Primal-Dual Algorithms for Weighted Abstract Path and Cut Packing

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1. Combinatorial Optimization
   - Integral LPs
Outline

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2. Hoffman’s Models
   - Packing problems
   - Path models
   - Cut models
   - Blocking

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   - P-D for path and cut packing
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   - Flows over Time
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Much combinatorial optimization is around which LPs have guaranteed integer optimal solutions.

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  - . . .to show that a general model of max flow is still integral.
- Here we proceed in this same spirit.
Non-TUM but Integral Network LPs

There are non-TUM ways to formulate some network flow problems that still guarantee integrality:

- Max flow via flows on paths — packing paths into arcs.

E.g., if we put general "rewards" on paths, then Max Weighted Path Flow is NP Hard.
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Natural Questions

**Question 1:** Can we generalize to abstract versions of path and cut packing while maintaining integrality?
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Question 2: Both max flow-type path packing and dual-Dijkstra cut packing have all-one objective vectors, and are known to be fractional and NP Hard with general objectives. For which more general objectives are we still guaranteed integrality?
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**Question 2:** Both max flow-type path packing and dual-Dijkstra cut packing have all-one objective vectors, and are known to be fractional and NP Hard with general objectives. For which more general objectives are we still guaranteed integrality?

**Question 3:** Can we find polynomial algorithms for these abstract weighted path and cut packing problems?
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- The decision is to choose a weight $y_D$ to put on each $D \in \mathcal{D}$ such that the total weight packed into $e$ is at most $u_e \ \forall \ e \in E$. 

We are usually interested in finding integer optimal solutions. 

This generic problem has many applications, e.g., flow is packing paths into arcs, connectivity is packing trees into edges, etc.
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- And among such feasible packings, find one that maximizes $r^T y$. 

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Packing as an LP

- Now formulate a packing problem as an LP (it’s more natural to make packing the dual):

\[
\begin{align*}
\text{(D)} & \quad \max \sum_{D \in D} r_D y_D \\
\text{(P)} & \quad \min \sum_{e \in E} u_e x_e \\
\text{s.t.} & \quad \sum_{D \ni e} y_D \leq u_e \forall e \in E \\
& \quad \sum_{e \in D} x_e \geq r_D \forall D \in D \\
& \quad y \geq 0 \\
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The dual linear programs are:

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“packing subsets into elements” \hfill “covering subsets by elements”
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Big Question: When do these LPs have guaranteed integer optimal solutions?
An example packing LP

Consider:

\[
\begin{align*}
\text{max } & \ 1^T y \\
\text{s.t. } & \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} y \leq \begin{pmatrix} 1 \\ 5 \\ 5 \\ 8 \\ 4 \\ 7 \\ 9 \\ 3 \\ 6 \end{pmatrix} \\
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- Does this LP have an integer optimal solution?
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Consider:

\[ \text{max } 1^T y \]

Subject to:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
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\end{pmatrix}
\begin{pmatrix} y \end{pmatrix} \leq 
\begin{pmatrix}
1 \\
5 \\
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\]

\[ y \geq 0. \]

- Does this LP have an integer optimal solution?
- What if we change the RHS \( u \)? The objective \( r \)?
More on the example

- This LP has an integer optimal solution: \( y^* = (1 4 0 4 0 0 3 0 0) \) of value 12.
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The same holds true for some objectives \( r \):

- E.g., \( r = (4 \ 3 \ 2 \ 3 \ 1 \ 1 \ 3 \ 2 \ 4) \) has integer optimal solution \( y^* = (1 \ 4 \ 0 \ 4 \ 0 \ 0 \ 0 \ 0 \ 3) \) of value 40 for the given RHS \( u \), and this is true for any integral \( u \).
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- E.g., $r = (0 9 0 0 9 0 0 9 0)$ has fractional optimal solution $y^* = (0 4.5 0 0 0.5 0 0 3.5 2.5)$ with value 76.5 for the given RHS $u$. 
More on the example

- This LP has an integer optimal solution: \( y^* = (1 \, 4 \, 0 \, 4 \, 0 \, 0 \, 3 \, 0 \, 0) \) of value 12.
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- How do I know that the first two objectives are “good” for all RHS?
How the example was constructed

Consider the following graph:

There is a 1–1 correspondence between $E$ and the nine edges of this graph.
There is a 1–1 correspondence between the 9 interesting $s$–$t$ cuts in this graph and the columns of the constraint matrix.

Why does this lead to integer optimal LP solutions?
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- Recall that the primal covering LP has variables $x_e \ldots$
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- ... and constraints $\sum_{e \in D} x_e \geq 1$ for all $D \in \mathcal{D}$. 

Imagine that $x$ is 0–1, so that it picks out a subset of edges.

What subsets of edges hit every $s$–$t$ cut?

The $s$–$t$ paths are the minimal edge subsets hitting every $s$–$t$ cut, i.e., the $s$–$t$ paths are the blocker of the $s$–$t$ cuts.

Therefore the primal LP is just Shortest Path.

And in fact Dijkstra’s Algorithm gives an integer optimal solution to this form of Shortest Path.
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Going back to the dual packing LP

- Here is the Dijkstra solution with its shortest path tree:
Going back to the dual packing LP

- Here is the Dijkstra solution with its shortest path tree:

- Recall that we can greedily construct a tight cut packing that proves that this shortest path tree is optimal:
Generalizing this behavior

- Since we know that Dijkstra, and this greedy cut packing, work for any non-negative capacities $u$, we know that we get integer optimal solutions for all RHS $u$. 

It is very cool that this random-looking constraint matrix always has an integer optimal solution with the special objective vector $1$. LPs such as this where you get guaranteed integer optimal solutions for all RHSs, but only for some special objective vectors, are called Totally Dual Integral, or TDI. A natural question here is whether we can generalize this sort of example to a broader class of packing LPs with 0–1 constraint matrices. Hoffman did it...
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Picture of Crossing Axiom

$P \times_e Q$

$P \times_e Q$

$P$

$Q$

$e$
Possible that $e \notin P \times_e Q$
Picture of Crossing Axiom

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  - Alan remarked in a 2010 email to me “when I first wrote the paper with the [super]modular $r$ (rather than all 1’s), I put in the $r$ because it came free”.
  - Alan earlier verbally told me that he put in the supermodular $r$ because he wanted to imitate the nice things that Jack Edmonds was doing.
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As a bonus, Bill relayed to us Alan’s concrete suggestion for an oracle for the max flow \((r \equiv 1)\) version: You send the oracle a subset \(S\) of the elements, and it tells you whether there is a path \(P\) with \(P \subseteq S\) (and \(<_P\)) or not.
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Ordinary cuts are consecutive \( (e \in R \cap T \implies e \in S) \):
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**Theorem (Hoffman & Schwartz ’76)**

*When $r$ and $u$ are integral, (P) and (D) have integral optimal solutions.*
Other applications

Lattice polyhedra would not be so interesting unless they included interesting applications other than Shortest Path:

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- Etc, etc . . .
Blocking Carries Over

Suppose that $\mathcal{L}$ is a clutter, i.e., if $R, S \in \mathcal{L}$, then $R \not\subset S$ and $S \not\subset R$ (edge sets of ordinary cuts are a clutter). Then
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**Theorem (Hoffman ’78)**

*If $\mathcal{L}$ is a submodular clutter, then the blocker of $\mathcal{L}$ is an abstract path system.*
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*If $\mathcal{L}$ is a submodular clutter, then the blocker of $\mathcal{L}$ is an abstract path system.*

Thus Weighted Abstract Flow and Weighted Abstract Cut Packing carry over the blocking relationship of ordinary $s-t$ paths and cuts.
Blocking Carries Over

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What remains now is Q3:

*Are there polynomial algorithms for solving Weighted Abstract Flow and Cut Packing?*
Outline

1. Combinatorial Optimization
   - Integral LPs

2. Hoffman’s Models
   - Packing problems
   - Path models
   - Cut models
   - Blocking

3. Algorithms
   - Primal-Dual Algorithm
   - P-D for path and cut packing

4. Extensions
   - Flows over Time
   - Parametric Capacities

5. Conclusion
   - Open questions
The Primal-Dual Algorithm

- Recall the Primal-Dual (Successive Shortest Path, SSP) Algorithm for max flow at min cost.
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- It greedily pushes flow on the cheapest (shortest) augmenting path.

**Primal-Dual Algorithm:**

- Set $x = 0$, $\pi = 0$.
- While augmenting paths remain do
  - Use Shortest Path to compute the subnetwork $S$ of min-cost augmenting paths (dual change).
  - Use Max Flow to augment all paths in $S$ (primal change).
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P-D for Path and Cut Packing 1

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\[ \max \text{ instead of } \min = \Rightarrow \text{must start with max weight paths.} \]

Define \( \lambda \) as the weight of the current highest-reward path; initially \( \lambda = \max \mathcal{P} \).

Relax \( y(P) \geq r \mathcal{P} \) to \( y(P) \geq r \mathcal{P} - \lambda \).

[When \( \lambda = r_{\text{max}} \), \( x = y = 0 \) is optimal.]

Now decrease \( \lambda \) to 0, keeping optimality \( \Rightarrow \) when \( \lambda = 0 \) we are optimal.

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For fixed $\lambda$, focus on subnetwork of paths with $\text{gap}(P) = y(P) - r_P + \lambda = 0$.

Lemma: this subnetwork still satisfies the axioms.

But $R = \{e | y_e > 0\}$ is restricted to be tight, i.e., $\sum_{P \ni e} x_P = u_e$.

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Abstract MF.
Since restr. subnetwork is MF, it's blocked by a Min Cut \( l \).
Here \( l \) is 0, ±1:

\[
L_+ \subseteq R_-
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\[
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Since restr. subnetwork is cut pack, it's blocked by a SP \( l \).
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- Solve $\text{gap}(P) = 0$ subnetwork using extension of Mc ’95
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Update

\[
y' \leftarrow y + \theta l
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\]

\[
\Rightarrow \text{gap}'(P) \leftarrow \text{gap}(P) + \theta(l(P) - 1)
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Lemma: \( \theta \) is always an integer.

If \( \theta \) is determined by \( \text{gap}'(P) \geq 0 \) as

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P-D for Path and Cut Packing 5

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Outline

1. Combinatorial Optimization
   - Integral LPs

2. Hoffman’s Models
   - Packing problems
   - Path models
   - Cut models
   - Blocking

3. Algorithms
   - Primal-Dual Algorithm
   - P-D for path and cut packing

4. Extensions
   - Flows over Time
   - Parametric Capacities

5. Conclusion
   - Open questions
Abstract Flows over Time

Question: can we extend the max flow version of WAF to flows over time ("dynamic flows") with time horizon $T$?

Answer (J. Matuschke): Yes, via extending Ford and Fulkerson's ideas about flows over time. For ordinary networks, can compute flows over time via a time-expanded network.

McCormick et al (UBC et al)
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- Same idea works for abstract networks, but need to repeat path flows over time.
The Static Abstract Network

- Now each element $e$ has a time delay $\tau_e$, so it takes time $\tau(P) = \sum_{e \in P} \tau_e$ for flow to traverse path $P$. 

**Lemma**

This $r(P)$ is supermodular. Thus we can solve max abstract flow over time in polynomial time (modulo lots of details).

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Abstract Path & Cut Packing

Hoffman-Fest Sept 2014
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  - It is needed for transportation problems, but they use modular $r$.
- This application to max abstract flow finally gives us an application where the supermodularity was really necessary.
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Extensions

Parametric Capacities

Parametric Abstract Flow?

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- General result by Topkis: Consider $\min f(x, \lambda)$ with $x$ on a lattice. Suppose that

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\text{2} \quad \text{For all } x \preceq y \text{ and } \lambda_1 \leq \lambda_2 \text{ we have Decreasing Differences}
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Gallo, Grigoriadis, and Tarjan (GGT) considered such a class, and showed that you can compute all min cuts in $O^*(1)$ Push-Relabel time.

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- Then parametric abstract flows over time :-)?
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6. One can make a good career out of answering open questions in Alan’s papers :-)
I dedicate this talk to Alan Hoffman’s 90th birthday, and to his long and fruitful career.

Questions?

Comments?