Graph ensemble design for channel coding

A. Montanari[1]
A. Amraoui[2] T. Richardson[3] R. Urbanke[2]
A. Dembo[4]

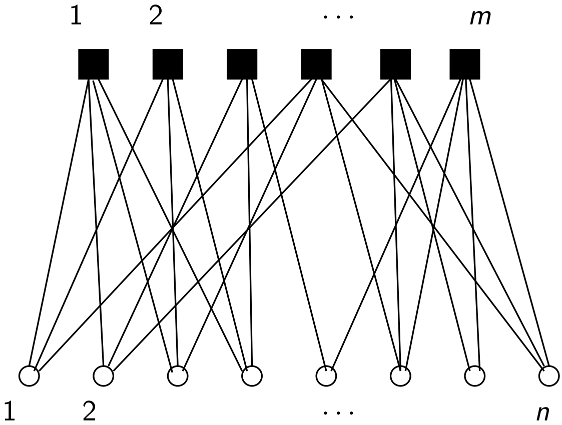
[1]ENS, France → Stanford, USA
[2]EPFL, Switzerland,
[3]Flarion, USA
[4]Stanford, USA

October 18, 2006

Outline

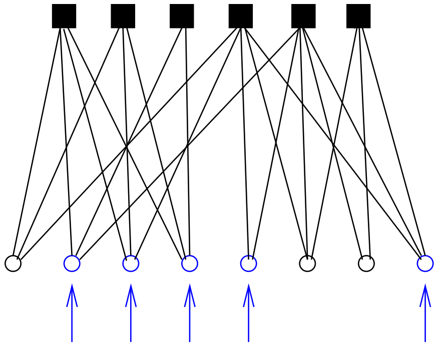
The optimization problem

The object to be optimized: A code, i.e. a graph



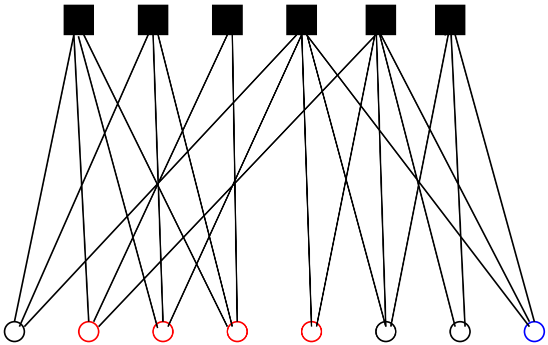
The objective function

Let $\mathcal{S} \subseteq [n]$ be random with density $\epsilon \in [0, 1]$...



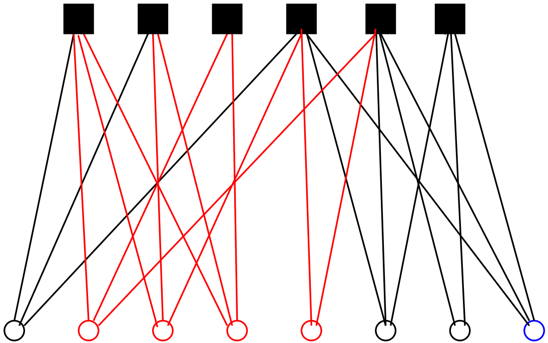
$$\mathrm{P_B}(G) = \mathbb{P}_\epsilon \{\mathcal{S} \text{ contains a 'stopping set'}\}$$

The objective function



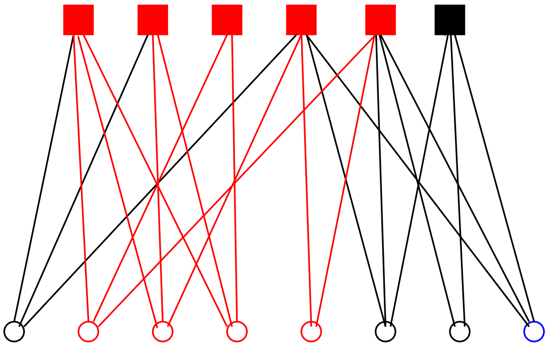
up-degree ≥ 2

The objective function



up-degree ≥ 2

The objective function



up-degree ≥ 2

If you did not get it

Bipartite graph $G \leftrightarrow$ hypergraph H

$$\mathrm{P_B} = \mathbb{P}_\epsilon\{\text{a random subgraph of } H \text{ contains a 2-core}\}$$

A probabilistic strategy

General strategy

1. Define a graph ensemble with parameters n, m, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$.

2. Prove an pproximate formula $\mathbb{E}_{\lambda,\rho} \mathrm{P_B} \simeq Q(\lambda, \rho)$. [*]

3. Find $(\lambda_*, \rho_*) = \arg\min Q(\lambda, \rho)$.

4. Sample G from the (λ_*, ρ_*)-ensemble and check it. *[Concentration]*

General strategy

1. Define a graph ensemble with parameters n, m, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$.
2. Prove an pproximate formula $\mathbb{E}_{\lambda,\rho} \mathrm{P}_{\mathrm{B}} \simeq Q(\lambda, \rho)$. [*]
3. Find $(\lambda_*, \rho_*) = \arg\min Q(\lambda, \rho)$.
4. Sample G from the (λ_*, ρ_*)-ensemble and check it. *[Concentration]*

General strategy

1. Define a graph ensemble with parameters n, m, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$.

2. Prove an pproximate formula $\mathbb{E}_{\lambda,\rho} \mathrm{P_B} \simeq Q(\lambda, \rho)$. [*]

3. Find $(\lambda_*, \rho_*) = \arg\min Q(\lambda, \rho)$.

4. Sample G from the (λ_*, ρ_*)-ensemble and check it. *[Concentration]*

General strategy

1. Define a graph ensemble with parameters n, m, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$.

2. Prove an pproximate formula $\mathbb{E}_{\lambda,\rho} \mathrm{P}_{\mathrm{B}} \simeq Q(\lambda, \rho)$. [*]

3. Find $(\lambda_*, \rho_*) = \arg\min Q(\lambda, \rho)$.

4. Sample G from the (λ_*, ρ_*)-ensemble and check it. *[Concentration]*

General strategy

1. Define a graph ensemble with parameters n, m, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, $\rho = (\rho_1, \rho_2, \dots, \rho_k)$.
2. Prove an pproximate formula $\mathbb{E}_{\lambda,\rho} \mathrm{P_B} \simeq Q(\lambda, \rho)$. [*]
3. Find $(\lambda_*, \rho_*) = \arg\min Q(\lambda, \rho)$.
4. Sample G from the (λ_*, ρ_*)-ensemble and check it. *[Concentration]*

A 'standard' ensemble

G random (configuration model) with

→Up-degree distribution: $\rho = (\rho_1, \ldots, \rho_k)$

→Down-degree distribution: $\lambda = (\lambda_1, \ldots, \lambda_k)$

Good for $n = \infty$! [Luby et al.]

Approximate formula for $\mathbb{E}_{\lambda,\rho}\mathrm{P}_\mathrm{B}$

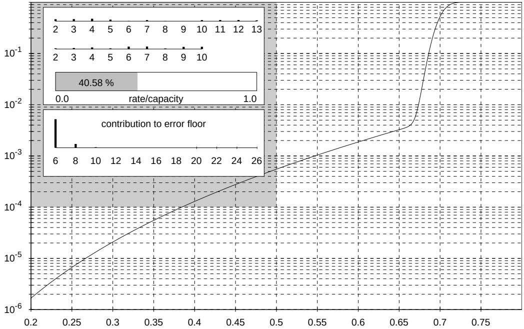
LATER!!!

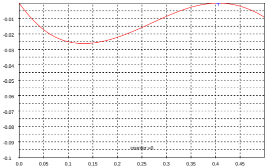
Implementation [Amraoui/Montanari/Urbanke]

Sample Run:
Minimize m/n, given $\mathrm{P_B} = 10^{-4}$

- $\epsilon = 0.5$
- $n = 5000$
- Largest degree 13
- Expurgation 6 *[I did not explain this]*

Example Run



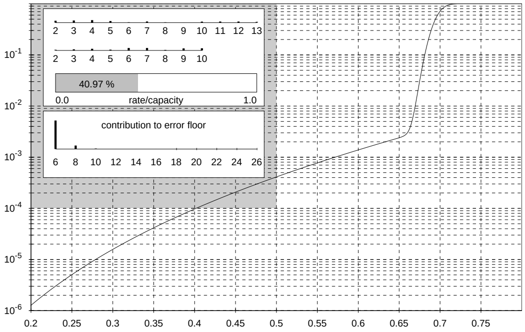


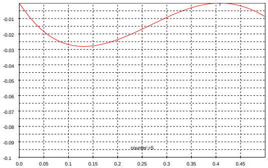
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

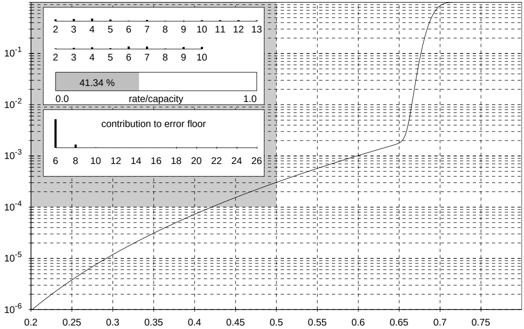


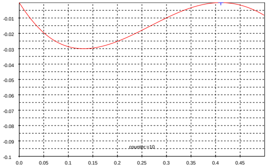


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



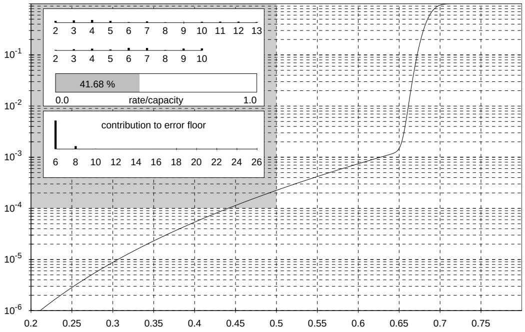


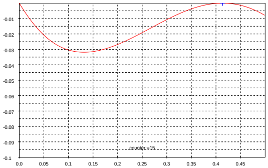
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



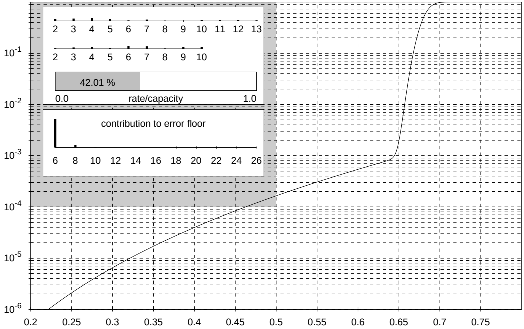


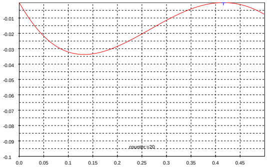
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

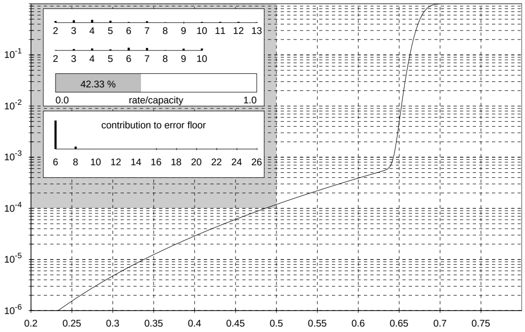


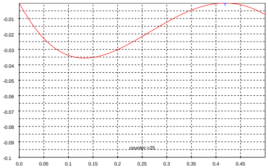


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



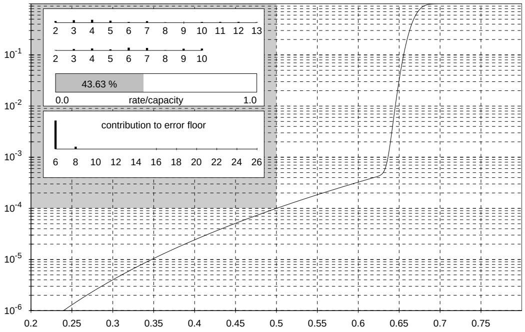


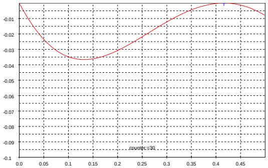
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



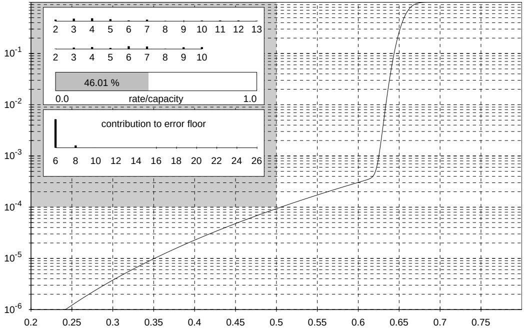


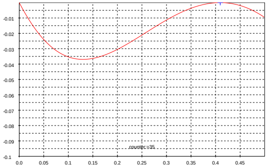
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



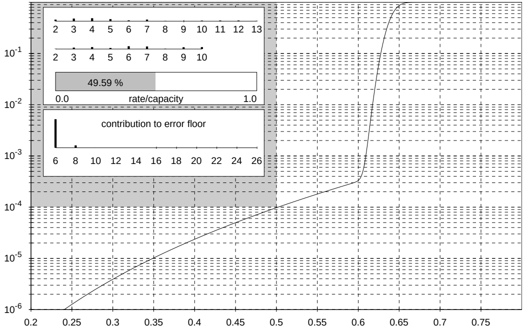


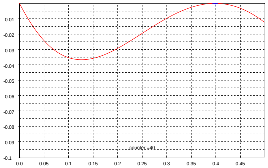
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



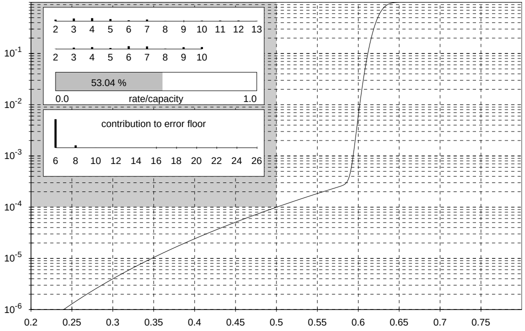


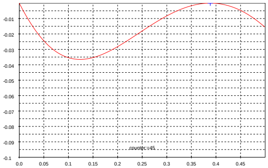
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

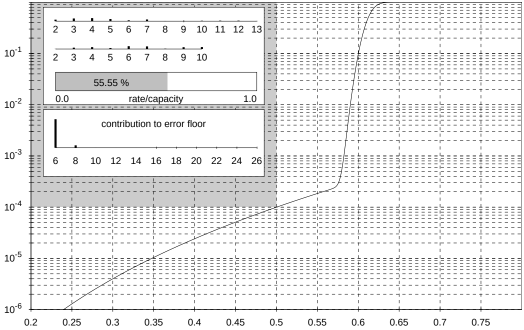


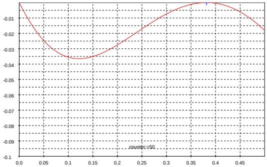


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



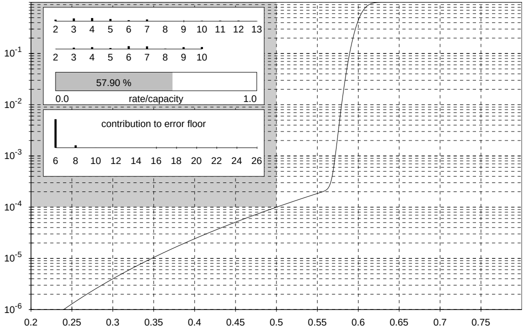


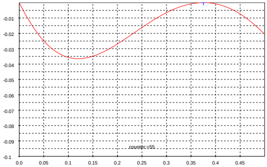
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

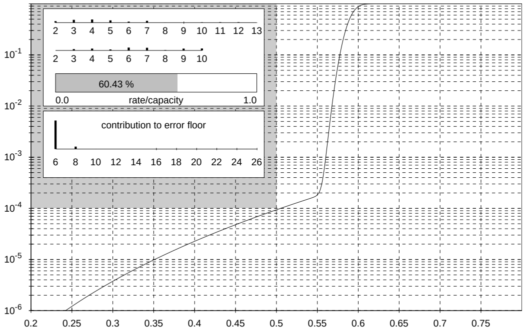


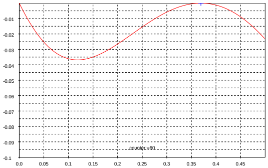


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



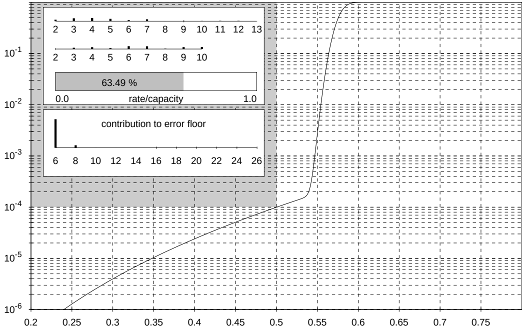


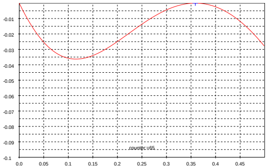
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



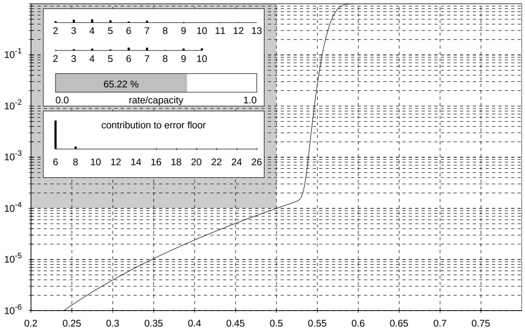


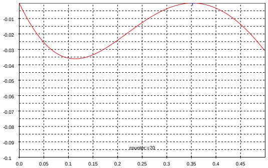
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



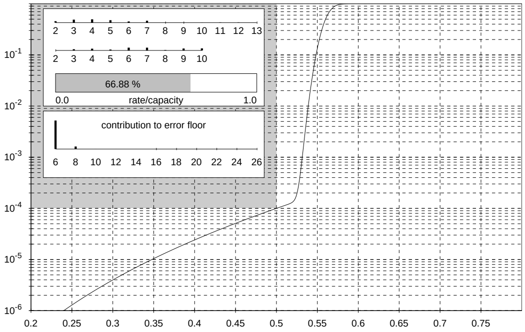


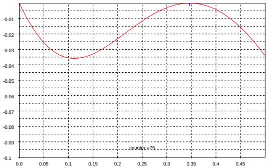
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



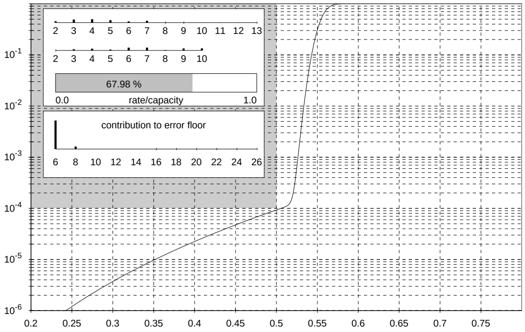


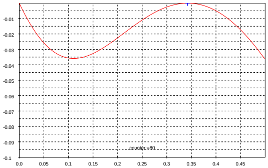
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



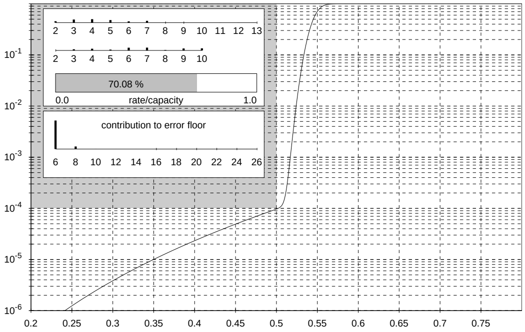


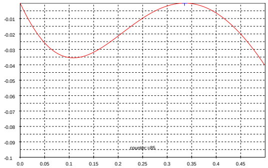
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



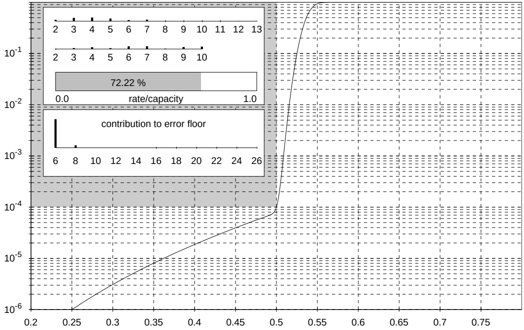


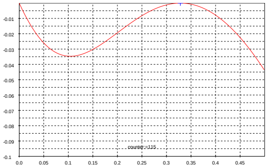
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

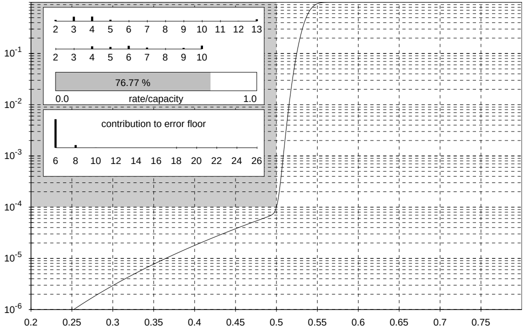


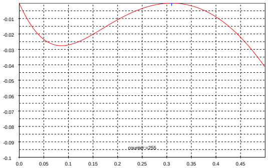


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



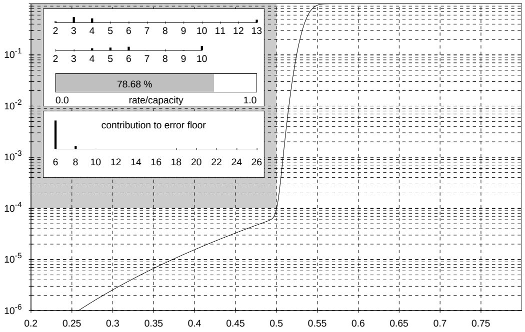


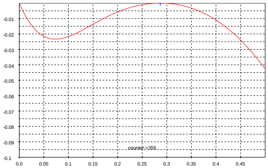
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

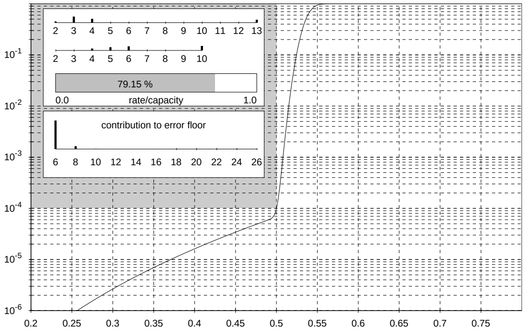


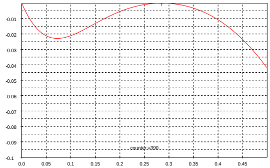


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



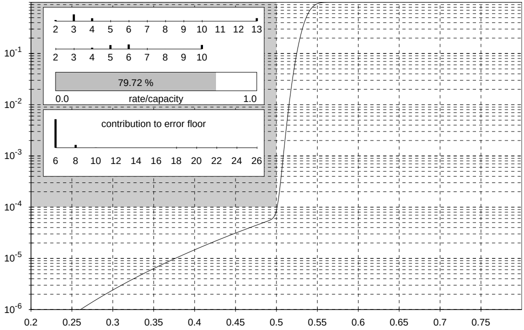


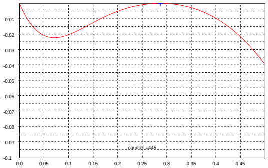
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



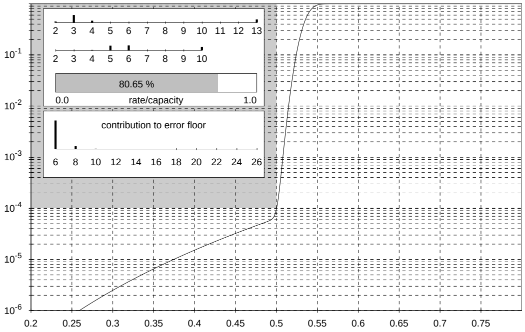


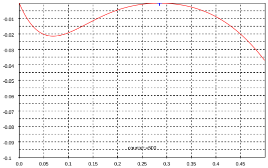
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



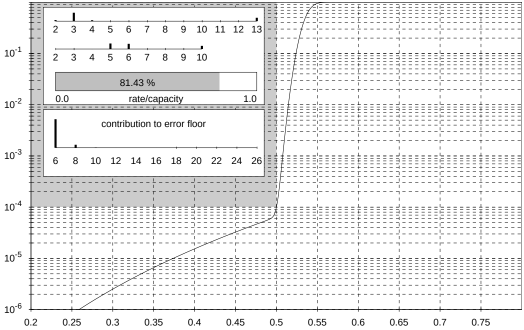


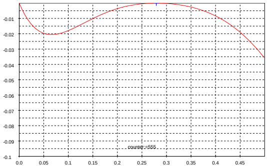
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

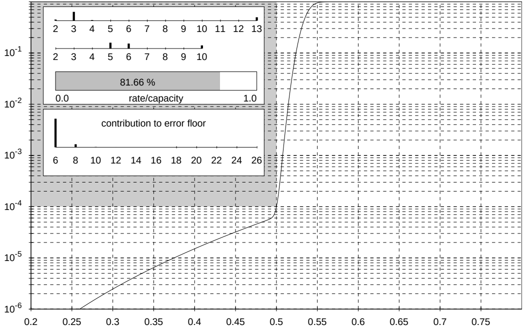


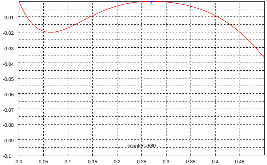


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



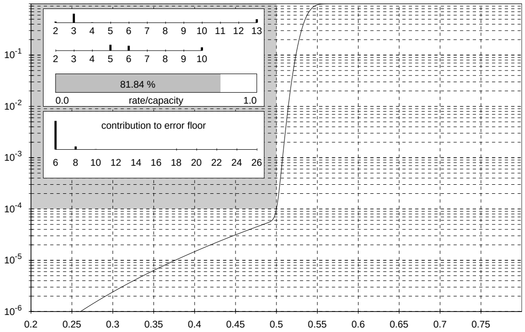


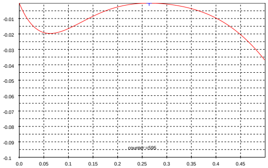
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



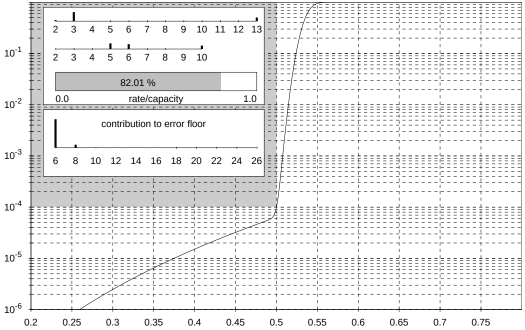


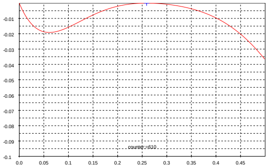
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



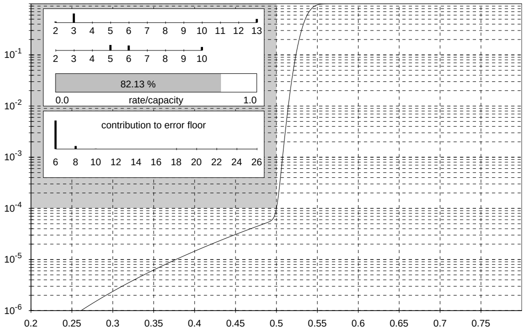


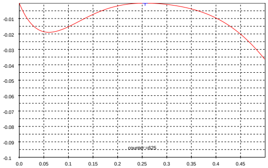
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



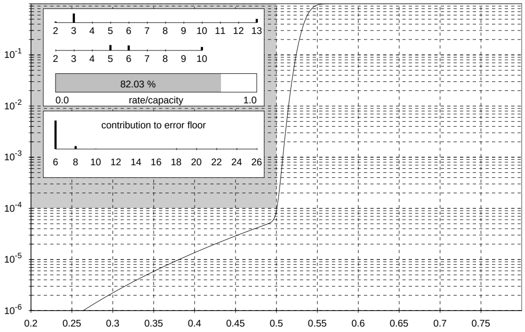


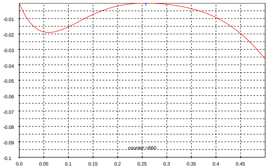
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run



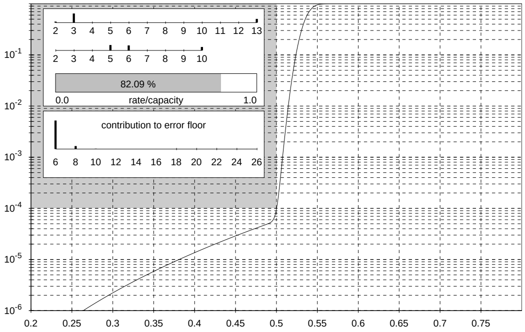


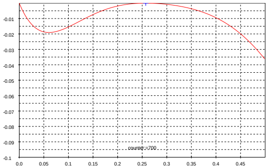
$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

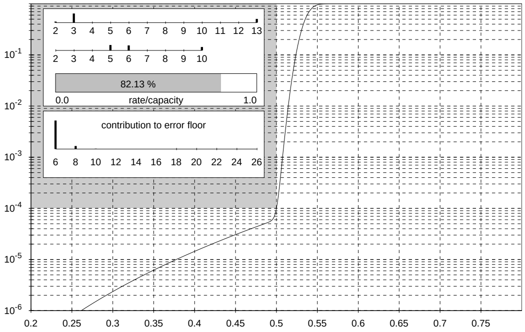


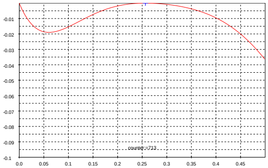


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run

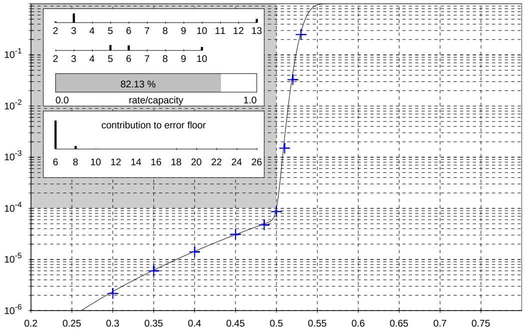


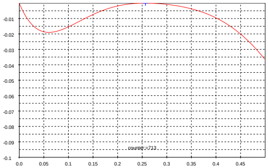


$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$
$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

Example Run





$$\lambda = 0.0739196x + 0.65789x^2 + 0.2681x^{12},$$

$$\rho = 0.390753x^4 + 0.361589x^5 + 0.247658x^9.$$

Play it Again Sam!

The approximate formula

For the sake of simplicity

I'll just consider the interesting part of the formula.

Up-degree Poisson, down regular

Want to approximate the probability that a random l-hypergraph contains a 2-core

For the sake of simplicity

I'll just consider the interesting part of the formula.

Up-degree Poisson, down regular

Want to approximate the probability that a random l-hypergraph contains a 2-core

Theorem (Luby et al., 1997, Darling/Norris, 2003)

Consider a random l-hypergraph over n edges and $m = n/\gamma$ vertices.
There exists $\gamma_{\rm d} > 0$, such that $P_B(\gamma, n) \to 0$ if $\gamma < \gamma_{\rm d}$ and $P(\gamma, n) \to 1$ if $\gamma > \gamma_{\rm d}$. Furthermore

$$\gamma_{\rm d} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0, 1]\}.$$

Theorem (Luby et al., 1997, Darling/Norris, 2003)

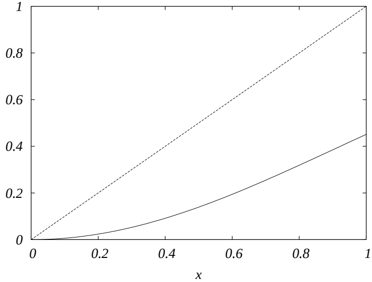
Consider a random l-hypergraph over n edges and $m = n/\gamma$ vertices.
There exists $\gamma_{\rm d} > 0$, such that $P_B(\gamma, n) \to 0$ if $\gamma < \gamma_{\rm d}$ and $P(\gamma, n) \to 1$ if $\gamma > \gamma_{\rm d}$. Furthermore

$$\gamma_{\rm d} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0,1]\}\,.$$

Theorem (Luby et al., 1997, Darling/Norris, 2003)

Consider a random l-hypergraph over n edges and $m = n/\gamma$ *vertices.*
There exists $\gamma_{\rm d} > 0$*, such that* $P_B(\gamma, n) \to 0$ *if* $\gamma < \gamma_{\rm d}$ *and* $P(\gamma, n) \to 1$ *if* $\gamma > \gamma_{\rm d}$*. Furthermore*

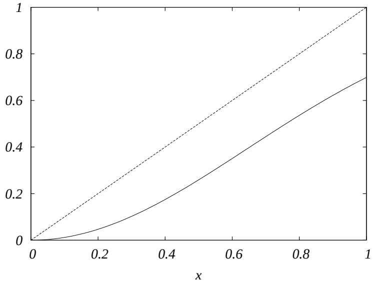
$$\gamma_{\rm d} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0, 1]\}\,.$$



Theorem (Luby et al., 1997, Darling/Norris, 2003)

Consider a random l-hypergraph over n edges and $m = n/\gamma$ *vertices.*
There exists $\gamma_{\rm d} > 0$*, such that* $P_B(\gamma, n) \to 0$ *if* $\gamma < \gamma_{\rm d}$ *and* $P(\gamma, n) \to 1$ *if* $\gamma > \gamma_{\rm d}$*. Furthermore*

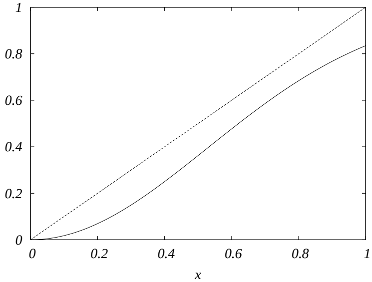
$$\gamma_{\rm d} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0, 1]\}\,.$$



Theorem (Luby et al., 1997, Darling/Norris, 2003)

Consider a random l-hypergraph over n edges and $m = n/\gamma$ *vertices.*
There exists $\gamma_{\mathrm{d}} > 0$*, such that* $P_B(\gamma, n) \to 0$ *if* $\gamma < \gamma_{\mathrm{d}}$ *and* $P(\gamma, n) \to 1$ *if* $\gamma > \gamma_{\mathrm{d}}$*. Furthermore*

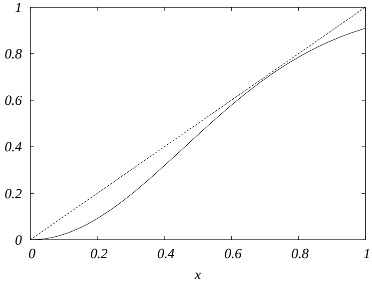
$$\gamma_{\mathrm{d}} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0, 1]\}.$$



Theorem (Luby et al., 1997, Darling/Norris, 2003)

Consider a random l-hypergraph over n edges and $m = n/\gamma$ vertices.
There exists $\gamma_{\mathrm{d}} > 0$, such that $P_B(\gamma, n) \to 0$ if $\gamma < \gamma_{\mathrm{d}}$ and $P(\gamma, n) \to 1$ if $\gamma > \gamma_{\mathrm{d}}$. Furthermore

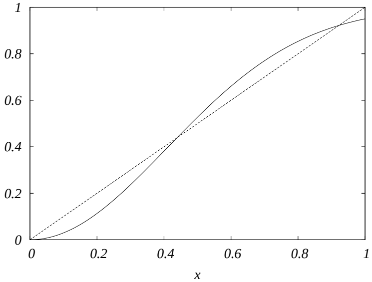
$$\gamma_{\mathrm{d}} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0, 1]\}\,.$$



Theorem (Luby et al., 1997, Darling/Norris, 2003)

Consider a random l-hypergraph over n edges and $m = n/\gamma$ vertices.
There exists $\gamma_{\rm d} > 0$, such that $P_B(\gamma, n) \to 0$ if $\gamma < \gamma_{\rm d}$ and $P(\gamma, n) \to 1$ if $\gamma > \gamma_{\rm d}$. Furthermore

$$\gamma_{\rm d} = \sup\{\gamma \mid x > 1 - e^{-l\gamma x^{l-1}} \quad \forall x \in (0,1]\}\,.$$



Pittel, Spencer, Wormald, 1996:
Analogous [classic!] theorem for k-cores of random graphs.

Finite-size scaling

Theorem (Dembo/Montanari 2006)

Let $\gamma_n = \gamma_{\rm d} + r\, n^{-1/2}$. *Then*

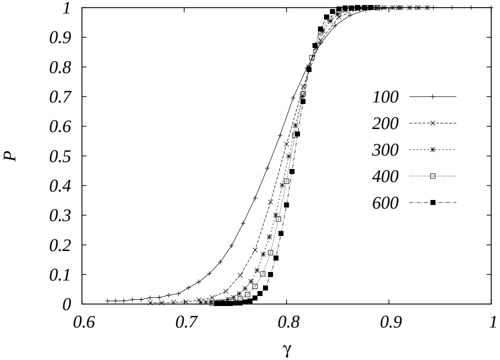
$$P(\gamma_n, n) = \Phi(r/\alpha) + \beta\Omega\, \Phi'(r/\alpha)\, n^{-1/6} + o(n^{-1/6})\,,$$

where $\Phi(x)$ *is the error function,* $\Omega \equiv \int_0^\infty [1 - \mathcal{K}(z)^2]\, dz$*, and*

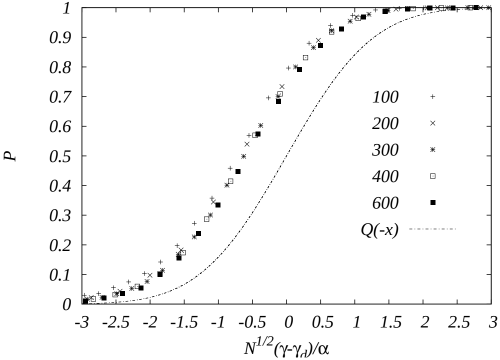
$$\mathcal{K}(z) \equiv \frac{1}{2} \int \frac{{\rm Ai}(iy){\rm Bi}(2^{1/3}z + iy) - {\rm Ai}(2^{1/3}z + iy){\rm Bi}(iy)}{{\rm Ai}(iy)}\, dy\,.$$

Finally α *and* $\beta\ldots$ *[solution ODE's]*

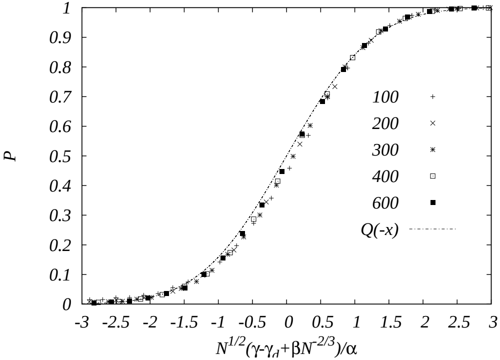
Data



Scaling: Without correction



Scaling: With correction



Analytical (!) expression for the scaling parameters

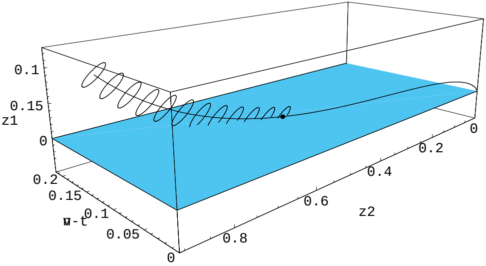
$$\alpha = \left(\frac{\rho(\bar{x})^2 - \rho(\bar{x}^2) + \rho'(\bar{x})(1 - 2x\rho(\bar{x})) - \bar{x}^2\rho'(\bar{x}^2)}{L'(1)\lambda(y)^2\rho'(\bar{x})^2} + \frac{\epsilon^2\lambda(y)^2 - \epsilon^2\lambda(y^2) - y^2\epsilon^2\lambda'(y^2)}{L'(1)\lambda(y)^2} \right)^{1/2},$$

$$\beta = \ldots$$

For those who have read Wormald et al.

Basic ideas in the proof

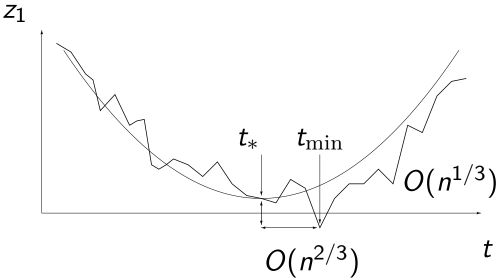
- Graph-pruning procedure as a Markov process in $\mathbb{R}^{2k}$.
- Prove a local CLT for the state.
- Approximate by parabola + Brownian near critical point (KMT).



$(z_1(t), z_2(t))$ discrete time Markov process

$$\mathrm{P_B} = \mathbb{P}_{\rm gauss}\left\{\min_t z_1(t) \le 0\right\}\,.$$

Correction term



Model: $z_1(t) = t^2/2n + \text{BM}(t)$

$\Rightarrow O(n^{-1/2}) \cdot O(n^{1/3}) = O(n^{-1/6})$ corrections.

Future directions

Future directions

- Other ensembles (Turbo, RA, multi-edge, etc.)
- Code dependency whithin the ensemble.
- *General memoryless channels.*