Various Techniques for Nonlinear Energy-Related Optimizations

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Acknowledgements

**Caltech**: Steven Low, Somayeh Sojoudi

**Columbia University**: Ramtin Madani

**UC Berkeley**: David Tse, Baosen Zhang

**Stanford University**: Stephen Boyd, Eric Chu, Matt Kraning

Optimizations:
- Resource allocation
- State estimation
- Scheduling

Issue: Nonlinearities

Transition from traditional grid to smart grid:
- More variables (10X)
- Time constraints (100X)
**OPF:** Given constant-power loads, find optimal $P$'s subject to:
- Demand constraints
- Constraints on $V$'s, $P$'s, and $Q$'s.

**Complex power:** $VI^* = P + Qi$

**Current $I$:**

**Voltage $V$:**
Summary of Results

Project 1: How to solve a given OPF in polynomial time? (joint work with Steven Low)

- A sufficient condition to globally solve OPF:
  - Numerous randomly generated systems
  - IEEE systems with 14, 30, 57, 118, 300 buses
  - European grid

- Various theories: It holds widely in practice

Project 2: Find network topologies over which optimization is easy? (joint work with Somayeh Sojoudi, David Tse and Baosen Zhang)

- Distribution networks are fine.
- Every transmission network can be turned into a good one.
Summary of Results

Project 3: How to design a parallel algorithm for solving OPF? (joint work with Stephen Boyd, Eric Chu and Matt Kranning)

- A practical (infinitely) parallelizable algorithm
- It solves 10,000-bus OPF in 0.85 seconds on a single core machine.

Project 4: How to do optimization for mesh networks? (joint work with Ramtin Madani)

Project 5: How to relate the polynomial-time solvability of an optimization to its structural properties? (joint work with Somayeh Sojoudi)

Project 6: How to solve generalized network flow (CS problem)? (joint work with Somayeh Sojoudi)
Convexification

Flow: \[ P_{ij} + Q_{ij} \sqrt{-1} = V_i (V_i - V_j) \ast \frac{1}{Z_{ij}^*} \]

Injection: \[ P_i = \sum_{j \in \mathcal{N}(i)} P_{ij} \]

Polar: \[ V_i \implies (|V_i|, \theta_i) \]

Rectangular: \[ V_i \implies (\text{Re}\{V_i\}, \text{Im}\{V_i\}) \]
Convexification in Polar Coordinates

\[ P_{ij} = |V_i|^2 G_{ij} - |Y_{ij}| |V_i||V_j| \cos(\theta_{ij} + \angle Z_{ij}) \]

\[ Q_{ij} = |V_i|^2 (-B_{ij}) - |Y_{ij}| |V_i||V_j| \sin(\theta_{ij} + \angle Z_{ij}) \]

\[ P_{ji} = |V_j|^2 G_{ij} - |Y_{ij}| |V_i||V_j| \cos(-\theta_{ij} + \angle Z_{ij}) \]

\[ Q_{ji} = |V_j|^2 (-B_{ij}) - |Y_{ij}| |V_i||V_j| \sin(-\theta_{ij} + \angle Z_{ij}) \]

**Theorem**

*Having fixed \(|V_1|, \ldots, |V_n|\), the functions \(P_{ij}, Q_{ij}, P_i\) and \(Q_i\)'s are all convex in \(\theta_1, \ldots, \theta_n\) if*

\[ 0 \leq \pm \theta_{ij} + \angle Z_{ij} \leq 90^\circ \]

Similar to the condition derived in Ross Baldick's book

<table>
<thead>
<tr>
<th>(\frac{X_{ij}}{R_{ij}})</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\max</td>
<td>\theta_{ij}</td>
<td>)</td>
<td>18.43°</td>
<td>11.30°</td>
</tr>
</tbody>
</table>

- ☑ Imposed implicitly (thermal, stability, etc.)
- ☑ Imposed explicitly in the algorithm
Convexification in Polar Coordinates

\[
P_{ij} = |V_i|^2 G_{ij} - |Y_{ij}| |V_i||V_j| \cos(\theta_{ij} + \angle Z_{ij})
\]

\[
Q_{ji} = |V_i|^2 (-B_{ij}) - |Y_{ij}| |V_i||V_j| \sin(\theta_{ij} + \angle Z_{ij})
\]

\[
P_{ji} = |V_j|^2 G_{ij} - |Y_{ij}| |V_i||V_j| \cos(-\theta_{ij} + \angle Z_{ij})
\]

\[
Q_{ij} = |V_j|^2 (-B_{ij}) - |Y_{ij}| |V_i||V_j| \sin(-\theta_{ij} + \angle Z_{ij})
\]

**Theorem**

*Having fixed \(\theta_{ij}\)'s satisfying*

\[
0 \leq \pm \theta_{ij} + \angle Z_{ij} \leq 90^\circ,
\]

the functions \(P_{ij}, Q_{ij}, P_i\) and \(Q_i\)'s are all convex in \(\sqrt{|V_1|}, \ldots, \sqrt{|V_n|}\).

**Idea:**

\[
|V_i|^2 \Rightarrow X_i
\]

\[
-|V_i||V_j| \Rightarrow -\sqrt{X_i} \sqrt{X_j}
\]

**Algorithm:**

- Fix magnitudes and optimize phases
- Fix phases and optimize magnitudes
Convexification in Polar Coordinates

\[ P_{ij} = |V_i|^2 G_{ij} - |Y_{ij}| |V_i||V_j| \cos(\theta_{ij} + \angle Z_{ij}) \]

\[ Q_{ji} = |V_i|^2 (-B_{ij}) - |Y_{ij}| |V_i||V_j| \sin(\theta_{ij} + \angle Z_{ij}) \]

\[ P_{ji} = |V_j|^2 G_{ij} - |Y_{ij}| |V_i||V_j| \cos(-\theta_{ij} + \angle Z_{ij}) \]

\[ Q_{ji} = |V_j|^2 (-B_{ij}) - |Y_{ij}| |V_i||V_j| \sin(-\theta_{ij} + \angle Z_{ij}) \]

- Can we jointly optimize phases and magnitudes?

**Change of variables:**

\[ |V_i| \Rightarrow X_i^{\frac{1}{m}} \]

**Assumption (implicit or explicit):**

\[ 45^\circ < \pm \theta_{ij} + \angle Z_{ij} < 90^\circ \]

- **Observation 1:** Bounding \( |V_i| \) is the same as bounding \( X_i \).
- **Observation 2:** \(-|V_i||V_j| \sin(\pm \theta_{ij} + \angle Z_{ij})\) is convex for a large enough \( m \).
- **Observation 3:** \(-|V_i||V_j| \cos(\pm \theta_{ij} + \angle Z_{ij})\) is convex for a large enough \( m \).
- **Observation 4:** \( |V_i|^2 \) is convex for \( m \leq 2 \).
Convexification in Polar Coordinates

**Strategy 1:** Choose $m=2$.

\[ P_{ij} = |V_i|^2 G_{ij} - |Y_{ij}||V_i||V_j| \cos(\theta_{ij} + \angle Z_{ij}) \quad \rightarrow \quad P_{ij} \approx |V_i|^2 G_{ij} - |Y_{ij}| \cos(\theta_{ij} + \angle Z_{ij}) \]

**Strategy 2:** Choose $m$ large enough

- $P_{ij}$, $Q_{ij}$, $P_i$ and $Q_i$ become convex after the following approximation:

Replace $|V_j|^2$ with its nominal value.
**Example 1**

\[
\begin{aligned}
\min_{x_1, x_2} & \quad x_1^4 + ax_2^2 + bx_1^2 x_2 + cx_1 x_2 \\
\text{Trick:} & \quad x_1^4 = (x_1^2)^2 = x_3^2
\end{aligned}
\]

\[
\begin{aligned}
\min_{x \in \mathbb{R}^4} & \quad x_3^2 + ax_2^2 + bx_3 x_2 - cx_1 x_2 \\
\text{s.t.} & \quad x_1^2 - x_3 x_4 = 0 \\
& \quad x_2^2 - 1 = 0
\end{aligned}
\]

SDP relaxation: 
\[
xx^* \rightarrow W \\
x_i x_j \rightarrow W_{ij}
\]

\[
\begin{aligned}
\min_{W \in \mathcal{S}^4} & \quad W_{33} + aW_{22} + bW_{32} + cW_{12} \\
\text{s.t.} & \quad W_{11} - W_{34} \leq 0 \\
& \quad W_{44} - 1 = 0 \\
& \quad W \succeq 0
\end{aligned}
\]

- Guaranteed rank-1 solution!
Example 1

Opt: \[
\min_{x_1, x_2} \ x_1^4 + a_0 x_2^2 + b_0 x_1^2 x_2 + c_0 x_1 x_2 \\
\text{s.t.} \quad x_1^4 + a_j x_2^2 + b_j x_1^2 x_2 + c_j x_1 x_2 \leq \alpha_j \quad j = 1, \ldots, m
\]

- **Sufficient condition for exactness:** Sign definite sets.
- **What if the condition is not satisfied?** Rank-2 W (but hidden)

Complex case:
Formal Definition: Optimization over Graph

Optimization of interest: \[
\min_{x \in \mathbb{D}^n} f_0(y, z)
\]
(real or complex)\[
\text{s.t. } f_j(y, z) \leq 0, \quad j = 1, 2, \ldots, m
\]

Define:\[
y = \big\{ |x_i|^2 \mid \forall i \in \mathcal{G} \big\}
\]
\[
z = \left\{ \text{Re}\left\{ c_{ij}^1 x_i x_j^* \right\}, \ldots, \text{Re}\left\{ c_{ij}^k x_i x_j^* \right\} \mid \forall (i, j) \in \mathcal{G} \right\}
\]

- SDP relaxation for \(y\) and \(z\) (replace \(xx^*\) with \(W\)).
- \(f(y, z)\) is increasing in \(z\) (no convexity assumption).

- **Generalized weighted graph**: weight set \(\{ c_{ij}^1, \ldots, c_{ij}^k \}\) for edge \((i,j)\).
Theorem (Real Case)

Exact relaxation if

\[ \prod_{(i,j) \in \mathcal{O}_r} \sigma_{ij} = (-1)^{|\mathcal{O}_r|}, \quad r \in \{1, \ldots, p\} \]

(i, j) \in \mathcal{G}

\[ \sigma_{ij} \neq 0, \]

Edge

Cycle

Theorem (Complex Case)

Exact relaxation for acyclic graphs with sign-definite weight sets.

Theorem (Imaginary Case)

Exact relaxation for weakly cyclic graphs with homogeneous weight sets.
Convexification in Rectangular Coordinates

\[
\begin{align*}
\min_{V,P_G,Q_G} & \quad \sum_{k \in G} f_k(P_{G_k}) \\
\text{Subject to} & \quad P_{k}^{\text{min}} \leq P_{G_k} \leq P_{k}^{\text{max}} \quad (1b) \\
& \quad Q_{k}^{\text{min}} \leq Q_{G_k} \leq Q_{k}^{\text{max}} \quad (1c) \\
& \quad V_{k}^{\text{min}} \leq |V_k| \leq V_{k}^{\text{max}} \quad (1d) \\
& \quad \text{Re} \left\{ V_l(V_l - V_m)^* y_{lm}^* \right\} \leq P_{lm}^{\max} \quad (1e) \\
& \quad \text{trace}\{V V^* Y (e_k e_k^*)\} = P_{G_k} - P_{D_k} + (Q_{G_k} - Q_{D_k}) i \quad (1f)
\end{align*}
\]

- Express the last constraint as an inequality.

**Trick:** Replace \( V V^* \) with a matrix \( W \succeq 0 \) subject to \( \text{rank}\{W\} = 1 \).
**Convexification in Rectangular Coordinates**

\[
\begin{align*}
\min_{\mathbf{V}} & \quad h_0(\mathbf{P}, \mathbf{Q}, |\mathbf{V}|) \\
\text{s.t.} & \quad h_j(\mathbf{P}, \mathbf{Q}, |\mathbf{V}|) \leq 0, \quad j = 1, \ldots, m
\end{align*}
\]

**Theorem**

*Exact relaxation for DC/AC distribution and DC transmission networks.*

- Partial results for AC lossless transmission networks.
Phase Shifters

Theorem

Exact relaxation for AC networks with virtual phase shifters.

Practical approach: Add phase shifters and then penalize their effects.

\[
\sum_{i \in G} f_i(P_i) \quad \rightarrow \quad \sum_{i \in G} f_i(P_i) + \lambda(\phi_1 + \cdots + \phi_k)
\]

Stephen Boyd’s function for PF:

\[
\cos(\theta - 4\sin(\theta))
\]
Synchronous machine with interval voltage $|E|e^{j\delta}$ and terminal voltage $|V|e^{j\theta}$.

Swing equation:

$$\frac{d\delta(t)}{dt} = \omega(t)$$

$$M\frac{d\omega}{dt} = -D\omega(t) + P_M(t) - \frac{|E||V(t)|\sin(\delta(t)) - \theta(t))}{\alpha}$$

Define: $x(t) = \begin{bmatrix} 1 & \omega(t) & \text{Re}\{E\} & \text{Im}\{E\} & \text{Re}\{V(t)\} & \text{Im}\{V(t)\} \end{bmatrix}^H$

Linear system:

$$\frac{dW_{14}(t)}{dt} = W_{32}(t)$$

$$\frac{dW_{12}(t)}{dt} = -\frac{D}{M}W_{12}(t) - \frac{1}{M\alpha}(W_{45}(t) - W_{36}(t)) + \frac{1}{M}P_M(t)$$
Sparse Solution to OPF

- Unit commitment:
  1. \( \alpha_i P_i^{\text{min}} \leq P_i \leq \alpha_i P_i^{\text{max}} \)
  2. \( \alpha_i \in \{0, 1\} \)

- Sparse solution to OPF:
  1. \( 0 \leq P_i \leq P_i^{\text{max}} \)
  2. Sparse vector \([P_1 \ P_2 \ \cdots \ P_n]\)

- Minimize:
  \[ \sum_{i=1}^{n} f_i(P_i) + \sum_{i=1}^{n} w_i P_i \]

<table>
<thead>
<tr>
<th>IEEE system</th>
<th>14 bus</th>
<th>30 bus</th>
<th>118 bus</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of ”on” generators</td>
<td>4-1</td>
<td>6-3</td>
<td>54-9</td>
</tr>
</tbody>
</table>

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Lossy Networks

- Relationship between polar and rectangular?

- Assumption (implicit or explicit): \[ 45^\circ < \pm \theta_{ij} + \angle Z_{ij} < 90^\circ \]

- Conjecture: This assumptions leads to convexification in rectangular coordinates.

- Partial Result: Proof for optimization of reactive powers.
Consider a lossless AC transmission network.

\[(P_{12}, P_{23}, P_{31})\]

\[(P_1, P_2)\]

\((P_{1}, P_{2}, P_{3})\) for a 4-bus cyclic Network:

Theorem: The injection region is never convex for \(n \geq 5\) if

\[|\theta_{ij}| \leq \theta_{ij}^{\text{max}} < 90^\circ, \quad (i, j) \in \mathcal{E}\]

Current approach: Use polynomial Lagrange multiplier (SOS) to study the problem
OPF With Equality Constraints

- Injection region under fixed voltage magnitudes:

- When can we allow equality constraints? Need to study Pareto front

\[ \theta_{ij}^{\text{max}} \leq \tan^{-1}\left(\frac{X_{ij}}{r_{ij}}\right) \]
Generalized Network Flow (GNF)

Goal:
\[ \min \sum_{i \in N} f_i(p_i) \]

Assumption:
- \( f_i(p_i) \): convex and increasing
- \( f_{ij}(p_{ij}) \): convex and decreasing

Injections:
\[ p_i = \sum_{j \in \mathcal{N}(i)} p_{ij} \]

Flows:
\[
\begin{align*}
p_{ji} &= f_{ij}(p_{ij}), \\
p_{ij} &\in [p_{ij}^{\min}, p_{ij}^{\max}] \end{align*}
\]

Limits:
\[ p_i^{\min} \leq p_i \leq p_i^{\max} \]
Convexification of GNF

Feasible set without box constraint

Convexification: \[ p_{ji} = f_{ij}(p_{ij}) \quad \longrightarrow \quad p_{ji} \geq f_{ij}(p_{ij}) \]

- It finds correct injection vector but not necessarily correct flow vector.
Conclusions

- **Motivation:** OPF with a 50-year history
- **Goal:** Find a good numerical algorithm

- Convexification in polar coordinates
- Convexification in rectangular coordinates
- Exact relaxation in several cases
- Some problems yet to be solved.