K$ is described in the Figure below. 

$\begin{array}{ccc}
E & L & R \\
T & & \\
B & & \\
W & H & (i) = 1 = H (j) = 1;
\end{array}$
Example 1

Consider, for instance, \( I = \{E, W\} \), \( J = \{T, B\} \), and \( K = \{L, R\} \), and the correlated distribution \( Q \) on \( I \times J \times K \) described in the Figure below,
Example 1

Consider, for instance, $I = \{E, W\}$, $J = \{T, B\}$, and $K = \{L, R\}$, and the correlated distribution $Q$ on $I \times J \times K$ described in the Figure below, $P_2$ chooses the rows (Top or Bottom),

\[
\begin{array}{c|c|c}
T & & T \\
B & & B \\
\end{array}
\]
Example 1

Consider, for instance, $I = \{E, W\}$, $J = \{T, B\}$, and $K = \{L, R\}$, and the correlated distribution $Q$ on $I \times J \times K$ described in the Figure below, $P2$ chooses the rows (Top or Bottom), $P3$ chooses the columns (Left or Right),
Example 1

Consider, for instance, $I = \{E, W\}$, $J = \{T, B\}$, and $K = \{L, R\}$, and the correlated distribution $Q$ on $I \times J \times K$ described in the Figure below, $P2$ chooses the rows (Top or Bottom), $P3$ chooses the columns (Left or Right), Temporal state of nature is East or West iid 1/2,1/2.
Example 1

Consider, for instance, $I = \{E, W\}$, $J = \{T, B\}$, and $K = \{L, R\}$, and the correlated distribution $Q$ on $I \times J \times K$ described in the Figure below, $P_2$ chooses the rows (Top or Bottom), $P_3$ chooses the columns (Left or Right), Temporal state of nature is East or West iid $1/2, 1/2$. The matrix entries are the desired probabilities of the action profile.
Example 1

Consider, for instance, \( I = \{E, W\} \), \( J = \{T, B\} \), and \( K = \{L, R\} \), and the correlated distribution \( Q \) on \( I \times J \times K \) described in the Figure below, \( P_2 \) chooses the rows (Top or Bottom), \( P_3 \) chooses the columns (Left or Right), Temporal state of nature is East or West iid 1/2, 1/2. The matrix entries are the desired probabilities of the action profile.

\[
\begin{array}{cc}
T & L & R \\
B & .1 & .1 \\
\end{array}
\quad
\begin{array}{cc}
T & L & R \\
B & .1 & .2 \\
\end{array}
\quad
\begin{array}{cc}
E &  \\
W & \\
\end{array}
\]

\[ H(i) = 1 = H(k) \quad \text{and} \quad H(i, j, k) = 1 + H(.4, .6) + .6 \log 3 > 2 \]
Example 2
Example 2

\( Q \) is described in the Figure below,

\[
\begin{array}{ccc}
L & R & T \\
0 & 5 & 1 \quad | \quad 0 & 5 & 1 \\
5 & 0 & 1 \quad | \quad 5 & 0 & 1 \\
B & & \\
\end{array}
\]
Example 2

$Q$ is described in the Figure below, $P_2$ chooses the rows ($T$ or $B$),

\begin{align*}
T & \quad B \\
T & \quad B
\end{align*}
Example 2

$Q$ is described in the Figure below, $P2$ chooses the rows ($T$ or $B$), $P3$ chooses the columns ($L$ or $R$),

\[
\begin{array}{cc}
T & B \\
L & \qquad & R \\
\end{array}
\begin{array}{cc}
T & B \\
L & \qquad & R \\
\end{array}
\begin{array}{cc}
T & B \\
L & \qquad & R \\
\end{array}
\begin{array}{cc}
T & B \\
L & \qquad & R \\
\end{array}
\]
Example 2

$Q$ is described in the Figure below, $P_2$ chooses the rows ($T$ or $B$), $P_3$ chooses the columns ($L$ or $R$), Temporal state of Nature is $E$ or $W$ iid $1/2,1/2$.

\[
\begin{array}{c|c|c|c|c}
 & L & R \\
\hline
T &  &  & &  \\
\hline
B &  &  & &  \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
 & L & R \\
\hline
T &  &  & &  \\
\hline
B &  &  & &  \\
\hline
\end{array}
\]
Example 2

$Q$ is described in the Figure below, $P_2$ chooses the rows ($T$ or $B$), $P_3$ chooses the columns ($L$ or $R$), Temporal state of Nature is $E$ or $W$ iid $1/2,1/2$. The matrix entries are the desired probabilities of the action profile.

\begin{align*}
\begin{array}{c|cc}
 & L & R \\
\hline
T & .35 & .05 \\
B & .05 & .05 \\
\end{array}
\quad
\begin{array}{c|cc}
 & L & R \\
\hline
T & .05 & .05 \\
B & .05 & .35 \\
\end{array}
\end{align*}
Example 2

Q is described in the Figure below, P2 chooses the rows (T or B), P3 chooses the columns (L or R), Temporal state of Nature is E or W iid 1/2, 1/2. The matrix entries are the desired probabilities of the action profile.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>.35</td>
<td>.05</td>
</tr>
<tr>
<td>B</td>
<td>.05</td>
<td>.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>.05</td>
<td>.05</td>
</tr>
<tr>
<td>B</td>
<td>.05</td>
<td>.35</td>
</tr>
</tbody>
</table>

\[ H(i) = 1 = H(k) \text{ and } H(i, j, k) = 1 + H(.7, .3) + .3 \log 3 > 2 \]
Example 3

Described in the Figure below, P2 chooses the rows (T or B), P3 chooses the columns (L or R), Temporal state of nature is E or W iid 1/2, 1/2. The matrix entries are the desired probabilities of the action profile.
Example 3

$q$ described in the Figure below,
Example 3

As described in the Figure below, $P_2$ chooses the rows ($T$ or $B$),

\[
\begin{array}{c|c|c}
T & & \ \\
\hline
B & & \ \\
\end{array}
\quad \begin{array}{c|c|c}
T & & \ \\
\hline
B & & \ \\
\end{array}
\]

$H(i) = 1 = H(k)$ and $H(i; j; k) = 1 + H(41; 59) + \frac{1}{3} \log 3 < 2$.
Example 3

As described in the Figure below, \( P2 \) chooses the rows (\( T \) or \( B \)), \( P3 \) chooses the columns (\( L \) or \( R \)),

\[
\begin{array}{cc}
T & L \quad R \\
B & \quad \quad \\
\end{array}
\]

\[
\begin{array}{cc}
T & L \quad R \\
B & \quad \quad \\
\end{array}
\]

\[
\begin{array}{cc}
T & \quad \quad \\
B & \quad \quad \\
\end{array}
\]

\[
\begin{array}{cc}
T & \quad \quad \\
B & \quad \quad \\
\end{array}
\]

\[
H(i) = 1 = H(k) \]

and

\[
H(i; j; k) + H(41; 59) + 18 \log 3 < 2
\]
Example 3

$q$ described in the Figure below, $P_2$ chooses the rows ($T$ or $B$), $P_3$ chooses the columns ($L$ or $R$), Temporal state of nature is $E$ or $W$ iid $1/2,1/2$.

\[ H(i) = 1 = H(k) \]
\[ H(i,j,k) = 1 + H(41;59) + 18 \log 3 < 2 \]
Example 3

Q described in the Figure below, P2 chooses the rows (T or B), P3 chooses the columns (L or R), Temporal state of nature is E or W iid 1/2, 1/2. The matrix entries are the desired probabilities of the action profile.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>.41</td>
<td>x_1</td>
</tr>
<tr>
<td>B</td>
<td>x_2</td>
<td>x_3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>x_3</td>
<td>x_2</td>
</tr>
<tr>
<td>B</td>
<td>x_1</td>
<td>.41</td>
</tr>
</tbody>
</table>

E  W
Example 3

As described in the Figure below, P2 chooses the rows (T or B), P3 chooses the columns (L or R), Temporal state of nature is E or W iid 1/2, 1/2. The matrix entries are the desired probabilities of the action profile.

\[
\begin{array}{ccc}
  & L & R \\
 T & .41 & x_1 \\
 B & x_2 & x_3 \\
\end{array}
\]

\[
\begin{array}{ccc}
  & L & R \\
 T & x_3 & x_2 \\
 B & x_1 & .41 \\
\end{array}
\]

\[
x_1 + x_2 + x_3 = .09
\]

\[
H(i) = 1 = H(k) \text{ and } H(i, j, k) \leq 1 + H(.41, .59) + .18 \log 3 < 2
\]
Basic model with a Markov law

- \( i_1, i_2, \ldots \) follow a Markov chain
- The Markov chain is irreducible
Basic model with a Markov law

- $i_1, i_2, \ldots$ follow a Markov chain
- The Markov chain is irreducible

Let $\mu \in \Delta(I)$ be the invariant distribution and $\hat{\mu} \in \Delta(I \times I)$ where the first coordinate has distribution $\mu$ and the transition from the first to the second is given by the transition of the Markov chain. As the distribution of $i_t$ conditional on $i_{t-1}$ is given by the Markov chain transitions we consider the implementation of distributions over $I \times I \times J \times K$ that represents the expected long-run average of $(i_{t-1}, i_t, j_t, k_t)$. 

Result: $Q_2(I \times I \times J \times K)$ is implementable iff $Q_I(I_I) = \hat{\mu}$ and $\mathbb{H}(Q(I_J, i_J, j_t, k_t)) = \mathbb{H}(Q(I_I, i_I))$.

An implicit conclusion that appears "between the lines" of this inequality is that the optimization of the forecaster and the agent needs 'banking' with entropy. Information/entropy banking appears also in Neyman and Okada 98 and Gossner and Tomala.
Basic model with a Markov law

- $i_1, i_2, \ldots$ follow a Markov chain
- The Markov chain is irreducible

**Result:** $Q \in \Delta(I \times I \times J \times K)$ is implementable
Basic model with a Markov law

- $i_1, i_2, \ldots$ follow a Markov chain
- The Markov chain is irreducible

**Result:** $Q \in \Delta(I \times I \times J \times K)$ is implementable iff $Q_{I \times I} = \hat{\mu}$ and
Basic model with a Markov law

- \( i_1, i_2, \ldots \) follow a Markov chain
- The Markov chain is irreducible

Result: \( Q \in \Delta(I \times I \times J \times K) \) is implementable iff \( Q_{I \times I} = \hat{\mu} \) and

\[
H_Q(j, i \mid k, i') \geq H_Q(i \mid i')
\]
Basic model with a Markov law

- $i_1, i_2, \ldots$ follow a Markov chain
- The Markov chain is irreducible

**Result:** $Q \in \Delta(I \times I \times J \times K)$ is implementable iff $Q_{I \times I} = \hat{\mu}$ and

$$H_Q(j, i \mid k, i') \geq H_Q(i \mid i')$$

An implicit conclusion that appears “between the lines” of this inequality is that the optimization of the forecaster and the agent needs ‘banking’ with entropy.
Basic model with a Markov law

- $i_1, i_2, \ldots$ follow a Markov chain
- The Markov chain is irreducible

Result: $Q \in \Delta(I \times I \times J \times K)$ is implementable iff $Q_{I \times I} = \hat{\mu}$ and

$$H_Q(j, i | k, i') \geq H_Q(i | i')$$

An implicit conclusion that appears “between the lines” of this inequality is that the optimization of the forecaster and the agent needs ‘banking’ with entropy

Information/entropy banking appears also in Neyman and Okada 98 and Gossner and Tomala.
We study repeated games where players strategies are implementable by finite state machines like finite automata or bounded recall strategies. We are interested in the analysis of such interaction where the power of the machines are differentiated.

In particular, we wish to study to what extent can a powerful machine that breaks a complicated code of a simple machine share its codes with a simple machine.
Repeated game strategies

\[ \sum_i \text{all pure strategies of player } i \]
Repeated game strategies

- $\Sigma_i$ all pure strategies of player $i$
- $\Sigma_i(m)$ all pure strategies of player $i$ that are implementable by an automaton of size $m$
Repeated game strategies

- $\Sigma_i$ all pure strategies of player $i$
- $\Sigma_i(m)$ all pure strategies of player $i$ that are implementable by an automaton of size $m$
- $\Sigma^*_i(m)$ all non-interactive pure strategies of player $i$ that are implementable by an automaton of size $m$. 

$X_i(m) := \Sigma_i(m)$
Repeated game strategies

- $\Sigma_i$ all pure strategies of player $i$
- $\Sigma_i(m)$ all pure strategies of player $i$ that are implementable by an automaton of size $m$
- $\Sigma_i^*(m)$ all non-interactive pure strategies of player $i$ that are implementable by an automaton of size $m$.
- $X_i(m) := \Delta(\Sigma_i(m))$
Repeated game strategies

- $\Sigma_i$ all pure strategies of player $i$
- $\Sigma_i(m)$ all pure strategies of player $i$ that are implementable by an automaton of size $m$
- $\Sigma_i^*(m)$ all non-interactive pure strategies of player $i$ that are implementable by an automaton of size $m$.
- $X_i(m) := \Delta(\Sigma_i(m))$
- $X_i^*(m) := \Delta(\Sigma_i^*(m))$
If $\mu$, $\sigma$, and $\tau$ are strategies of players 1, 2, and 3 respectively that are implementable by finite automata then the play of a repeated game enters a cycle and thus the expectation of the limiting average payoff is well defined and denoted by $g(\mu, \sigma, \tau)$. 
Main result: Finite state machines

\[ \bar{V}(m_1, m_2, m_3) = \min_{\mu \in X_1^*(m_1)} \max_{\sigma \in X_2^*(m_2)} \max_{\tau \in X_3^*(m_3)} G(\mu, \sigma, \tau) \quad (1) \]

\[ V(m_1, m_2, m_3) = \max_{\sigma \in X_2^*(m_2)} \min_{\mu \in X_1^*(m_1)} \min_{\tau \in X_3^*(m_3)} G(\mu, \sigma, \tau) \quad (2) \]

where \( G(\mu, \sigma, \tau) = g_2(\mu, \sigma, \tau) \). Note that
\( \bar{V}(m_1, m_2, m_3) \geq V(m_1, m_2, m_3) \). The main result specifies asymptotic conditions on \( m_1, m_2, m_3 \) for which the limits of \( \bar{V}(m_1, m_2, m_3) \) and \( V(m_1, m_2, m_3) \) exist and are equal. Moreover, we characterize the limit.
Given \( x \in \Delta(I) \) we denote by \( Q(x) \) the set of all probability measures \( Q \) on \( I \times J \times K \) such that

\[
H_Q(i, j, k) \geq H_Q(i) + H_Q(k).
\]

\[
v^* = \min_{x \in \Delta(I)} \max_{Q \in Q(x)} g_2(Q).
\]
Theorem 1

\[ \limsup_{\log m_3 = o(m_1) \to \infty} \bar{V}(m_1, m_2, m_3) \leq v^* \quad (3) \]

and

\[ \liminf_{m_2 > |I|^{2m_1^2} m_1 \to \infty, m_3 \to \infty} V(m_1, m_2, m_3) \geq v^* \quad (4) \]

Special cases of the result are of interest and generalize earlier known results. Consider for example the case where \(|J| = 1\). It follows that \(Q(x)\) consists of product distributions and thus \(v^* = \min_{x \in \Delta(I)} \max_{z \in \Delta(K)} g(x, z)\) and thus the result implies the result of Ben-Porath.