Generalized Trade Reductions: The Role of Competition in Designing Budget-Balanced Mechanisms

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Abstract

When designing a mechanism there are several desirable properties to maintain such as incentive compatibility (IC), individual rationality (IR), and budget balance (BB). It is well known [15] that it is impossible for a mechanism to maximize social welfare whilst also being IR, IC, and BB. There have been several attempts to circumvent [15] by trading welfare for BB, e.g., in domains such as double-sided auctions[13], distributed markets[3] and supply chain problems[2, 4].

In this paper we provide a procedure called a *Generalized Trade Reduction (GTR)* for *single-value players*, which given an IR and IC mechanism, outputs a mechanism that is IR, IC and BB with a loss of welfare. We bound the welfare achieved by our procedure for a wide range of domains. In particular, our results improve on existing solutions for problems such as double-sided markets with homogenous goods, distributed markets and several kinds of supply chains. Furthermore, our solution provides budget balanced mechanisms for several open problems such as combinatorial double-sided auctions and distributed markets with strategic transportation edges.

1 Introduction

When designing a mechanism there are several key properties that are desirable to maintain. Some of the more important ones are *individual rationality (IR)* - to make it worthwhile for

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all players to participate, *incentive compatibility (IC)* - to give incentive to players to report their true value to the mechanism, and *budget balance (BB)* - not to run the mechanism at a loss. In many of the mechanisms the goal function that a mechanism designer attempts to maximize is the *social welfare*[1] - the total benefit to society. However, it is well known from [15] that any mechanism that maximizes social welfare while maintaining individual rationality and incentive compatibility runs a deficit perforce, i.e., is *not* budget balanced.

Of course, for many applications of practical importance we lack the will and the capability to allow the mechanism to run a deficit and hence one must balance the payments made by the mechanism. To maintain the BB property in an IR and IC mechanism it is necessary to compromise on the optimality of the social welfare.

1.1 Related Work and Specific Solutions

There have been several attempts to design budget-balanced mechanisms for particular domains[2]. For instance, for double-sided auctions where both the buyers and sellers are strategic and the goods are homogeneous [13] (or when the goods are heterogeneous [5]). [13] developed a mechanism that given valuations of buyers and sellers produces an *allocation* (which are the trading players) and a *matching* between buyers and sellers such that the mechanism is IR, IC, and BB while retaining most of the social welfare. For matched pairs of buyers and sellers encounter a transaction cost [9] developed a mechanism that is IR, IC and BB that also retains most of the social welfare. In the distributed markets problem (and closely related problems) goods are transported between geographic locations while incurring some constant cost for transportation. [16, 3] present mechanisms that approximate the social welfare while achieving an IR, IC and BB mechanism. For supply chain problems [2, 4] bound the loss of social welfare that is sufficient to inflict on the mechanism in order to achieve the desired combination of IR, IC, and BB.[3]

Despite the work discussed above, the question of how to design a general mechanism that achieves IR, IC, and BB independent of the problem domain remains open. Furthermore, there are several domains where the question of how to design an IR, IC and BB mechanism that approximates the social welfare remains an open problem. For example, in the important domain of multi-minded combinatorial double-sided auctions (or even for unknown single-minded combinatorial double-sided auctions) there is no known result that bounds the loss of social welfare needed to achieve budget balance. Another interesting example is the open question left by [3]:How can one bound the loss in social welfare that is needed to achieve budget balance in an IR and IC distributed market where the transportation edges are strategic. Naturally an answer to the BB distributed market with strategic edges has vast practical implications, for example to transportation networks.

[1]Social Welfare is also referred to as efficiency in the economics literature.

[2]A servay of all of the particular domains used in this paper can be found in Appendix B

[3]A through discussion of the related work and the implications of this work on the related work can be found in section 5

1.2 Our Contribution

In this paper we unify all the problems discussed above (both the solved as well as the open ones) into one solution concept procedure. The solution procedure called the *Generalized Trade Reduction (GTR)*. GTR accepts an IR and IC mechanism for *single-valued players* and outputs an IR, IC and BB mechanism. The output mechanism may suffer some welfare loss as a tradeoff of achieving BB. There are problem instances in which no welfare loss is necessary but by [15] there are problem instances in which there is welfare loss. Nevertheless for a wide class of problems we are able to bound the loss in welfare. A particularly interesting case is one in which the input mechanism is an efficient allocation.

In addition to unifying many of the BB problems under a single solution concept, the GTR procedure improves on existing results and solves several open problems in the literature. The existing solutions our GTR procedure improves are homogeneous double-sided auctions, distributed markets [3], and supply chain [2, 4]. For the homogeneous double-sided auctions the GTR solution procedure improves on the well known solution [13] by allowing for more cases where no trade reduction takes place. For the distributed markets in [3] and the supply chain [2, 4] the GTR solution procedure improves the bound on the welfare loss, i.e., allows one to achieve an IR, IC and BB mechanism with smaller loss on the social welfare. Recently we also learned that the GTR procedure allows one to turn the model newly presented [6] into a BB mechanism. The open problems that are answered by GTR are distributed markets with strategic transportation edges and bounded paths, unknown single-minded combinatorial double-sided auctions, multi-minded combinatorial double-sided auctions with bounded size of the trading group i.e., a buyer and the sellers of his bundled goods, and multi-minded combinatorial double-sided auctions with a bounded number of possible trading groups.

The GTR procedure succeeded in achieving all the improvements described above by identifying the key element in maintaining BB: competition. Two types of competition are defined; internal competition and external competition. Another important contribution of the paper is defining two general classes of problem domains; class based domains and procurement-class based domains. The classification of the problem domains is made possible using the newly defined competition concepts. Most of the studied problem domains are of the more restrictive domains, the procurement class based domains. We believe that the more general setting will inspire further research.

An early version of this paper was published at the EC'07 conference. This version elaborates on the conference version in number of ways:

(1)A Number of examples were added to illustrate the new concepts of internal competition, external competition, class based domains and procurement class based domains. The added examples illustrate the new concepts on known problems domains. (2) The paper presents two procedures of generalized trade reduction GTR-1 and GTR-2. Some of the proofs of GTR-2's properties were omitted from the conference version and added to this version. (3) Section 5 is added in this version and did not appear in the conference version. Section 5 thoroughly introduces the existing literature that achieves IR, IC and BB in specific domains. The section explains how BB was achieved in the related work and shows how the GTR procedure changes the existing mechanisms in the literature and how it would improve results if applied. Section 5 is important to the understanding of the generality and implications of this work.

2 Preliminaries

2.1 The Model

In this paper we design a method which given any IR and IC mechanism outputs a mechanism that maintains the IC and IR properties while achieving BB. For some classes of mechanisms we bound the competitive approximation of welfare.

In our model there are N players divided into sets of trade. The sets of trade are called *procurement sets* and are defined (as per [2]) as follows:

DEFINITION 2.1. *A* procurement set $\mathbf{s}$ *is the smallest set of players that is required for trade to occur.*

For example, in a double-sided auction, a procurement set is a pair consisting of a buyer and a seller. In a combinatorial double-sided auction a procurement set can consist of a buyer and several sellers. We mark the set of all procurement sets as $\mathbf{S}$ and assume that any allocation is a disjoint union of procurement sets.

Each player i, $1 \leq i \leq n$, assigns a real value $v_i(\mathbf{s})$ to each possible procurement set $\mathbf{s} \in \mathbf{S}$. Namely, $v_i(\mathbf{s})$ is the valuation of player i on procurement set $\mathbf{s}$. We assume that for each player i $v_i(\mathbf{s})$ is i's private value and that i is a *single value player*, meaning that if $v_i(\mathbf{s}_j) > 0$ then for every other $\mathbf{s}_k$, $k \neq j$, either $v_i(\mathbf{s}_k) = v_i(\mathbf{s}_j)$ or $v_i(\mathbf{s}_k) = 0$. For ease of notation we denote by v_i the value of player i for any procurement set $\mathbf{s}$ such that $v_i(\mathbf{s}) > 0$. The set $V_i \subseteq \mathbb{R}$ is the set of all possible valuations v_i. The set of all possible valuations of all the players is denoted by $V = V_1 \times ... \times V_n$. Let $v_{-i} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$ be the vector of valuations of all the players beside player i, and let V_{-i} be the set of all possible vectors v_{-i}.

We denote by $W(\mathbf{s})$ the *value of a procurement set* $\mathbf{s} \in \mathbf{S}$ such that $W(\mathbf{s}) = \sum_{i \in \mathbf{s}} v_i(\mathbf{s}) + F(\mathbf{s})$, where F is some function that assigns a constant to procurement sets. For example, F can be a (non-strategic) transportation cost in a distributed market problem. Let the *size of a procurement set* $\mathbf{s}$ be denoted as $|\mathbf{s}|$.

It is assumed that any allocation is a disjoint union of procurement sets and therefore one can say that an allocation partitions the players into two sets; a set of players who trade and a set of players who do not trade.

The paper denotes by O the set of possible partitions of an allocation A into procurement sets. The *value* $W(A)$ *of an allocation* A is the sum of the values of its most efficient partition to procurement sets, that is $W(A) = \max_{S \in O} \sum_{\mathbf{s} \in S} W(\mathbf{s})$. This means that $W(A) = \sum_{i \in A} v_i + \max_{S \in O} \sum_{\mathbf{s} \in S} F(\mathbf{s})$. In the case where F is identically zero, then $W(A) = \sum_{i \in A} v_i$.

An *optimal partition* $S^*(A)$ is a partition that maximizes the above sum for an allocation A. Let the value of A be $W(S^*(A))$ (note that the value can depend on F). We say that the allocation A is *efficient* if there is no other allocation with a higher value. The *efficiency* of the allocation $\hat{A}$ is $\frac{W(\hat{A})}{W(A)}$, where A is a maximal valued allocation. We assume w.l.o.g. that there are no two allocations with the same value[4].

[4]Ties can be broken using the identities of the players.

A *mechanism* M defines an allocation and payment rules, $M = (R, P)$. A *payment rule* P decides i's payment p_i where P is a function $P : V \rightarrow \mathbb{R}^N$. We work with mechanisms in which players are required to report their values. An example of such a mechanism is the VCG mechanism [17, 8, 10]. The reported value $b_i \in V_i$ of player i is called a *bid* and might be different from his private value v_i. Let $b \in V$ be the bids of all players. An *allocation rule* R decides the allocation according to the reported values $b \in V$. We make the standard assumption that players have *quasi-linear utility* so that when player i trades and pays p_i then his utility is $u_i(v_i, b_{-i}) = v_i - p_i$, $u_i : V \Rightarrow \mathbb{R}$. We also assume that players are *rational utility maximizers.*

Mechanism M is *Budget Balanced (BB)* if $\sum_{i \in N} p_i \geq 0$ for any bids $b \in V$. M is *Incentive-Compatible (IC) in dominant strategies* if for any player i, value v_i and any $b_{-i} \in V_{-i}$, $u_i(v_i, b_{-i}) \geq u_i(b)$ meaning that for any player i, bidding v_i maximized i's utility over all possible bids of the other players. M is (ex-post) *Individually Rational (IR)* if for any player i value v_i, and any $b_{-i} \in V_{-i}$ $u_i(v_i, b_{-i}) \geq 0$ meaning that for all possible bids of the other players, player's i utility is non-negative. Note that since our mechanisms are normalized IR, if a player does not trade then the player pays 0 and has utility 0.

The algorithm presented in the next section employs a commonly used payment scheme, the critical value payment scheme.

DEFINITION 2.2. *Critical value payment scheme: A mechanism $M(R, P)$ uses a critical value payment scheme if given an allocation A determent by allocation rule R it charges every player $i \in A$, p_i such that if $b_i < p_i$ then allocation rule R decides on allocation A' such that $i \neq A'$. I.e., the mechanism $M(R, P)$ charges players the minimum value they need to report to the mechanism in order to remain allocated.*

We denote by C_i the critical value price computed for player i.

2.2 Competitions and Domains

In this paper we present two generalized trade reduction algorithms. The two algorithms are such that given an IR and IC mechanism M that solves a problem in some domain (different domains are formally defined below), turns M into IR, IC and BB mechanism. The algorithm presented finds procurement sets and removes them in iterations until the "right conditions" are fulfilled and the mechanism M is turned into a BB one. The "right conditions" that need to be met are conditions of competition among the players in the given problem.

Before we dive into the competition condition definition let us first illustrate the connection between competition and the BB property. Consider a double sided auction mechanism with three buyers and three sellers. The buyers values are $v_1^b = 10$, $v_2^b = 9$, and $v_3^b = 7$. The sellers values are $v_1^s = 1$, $v_2^s = 3$ and $v_3^s = 5$. In the efficient allocation all the players trade: v_1^b is matched with v_1^s, v_2^b is matched with v_2^s etc. As the critical value for a buyer to trade is 5 and the critical value for a seller to trade is 7 the IC payment scheme yields that every buyer pays 5 and every seller expects a payment of 7. The budget deficit is then $3x2 = 6$. If we relax efficiency and reduce the most insignificant trade, i.e., the match v_3^b and v_3^s then the critical value for a buyer to trade is 7 and the critical value for a seller to trade is 5. The IC payment scheme in this case yields that every trading buyer pays 7 and every trading seller expects payment of 5. Therefore the budget has a surplus of $2x2 = 4$. Why does the reduction of the least significant trade turn the mechanism from creating a budget deficit to creating a surplus? By reducing the trade with the lowest value we induce competition to players who are not reduced. This competition means that players have to bid higher than a valid trade and hence the total payment are positive.

The following definitions allow us to create the competition conditions require.

Definition 2.3. *For any player $i \in N$, we say that the set $R_i \subseteq N \setminus \{i\}$ is a* ***replacing set*** *of i, if for any procurement set $\mathbf{s} \in \mathbf{S}$ such that $i \in \mathbf{s}$ and $R_i \cap \mathbf{s} = \emptyset$, $\mathbf{s} \setminus \{i\} \cup R_i \in \mathbf{S}$.*

For example, in a (homogeneous) double-sided auction (see problem B.1) the replacement set for any buyer is simply any other buyer. In an auction for transportation slots (see problem B.7), the replacement set of an edge is a path between the endpoints of the edge. Note that a set can replace a single player. Furthermore, this relationship is transitive but *not* necessarily symmetric. If i is a replacement set for j, it is not necessarily true that j is a replacement set for i.

Definition 2.4. *For any allocation A, procurement set $\mathbf{s} \subseteq A$, and any $i \in \mathbf{s}$ we say $R_i(A, \mathbf{s})$ is an* ***internal competition*** *for i with respect to A and $\mathbf{s}$, if $R_i(A, \mathbf{s}) \subseteq N \setminus A$ is a replacement set for i s.t. $T = \mathbf{s} \setminus \{i\} \cup R_i(A, \mathbf{s}) \in \mathbf{S}$ and $W(T) \geq 0$.*

Definition 2.5. *For any allocation A and procurement set $\mathbf{s} \subseteq A$ and any $i \in \mathbf{s}$ we say that $E_i(A, \mathbf{s})$ is an* ***external competition*** *for i with respect to A and $\mathbf{s}$, if $E_i(A, \mathbf{s}) \subseteq N \setminus A$ is a set s.t., $T = \{i\} \cup E_i(A, \mathbf{s}) \in \mathbf{S}$ and $W(T) \geq 0$.*

To illustrate the internal and external competition concepts recall our previous example of the three buyers and three sellers. When all three buyers and three sellers are trading, none of the players has any competition by definition. When we remove the least significant trade, i.e., v_3^b with v_3^s then:

- the buyer $v_3^b = 7$ becomes the external competition for the sellers v_2^s and v_1^s as their values are 3 and 1 and become the internal competition for the other trading buyers v_2^b and v_1^b as their values are 9 and 10.

- the seller $v_3^s = 5$ becomes the external competition for the buyers v_2^b and v_1^b as their values are 9 and 10 and become the internal competition for the other trading sellers v_2^s and v_1^s as their values are 3 and 1.

We assume without loss of generality, that there are no ties between the values of any allocations and in particular that there are no ties between values of procurement sets. Should a tie arise it can be broken using the identities of the players[5]. So for any allocation A, procurement set $\mathbf{s}$, and player i with external competition $E_i(A, \mathbf{s})$, there exists exactly one set representing the maximally valued external competition.

Definition 2.6. *A set $X \subset N$ is closed under replacement if $\forall i \in X$ then $R_i \subset X$*

[5]The details of how to break ties in allocations are standard and are omitted.

The following defines the required competition needed to maintain IC, IR and BB. The set X[6] denotes this competition and is closed under replacement. In the remainder of the paper we assume that all of our sets which define competition in a mechanism are closed under replacement.

DEFINITION 2.7. *Let $X \subset N$ be a set that is closed under replacement. We say that the mechanism is an X-**external mechanism** if*

1. *Each player $i \in X$ has an external competition.*
2. *Each player $i \notin X$ has an internal competition.*
3. *For all players $i_1, \ldots, i_t \in \mathbf{s} \setminus X$ there exist*
$$R_{i_1}(A, \mathbf{s}), \ldots, R_{i_t}(A, \mathbf{s})$$
such that for every $i_z \neq i_q$, $R_{i_z}(A, \mathbf{s}) \cap R_{i_q}(A, \mathbf{s}) = \varnothing$
4. *for every procurement set $\mathbf{s} \in \mathbf{S}$ it holds that $\mathbf{s} \cap X \neq \varnothing$*

For general domains the choice of X can be crucial. In fact even for the same domain the welfare (and revenue) can vary widely depending on how X is defined. In appendix C we give an example where two possible choices of X yield greatly different results. Although we show that X should be chosen as small as possible we do not give any characterization of the optimality of X, which is an important open problem.

Our two generalized trade reduction algorithms will ensure that for any allocation we have the desired types of competition. So given a mechanism M that is IC and IR with allocation A, the goal of the algorithms is to turn M into an X-external mechanism. The two generalized trade reduction algorithms utilize a *dividing function* D that divides allocation A into disjoint procurement sets. The algorithms order the procurements sets defined by D in order of increasing value. For any procurement set there is a desired type of competition that depends only on the players who compose the procurement set. The generalized trade reduction algorithms go over the procurement sets in order (from the smallest to the largest) and remove any procurement set that does not have the desired competition when the set is reached. The reduction of procurement sets will also be referred to as a trade reduction.

Formally:

DEFINITION 2.8. *D is a **dividing function** if for any allocation A and the players' value vector v, D divides the allocation into disjoint procurements sets $\mathbf{s}_1, \ldots, \mathbf{s}_k$ s.t. $\cup \mathbf{s}_j = A$ and for any player i with value v_i if $i \in \mathbf{s}_{j_1}$ and $t \in \mathbf{s}_{j_2}$ s.t. $j_1 \geq j_2$ then for any value $v_i^{'} > v_i$ of player i and division by D into $\mathbf{s}_1^{'}, \ldots, \mathbf{s}_{k'}^{'}$ such that $i \in \mathbf{s}_{j_1^{'}}$ and $t \in \mathbf{s}_{j_2}^{'}$ then $j_1^{'} > j_2^{'}$.*

In other words D is monotonic non-decreasing function.

There are two general domains of problems that our generalized trade reduction algorithms can accept as an input. The formal domain definitions follow:

DEFINITION 2.9. *A domain is a* class based domain *if for all $i \in N$ and all replacement sets of i, R_i, $|R_i| = 1$ and for all i, j, $i \neq j$ if $j = R_i$ then $i = R_j$.*

[6] We present some tradeoffs between the different possible sets in Appendix C.

Intuitively this means that replacement sets are of size 1 and the replacing relationship is symmetric.

We define the class of a player i as the set of the player's replacement sets and denote the class of player i by $[i]$. It is important to note that since replacement sets are transitive relations and since class based domains also impose symmetric relations on the replacement sets, the class of a player i, $[i]$ is actually an equivalence class for i.

Definition 2.10. *A domain is a* procurement-class based domain *if the domain is a class-based domain and if for any player i such that there exists two procurement sets* $\mathbf{s}_1, \mathbf{s}_2$ *(not necessarily trading simultaneously in any allocation) such that* $i \in \mathbf{s}_1$ *and* $i \in \mathbf{s}_2$ *then there exists a bijection* $f : \mathbf{s}_1 \to \mathbf{s}_2$ *such that for any* $j \in \mathbf{s}_1$, $f(j)$ *is a replacement set for* j *in* $\mathbf{s}_2$.

Example 2.1. *A (homogeneous) double-sided auction (see problem B.1) is a procurement-class based domain. For the (homogeneous) double-sided auction each procurement set consists of a buyer and a seller. The homogeneous double-sided auction is a class based domain as for every buyer a replacing set is another buyer and for every seller a replacing set is another seller. Therefore the buyers are an equivalent class and the sellers are an equivalent class. Thus if i is a buyer in* $\mathbf{s}_1$ *under one division of D and i is in* $\mathbf{s}_2$ *under another division of D then surely the seller in* $\mathbf{s}_1$ *can be mapped into a seller in* $\mathbf{s}_2$ *such that the seller in* $\mathbf{s}_2$ *is a replacing set for the seller in* $\mathbf{s}_1$ *as the sellers are an equivalent class.*

Example 2.2. *The double sided combinatorial auction consisting of a single multi-minded buyer and multiple sellers of heterogenous goods (see problem B.9), is a class based domain as the single buyer is in one equivalent class and each set of sellers of identical good are in an equivalent class. The buyer has no replacing set and for each seller his replacing sets are sellers in his equivalent class. So every seller has replacing set of size one and the replacing set relation is symmetric for all sellers in the same equivalent class. Nevertheless the double-sided combinatorial auction consisting of a single multi-minded buyer is not a procurement-class based domain as for the buyer there is no bijection between different bundles of goods the buyer is interested in acquiring. In this case every bundle may consist of a different number of goods or may include different goods. So for example if the buyer is interested in the bundle* $q_1 = \{1, 2, 3\}$ *or* $q_2 = \{1, 3, 5, 6\}$, *the dividing function D matches the buyer* b_1 *with sellers* s_1,s_2, *and* s_3 *in one division and with sellers* s'_1,s'_3,s'_5, *and* s'_6 *in the other division. Then* $s'_1 = f(s_1)$, $s'_3 = f(s_3)$ *but* $s'_5 \neq f(s_2)$ *and* $s'_6 \neq f(s_2)$ *as* s'_1, s_1 *are in equivalent class,* s'_3, s_3 *are also in equivalent class and* s'_5, s_2 *or* s'_6, s_2 *are not in a mutually equivalent class.*

Example 2.3. *The spatially distributed market with strategic edges (see problem B.6) is not a class-based domain (and therefore not a procurement-class domain). In this case the buyers are in one equivalent class, the sellers are in another equivalent class and each edge on the graph is by itself an equivalent class as it is a unique transportation provider. Therefore even for a fixed buyer and a fixed seller there are two different procurement sets consisting of different paths between the buyer and seller. Thus there does not exist a bijection function between the edges in the different paths.*

The next sections present two algorithms GTR-1 and GTR-2. GTR-1 accepts problems in procurement-class based domains, its properties are proved with a general dividing function D. The GTR-2 algorithm accepts problems in any domain. We prove the GTR-2's properties with specific dividing function D_0. The function will be defined in section 4. Since the dividing function can have a large practical impact on welfare (and revenue) the generality of $GTR-1$ (albeit in special domains) can be an important practical consideration.

3 Procurement-Class Based Domains

This section focuses on the problems that are procurement-class based domains. For this domain, we present an algorithm called GTR-1, which given a mechanism that is IR and IC outputs a mechanism with reduced welfare that is IR, IC and budget balanced.

Although procurement class domains appear to be a relatively restricted model, in fact many domains studied in the literature are procurement class domains.

EXAMPLE 3.1. *The following domains are procurement class based domains:*

- *double-sided auctions with homogenous goods [13](problem B.1). In this domain there are two classes. The class of buyers and the class of sellers. Each procurement set consists of a single buyer and a single seller. Since every pair of* $(buyer, seller)$ *is a valid procurement set (albeit possible with negative value) this is a procurement class domain as was shown in example 2.1. In this domain the constant assigned to the procurement sets is* $F = 0$.
- *Spatially distributed markets with non strategic edges [3, 9](problem B.3). Like the double-sided auctions with homogenous goods, there are two classes in the domain; A class of buyers and a class of sellers with procurement sets consisting of a single buyer and a single seller. The buyers and sellers are part of market nodes in a graph and the function* F *is the distance between two nodes (length of the edge) that represents transport costs. These costs differ between different* $(buyer, seller)$ *pairs that are matched from two different distributed markets. Similar to the double-sided (homogenous) auction case, every player* i *who is matched with player* j *in procurement set* $\mathbf{s}_1$ *and with player* j' *in procurement set* $\mathbf{s}_2$ *will have a bijection function between* j *and* j' *as* j *and* j' *are in the same equivalent class.*
- *Supply chains [2, 4] (problems B.5.1,B.5.2). The assumption of unique manufactory by [2, 4] can best be understood as turning general supply chains (which need not be a procurement class domain) into a procurement class domain. The assumption of unique manufacturing technology in [2, 4] leads to a tree structure of the supply chain. If every node in the tree is a market of a double-sided auction and every type of good can be manufactured only in a single node in the tree, then all the buyers in node* a *who manufactured good* j *can become sellers in all the nodes that are neighbors to* a *say* b, c, d*. In this case all the sellers of good* j *in markets* b, c, d *are in the same class and every buyer* i *who is interested in good* j *and matched to two different sellers of* j *has a bijection between them. If the production of* j *was not unique then there would not necessarily be a bijection between sellers of good* j*, who essentially do not belong to the same class of sellers.*

- *Single-minded combinatorial double auctions [11] (problem B.8). In this context each seller sells a single good and each buyer wants a set of goods. The classes are the sets of sellers selling the* same *good as well as the buyers who desire the same* bundle. *A procurement set consists of a single buyer as well as a set of sellers who can satisfy that buyer. A single-minded combinatorial auction is a procurement-class based domain as every buyer in procurement sets* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *is matched to sellers of the same goods and each seller in* $\mathbf{s}_1$ *has a bijection to a different seller in* $\mathbf{s}_2$ *who is from the same class of sellers.*

A definition of the mechanism follows:

Definition 3.1. **The GTR-1 algorithm** - *given a mechanism M, a set $X \subset N$ that is closed under replacement, a dividing function D, and allocation A, GTR-1 operates as follows:*

1. *Use the dividing function D to divide A into procurement sets $\mathbf{s}_1, \ldots, \mathbf{s}_k \in \mathbf{S}$.*
   for each $\mathbf{s}_j$, $\mathbf{s}_j \cap X \neq \emptyset$
2. *Order the procurement sets by increasing value.*
3. *For each $\mathbf{s}_j$, starting from the lowest value procurement set:*
   If for every $i \in \mathbf{s}_j \cap X$ there is external competition and every $i \in \mathbf{s}_j \setminus X$ there is internal competition then keep $\mathbf{s}_j$. Otherwise reduce the trade $\mathbf{s}_j$ (i.e., remove every $i \in \mathbf{s}_j$ from the allocation).[7]
4. *All trading players are charged the critical value for trading. All non trading players are charged nothing.*

Remark 3.1. *The special case where $X = N$ has received attention under different guises in various special cases, such as ([13, 3, 4]).*

3.1 The GTR-1 Produces an X-external Mechanism that is IR, IC and BB

In this subsection we prove that the GTR-1 algorithm produces an X-external mechanism that is IR, IC and BB.

To prove GTR-1's properties we make use of theorem 3.1 that is a well known result (e.g., [14, 11]). Theorem 3.1 characterizes necessary and sufficient conditions for a mechanism for single value players to be IR and IC:

Definition 3.2. *An allocation rule R is* Bid Monotonic *if for any player i, any bids of the other players $b_{-i} \in V_{-i}$, and any two possible bids of i, $\hat{b}_i > b_i$, if i trades under the allocation rule R when reporting b_i, then i also trades when reporting $\hat{b}_i$.*

[7] Although the definition of an X-external mechanism requires that X intersects every procurement set, this is not strictly necessary. It is possible to define an X that does not intersect every possible procurement set. In this case, any procurement set $\mathbf{s} \in \mathbf{S}$ s.t. $\mathbf{s} \cap X = \varnothing$ will be reduced.

Intuitively, a bid monotonic allocation rule ensures that no trading player can become a non-trading player by improving his bid.

THEOREM 3.1. *An IR mechanism M with allocation rule R is IC if and only if R is Bid Monotonic and each trading player i pays his critical value C_i ($p_i = C_i$).*

So for normalized IR[8] and IC mechanisms, the allocation rule which is bid monotonic uniquely defines the critical values for all the players and thus the payments.

OBSERVATION 3.1. *Let M_1 and M_2 be two IR and IC mechanisms with the same allocation rule. Then M_1 and M_2 must have the same payment rule.*

In the following we prove that the X-external GTR-1 algorithm produces a IR, IC and BB mechanism, but first a subsidiary lemma is shown to help us prove the X-external GTR-1 algorithm produce properties.

LEMMA 3.1. *For procurement-class based domains if there exists a procurement set $\mathbf{s}_j$ s.t. $i \in \mathbf{s}_j$ and i has external competition then all $t \neq i$ $t \in \mathbf{s}_j$, t has internal competition.*

Proof. This follows from the definition of procurement class domains. Suppose that i has external competition, then there exists a set of players $E_i(A, \mathbf{s})$ such that $\{i\} \cup E_i(A, \mathbf{s}) \in \mathbf{S}$. Let us denote by $\mathbf{s}'_j = \{i\} \cup E_i(A, \mathbf{s})$. Since the domain is a procurement-class based domain there exists a bijection function f between $\mathbf{s}_j$ and $\mathbf{s}'_j$. f defines the required internal competition. □

We start by proving IR and IC:

LEMMA 3.2. *For any X, the X-external mechanism with a critical value pricing scheme produced by the GTR-1 algorithm is an IR and IC mechanism.*

Proof. By the definition of a critical value pricing scheme 2.2 and the GTR-1 algorithm in definition 3.1 it follows that for every trading player i, $v_i \geq 0$. By the GTR-1 algorithm in definition 3.1 non-trading players i have a payment of zero. Thus for every player i, value v_i, and any $b_{-i} \in V_{-i}$ $u_i(v_i, b_{-i}) \geq 0$, meaning the produced X-external mechanism is IR.

As the X-external GTR-1 algorithm is IR and applies the critical value payment scheme according to theorem 3.1, in order to show that the produced X-external mechanism with the critical value payment scheme is IC it remains to show that the produced mechanism's allocation rule is bid monotonic.

Since GTR-1 orders the procurement sets according to increasing value, if player i increases his bid from b_i to $b'_i > b_i$ then for any division function D of procurement sets, the procurement set $\mathbf{s}$ containing i always appears later with the bid b'_i than with the bid b_i. So the likelihood of competition can only increase if i appears in later procurement sets. This follows as GTR-1 can reduce more of the lower value procurement sets which will result in more non-trading players.

Therefore if $\mathbf{s}$ has the required competition and is not reduced with b_i then $\mathbf{s}$ will have the required competition with b'_i and will not be reduced. □

[8] Note that this is not true for mechanisms that are not normalized e.g., [7, 12]

Finally we prove BB:

LEMMA 3.3. *For any X, the X-external mechanism with critical value pricing scheme produced by the GTR-1 algorithm is a BB mechanism.*

Proof. In order to show that the produced mechanism is BB we show that each procurement set that is not reduced has a non negative budget (i.e., the sum of payments is non negative).

Let $\mathbf{s} \in \mathbf{S}$ be a procurement set that is not reduced. Let $i \in \mathbf{s} \cap X$ then according to the definition of X-external definition 2.7 and the GTR-1 algorithm in definition 3.1 i has an external competition. Assume w.l.o.g.[9] that i is the only player with external competition in $\mathbf{s}$ and all other players $j \neq i$, $j \in \mathbf{s}$ have internal competition.

Let A be the allocation after the procurement sets reduction by the GTR-1 algorithm. According to the definition of external competition 2.5, there exists a set $E_i(A, \mathbf{s}) \subset N \setminus A$ such that $i \cup E_i(A, \mathbf{s}) \in \mathbf{S}$ and $W(i \cup E_i(A, \mathbf{s})) \geq 0$. Since $W(i \cup E_i(A, \mathbf{s})) = v_i + W(E_i(A, \mathbf{s}))$ then $v_i \geq -W(E_i(A, \mathbf{s}))$. By the critical value pricing scheme definition 2.2 if player i bids any less than $-W(E_i(A, \mathbf{s}))$ he will not have external competition and therefore will be removed from trading. Thus i pays no less than $\min -W(E_i(A, \mathbf{s}))$. Since all other players $j \in \mathbf{s}$ have internal competition their critical price can not be less than their maximal value internal competitor (set) i.e., $\max W(R_j(A, \mathbf{s}))$. If any player $j \in \mathbf{s}$ bids less then its maximal internal competitor (set) then he will not be in $\mathbf{s}$ but his maximal internal competitor (set) will.

As a possible $E_i(A, \mathbf{s})$ is $\cup_{j \in \mathbf{s}} R_j(A, \mathbf{s})$ one can bound the maximal value of i's external competition $W(E_i(A, \mathbf{s}))$ by the sum of the maximal values of the rest of the players in $\mathbf{s}$ internal competition i.e., $\sum_{j \in \mathbf{s}} \max W(R_j(A, \mathbf{s}))$. Therefore $\min -W(E_i(A, \mathbf{s})) = -(\sum_{j \in \mathbf{s}} \max W(R_j(A, \mathbf{s})))$. As the F function is defined to be a positive constant we get that $W(\mathbf{s}) = \min -W(E_i(A, \mathbf{s})) + (\sum_{j \in \mathbf{s}} \max W(R_j(A, \mathbf{s}))) + F(\mathbf{s}) \geq 0$ and thus $\mathbf{s}$ is at least budget balanced. As each procurement set that is not reduced is at least budget balanced, it follows that the produced X-external mechanism is BB. □

The above two lemmas yield the following theorem:

THEOREM 3.2. *For procurement-class based domains for any X, the X-external mechanism with critical value pricing scheme produced by the GTR-1 algorithm is an IR, IC, and BB mechanism.*

REMARK 3.2. *The proof of the theorem yields bounds on the payments any player has to make to the mechanism.*

4 Non Procurement-Class Based Domains

The main reason that GTR-1 accepts only procurement-class based domains is that each player's possibility of being reduced is monotonic. By the definition of a dividing function if

[9] since the domain is a procurement class based domain we can use lemma 3.1

a player $i \in \mathbf{s}_j$ increases his value, i can only appear in later procurement set $\mathbf{s}_j^{'}$ and hence has a higher chance of having the desired competition.

Therefore, the chance of i lacking the requisite competition is decreased. Since the domain is a procurement class based domain, all other players $t \neq i, t \in \mathbf{s}_j^{'}$ are also more likely to have competition since members of their class continue to appear before i and hence the likelihood that i will be reduced is decreased. Since by theorem 3.1 a necessary and sufficient condition for the mechanism to be IC is monotonicity. GTR-1 is IC for procurement-class based domains.

However, for domains that are *not* procurement-class based domains the monotonicity of the dividing function D does not suffice to maintain monotonicity in the trade reduction procedure, even if the domain is a class based domain. Although, all members of $\mathbf{s}_j$ continue to have the required competition it is possible that there are members of $\mathbf{s}_j^{'}$ who do not have analogues in $\mathbf{s}_j$ who do not have competition. Hence i might be reduced after increasing his value which by lemma 3.1 means the mechanism is not IC.

We therefore define a different algorithm for non procurement class domains. Our modified algorithm requires a special dividing function in order to maintain the IC property. Although our restriction on this special dividing function appears stringent, the dividing function we use is a generalization of the way that procurement sets are chosen in procurement-class based domains e.g., [13, 16, 9, 3, 2, 4].

For ease of presentation in this section we assume that $F = 0$.

The dividing function for general domains is defined by looking at all possible dividing functions. For each dividing function D_i and each set of bids, the GTR-1 algorithm yields a welfare that is a function of the bids and the dividing function[10]. We denote by D_0 the dividing function that divides the players into sets s.t. the welfare that GTR-1 finds is maximal[11].

Formally,

Let $\mathscr{D}$ be the set of all dividing functions D. Denote the welfare achieved by the mechanism produced by GTR-1 when using dividing function D and a set of bids $\bar{b}$ by $GTR1(D, \bar{b})$. Denote by $D_0(\bar{b}) = \text{argmax}_{D \in \mathscr{D}}(GTR1(D, \bar{b}))$. For ease of presentation we denote $D_0(\bar{b})$ by D_0 when the dependence on b is clear from the context.

Remark 4.1. *D_0 is an element of the set of dividing functions, and therefore is a dividing function.*

The second generalized trade reduction algorithm GTR-2 follows.

Definition 4.1. **The GTR-2 algorithm** - *Given mechanism M, allocation A, and a set $X \subset N$ closed under replacement, GTR-2 operates as follows:*

1. *Calculate the dividing function D_0 as defined above.*
2. *Use the dividing function D_0 to divide A into procurement sets $\mathbf{s}_1, \ldots, \mathbf{s}_k \in \mathbf{S}$. for each $\mathbf{s}_j$, $\mathbf{s}_j \cap X \neq \emptyset$.*

[10]Note that for any particular D_i this might not be IC as GTR-1 is IC only for procurement class based domains and not for general domains

[11]In Appendix A we show how to calculate D_0 in polynomial time for procurement-class based domains. Calculating D_0 in polynomial time for general domains is an important open problem.

3. *For each $\mathbf{s}_j$, starting from the lowest value procurement set, do the following: If for $i \in \mathbf{s}_j \cap X$ there is an external competition and there is at most one $i \in \mathbf{s}_j$ that does not have an internal competition then keep $\mathbf{s}_j$. Otherwise, reduce the trade $\mathbf{s}_j$.*

4. *All trading players are charged the critical value for trading. All non trading players are charged zero.*

We will prove that the mechanism produced by GTR-2 maintains the desired properties of IR, IC, and BB. The following lemma shows that the GTR-2 produced mechanism is IR, and IC.

LEMMA 4.1. *For any X, the X-external mechanism with critical value pricing scheme produced by the GTR-2 algorithm is an IR and IC mechanism.*

Proof. By theorem 3.1 it suffices to prove that the produced mechanism by the GTR-2 algorithm is bid monotonic for every player i. Suppose that i was not reduced when bidding b_i we need to prove that i will not be reduced when bidding $b'_i > b_i$. Denote by $D_1 = D_0(b)$ the dividing function used by GTR-2 when i reported b_i and the rest of the players reported b_{-i}. Denote by $D'_1 = D_0(b'_i, b_{-i})$ the dividing function used by GTR-2 when i reported b'_i and the rest of the players reported b_{-i}. Denote by $\bar{D}_1(b)$ a maximal dividing function that results in GTR-1 reducing i when reporting b_i. Assume to the contrary that GTR-2 reduced i from the trade when i reported b'_i then $GTR1(D'_1, (b'_i, b_{-i})) = GTR1(\bar{D}_1, b)$. Since $D_1 = \text{argmax}_{D\in\mathscr{D}}(GTR1(D, b))$ it follows that $GTR1(D_1, b) > GTR1(\bar{D}_1, b)$ and therefore $GTR1(D_1, b) > GTR1(D'_1, (b'_i, b_{-i}))$. However according to the definition $D'_1 = \text{argmax}_{D\in\mathscr{D}}(GTR1(D, \bar{b}))$, GTR-2 should not have reduced i with the dividing function D'_1 and gained a greater welfare than $GTR1(D_1, b)$. Thus a contradiction arises and GTR-2 does not reduce i from the trade when i reports $b'_i > b_i$. $\square$

LEMMA 4.2. *For any X, the X-external mechanism with critical value pricing scheme produced by the GTR-2 algorithm is a BB mechanism.*

Proof. First we show that the one player without internal competition is the one with external competition. Then we can prove the X-external mechanism with critical value pricing scheme produced by the GTR-2 algorithm is a BB mechanism in an identical manner to the way that the X-external mechanism with critical value pricing scheme produced by the GTR-1 algorithm is shown to be a BB mechanism.

Let i be a player with external competition, then by the external competition definition there exist $E_i(A, \mathbf{s})$. Assume to the contrary that there exist player j who does not have internal competition and $i \neq j$. So all other players z, $z \neq j$, $z \neq i$ has $R_z(A, \mathbf{s})$. As $E_i(A, \mathbf{s})$ is an internal set of competition to all z and j by definition there exist a set $r \in E_i(A, \mathbf{s})$ of players such that $\cup_{z\in\mathbf{s}} R_z(A, s) + r = \bar{E}_i(A, \mathbf{s})$ where $\bar{E}_i(A, \mathbf{s})$ is some external competition of i and thus $r = R_j(A, \mathbf{s})$ is an internal competition set for j which contradicts the assumption. The rest of the proof follows identically from lemma 3.3's proof. $\square$

Combining the two lemmas above we get:

THEOREM 4.1. *For any X closed under replacement, the X-external mechanism with critical value pricing scheme produced by the GTR-2 algorithm is an IR, IC and BB mechanism.*

Appendix A shows how to calculate D_0 for procurement class based domains in polynomial time, it is not generally known how to easily calculate D_0. Creating a general method for calculating the needed dividing function in polynomial time remains as an open question.

4.1 Bounding the Welfare for Procurement-Class Based Domains and other General Domains Cases

This section shows that in addition to producing a mechanism with the desired properties, GTR-2 also produces a mechanism that maintains high welfare. Since the GTR-2 algorithm finds a budget balanced mechanism in arbitrary domains we are unable to bound the welfare for general cases. However we can bound the welfare for procurement-class based domains, class based domains, and a wide variety of cases in general domains which includes many cases previously studied.

Definition 4.2. *Denote $freq_k([i], \mathbf{s}_j)$ to indicate that a class $[i]$ appears in a procurement set $\mathbf{s}_j$, k times and there are k members of $[i]$ in $\mathbf{s}_j$.*

Definition 4.3. *Denote by $freq_k([i], \mathbf{S})$ the maximal k s.t. there are k members of $[i]$ in $\mathbf{s}_j$. I.e., $freq_k([i], \mathbf{S}) = \max_{\mathbf{s}_j \in \mathbf{S}} freq_k([i], \mathbf{s}_j)$.*

Let the set of equivalence classes in procurement class based domain mechanism be e_c and $|e_c|$ be the number of those equivalence classes.

Using the definition of class appearance frequency we can bound the welfare achieved by the GTR-2 produced mechanism for procurement class based domains and class based domains[12]:

Lemma 4.3. *For procurement class domains with $F = 0$, the number of procurement sets that are reduced by GTR-2[13] is at most $|e_c|$ times the maximal frequency of each class. Formally, the maximal number of procurement sets that is reduced is $O(\sum_{[i] \in e_c} freq_k([i], \mathbf{S}))$*

Proof. Let D be an arbitrary dividing function. We note that by definition any procurement set $\mathbf{s}_j$ will not be reduced if every $i \in \mathbf{s}_j$ has both internal competition and external competition.

Every procurement set $\mathbf{s}$ that is reduced has at least one player i who has no competition. Once $\mathbf{s}$ is reduced all players of $[i]$ have internal competition. So by reducing the number of equivalence classes $|e_c|$ procurement sets we cover all the remaining players with internal competition.

If the maximal frequency of every equivalence classes was one then each remaining player t in procurement set $\mathbf{s}_k$ also has external competition as all the internal competitors of players $\bar{t} \neq t$, $\bar{t} \in \mathbf{s}_k$ are an external competition for t. If we have $freq_k([t], \mathbf{S})$ players from class $[t]$ who were reduced then there is sufficient external competition for all players in $\mathbf{s}_k$. Therefore it suffices to reduce $O(\sum_{[i] \in e_c} freq_k([i], \mathbf{S}))$ procurement sets in order to ensure that both the requisite internal and external competition exists. □

[12]The welfare achieved by GTR-1 can also be bounded for the cases presented in this section. However, we focus on GTR-2 as it always achieves at least the same welfare.

[13]or GTR-1

The next theorem follows as an immediate corollary for lemma 4.3.

THEOREM 4.2. *Given procurement-class based domain mechanisms with H procurement sets, the efficiency is at least a* $1 - O(\frac{O(\sum_{[i]\in e_c} freq_k([i],\mathbf{S}))}{H})$ *fraction of the optimal welfare.*

The following section does a detailed comparison of the performance of the GTR-1 and GTR-2 procedures in known specific domains with respect to the existing literature. In addition the section also examines the performance of GTR-1 and GTR-2 procedures in some new and unsolved specific domains.

5 The GTR Implications in Existing Specific Domains and in New Domains

In this section we examine all existing specific domains in detail and compare the performance of the GTR algorithm with the existing literature. Furthermore we examine the performance of the GTR algorithm on some new specific domains.

The first domain to be examined is the double-sided auction [13].

5.1 Double-Sided Auction

Let b_i denote the bid of buyer i and let s_j denote the bid of seller j. The McAfee scheme can be described in the following manner:

- Order the buyers' bids in decreasing order and order the sellers' bids in increasing order.
- Match the top-value bid buyer with the bottom-value bid seller and second top value bid buyer with the second bottom value bid seller etc.
- Find the first non trading pair, meaning the first (i,j) pair such that $b_i < s_j$, and compute $(b_i + s_j)/2$.
- Reduce the trading pair b_i, s_j only if $(b_i + s_j)/2 \notin [b_{i-1}, s_{j-1}]$.

In order to compare the performance of the GTR algorithm with the McAfee mechanism there are four points to consider:

1. All pairs (i, j) are such that $b_i > s_j$.
2. There are more sellers than buyers such that all buyers are matched in trading pairs, meaning for all i and pair (i, j) $b_i > s_j$.
3. There are more buyers than sellers such that all sellers are matched in trading pairs, meaning for all j and pair (i, j) $b_i > s_j$.
4. There exist a pair (i, j) in which $b_i < s_j$.

In the case of the first point case the GTR algorithm and the McAfee mechanism will perform the same and will reduce a single trade. In the case of the next two points the GTR algorithm may not reduce any trade in some cases and the McAfee mechanism will always reduce a trade. And finally in the case of the forth point, in some cases the McAfee mechanism will reduce a trade while in those same cases the GTR may not reduce the trade.

1. The McAfee mechanism will reduce a trade as no non-trading pair exists in this case. The GTR algorithm can not find external competition or internal competition for the least significant pair of players and therefore one trade will be reduced. After the reduction of the single pair the desired competition is found for the remainder of the players.

2. The McAfee mechanism will reduce a trade as no non-trading pair exists in this case, although there is at least one seller left who is not trading. In this case the GTR algorithm will find the smallest value non-trading seller to be external competition for the trading buyers and internal competition for the sellers if the smallest trading buyer's value is greater than the smallest non trading seller's value. As a result the GTR algorithm will not reduce any trade.

3. Is very similar to the previous case. The McAfee mechanism will reduce a trade as no non-trading pair exists in this case although there is at least one buyer left who is not trading. The GTR algorithm will find the largest value non-trading buyer to be external competition for the trading sellers and internal competition for the buyers if the largest trading seller's value is less than the largest non-trading buyer's value. As a result the GTR algorithm will not reduce any trade.

4. If there exist a pair (i, j) in which $b_i < s_j$ and $(b_i + s_j)/2 \in [b_{i-1}, s_{j-1}]$ then the McAfee mechanism will not reduce any trade. In this case at least one of b_i or s_j can provide the right competition for all the other trading pairs in GTR and the GTR algorithm will not reduce any trade either. The reason that at least one of b_i or s_j provides the competition is that if s_j is greater from b_{i-1} and can not provide external competition for b_{i-1} and b_i is less than s_{j-1} and can not provide the external competition for s_{j-1}, which also means that $(b_i + s_j)/2 \notin [b_{i-1}, s_{j-1}]$. If one of b_i or s_j provides external competition to s_{j-1} or b_{i-1} then it also provides the sufficient internal competition to b_{i-1} or s_{j-1}.

   If there exist a pair (i, j) in which $b_i < s_j$ and $(b_i + s_j)/2 \notin [b_{i-1}, s_{j-1}]$ then the McAfee mechanism will reduce a trade. If one of b_i or s_j provides the desired competition in GTR, meaning either $b_i > s_{j-1}$ or $s_j < b_{i-1}$ then the GTR algorithm will not reduce a trade.

The following corollaries are a result of theorem 4.2 and the above analysis.

Corollary 5.1. *The GTR-2 algorithm for homogenous double-sided auctions (problem B.1)* at most *reduces one procurement set.*

Corollary 5.2. *The GTR-2 for homogenous double-sided auctions (problem B.1) reduces at most as much as the McAfee mechanism does [13] and there exists many cases in whitc the McAfee mechanism reduces a trade while GTR-2 does not reduce any trade.*

The second domain to be examined is the double-sided auction with pair wise transaction costs [9].

5.2 Double-Sided Auction with Pair Wise Transaction Costs

The Chu Shen [9] model is an extension of the McAfee model [13]. In Chu Shen the double-sided auction is elevated by introducing a transaction cost between each buyer and seller. The transaction cost is a constant function and therefore does not introduce another set of players in addition to the two existing ones(the buyers set and the sellers set). We will follow some of the Chu Shen notations in this subsection to make it easer for readers familiar with the Chu Shen work to follow the comparison of our results.

Let f_i be a buyer bid and g_j be a seller bid. And let $d_{i,j}$ be the transaction cost of a trade between buyer i and seller j. A feasible trade is a pair of a buyer i and a seller j such that $f_i - g_j - d_{i,j} \geq 0$. Chu Shen denote by

- $V(I', J')$ the maximum feasible social welfare with respect to the bids of buyer set I' and seller set J'.
- $V_{-k}(I', J')$ the maximum feasible social welfare with respect to the bids of agent set $I' \cup J' \; \{k\}(k \in I' \cup J')$
- $V_k(I', J')$ the maximum feasible social welfare with respect to the bids of agent set $I' \cup J'$ and one more agent who is identical to the agent $k(k \in I' \cup J')$

Two more terms are defined by Chu and Shen using the terminology above.

- $p_-(k)(I', J')$ the infimum (supremum) of bid prices of buyer (seller) k satisfying $V(I', J') > V_{-k}(I', J')$
- $p_+(k)(I', J')$ the infimum (supremum) of bid prices of buyer (seller) k satisfying $V_k(I', J') > V(I', J')$

The last two terms can best be interpreted in our terminology as internal competition and external competition. $p_-(k)(I', J')$ is the internal competition as it is looking for the minimum (maximum) bid buyer (seller) k had to give and still be allocated. $p_+(k)(I', J')$ is the external competition as it is looking for the minimum (maximum) bid buyer (seller) k had to give and still be matched with another seller (buyer) i.e. the external competitor.

Chu Shen has two separate algorithms for creating BB. The first algorithm is called seller competition mechanism and the other is called buyer competition mechanism. The seller competition mechanism scheme can be described in the following manner:

- Remove sellers until all remaining sellers are such that $j|g_j \leq p_+(j)(I, J), j \in J$.
- Match the buyers (set I) and the remaining sellers (set $\tilde{J}$) in the most efficient way
- The trading buyer k pays $p_-(k)(I, \tilde{J})$, and the trading seller l receives $p_+(l)(I, J)$.

The buyer competition mechanism scheme can be described in the following manner:

- Remove buyers until all remaining buyers are such that $i|f_i \geq p_+(i)(I,J), i \in I$.
- Match the remaining buyers (set $\tilde{I}$) and the sellers (set J) in the most efficient way
- The trading buyer k pays $p_+(k)(I,J)$, and the trading seller l receives $p_-(l)(\tilde{I},J)$.

In order to compare the performance of the GTR algorithm with the Chu Shen mechanism there are four points to consider:

1. All pairs (i,j) are such that $f_i > g_j$.
2. There are more sellers than buyers such that all buyers are matched in trading pairs meaning for all i and pair (i,j) $f_i > g_j$.
3. There are more buyers than sellers such that all sellers are matched in trading pairs meaning for all j and pair (i,j) $f_i > g_j$.
4. There exist a pair (i,j) in which $f_i < g_j$.

1. If all pairs (i,j) are such that $f_i > g_j$ and the transaction costs $d_{i,j}$ are all zero then the Chu Shen seller competition mechanism scheme will remove a single seller and the Chu Shen buyer competition mechanism scheme will remove a single buyer. Similarly in this case the GTR algorithm will remove a single trade. When the transaction costs $d_{i,j}$ are not all zero for all pairs (i,j) the Chu Shen seller/buyer competition mechanism scheme will remove more sellers/buyers. As the sellers/buyers in the Chu Shen scheme are removed until all remaining sellers/buyers have external competition the GTR algorithm will remove the same number of trades to create identical external competition.

2. If there are more sellers than buyers such that all buyers are matched in trading pairs, meaning that for all i and pair (i,j) such that $f_i > g_j$, and the transaction costs $d_{i,j}$ are all zero then the Chu Shen buyer competition mechanism scheme will not remove any buyers. On the other hand the Chu Shen seller competition mechanism scheme will have to remove a trading seller to provide external competition to the trading buyers. Unlike the Chu Shen seller Competition mechanism scheme the GTR algorithm will not remove any trade in that case as the extra sellers are already providing the required external competition (depending on the X set including all and only the buyers). Similar comparison can be concluded when the transaction costs $d_{i,j}$ are not all zero.

3. If there are more buyers than sellers such that all sellers are matched in trading pairs, meaning that for all i and pair (i,j) such that $f_i > g_j$, and the transaction costs $d_{i,j}$ are all zero then the Chu Shen seller competition mechanism scheme will not remove any sellers. On the other hand the Chu Shen buyer competition mechanism scheme will have to remove a trading buyer to provide external competition to the trading sellers. Unlike the Chu Shen buyer Competition mechanism scheme the GTR algorithm will not remove any trade in this case as the extra buyers are already providing the required external competition (depending on the X set including all and only the sellers). Similar comparison can be concluded when the transaction costs $d_{i,j}$ are not all zero.

4. If there exist a pair (i, j) in which $f_i < g_j$ and $g_{j-1} > f_i$ then the Chu Shen seller competition scheme will remove seller g_j. If the GTR algorithm's X set includes only and all sellers then the GTR algorithm will do the same. Otherwise GTR might not reduce seller s_j. If there exist a pair (i, j) in which $f_i < g_j$ and $f_{i-1} < g_j$ then the Chu Shen buyer competition scheme will remove buyer f_i. If the GTR algorithm's X set includes only and all buyers then the GTR algorithm will do the same. Otherwise it might not reduce buyer f_i.

It is interesting to note that when the X set of our GTR algorithm includes all and only the sellers, it performs essentially equivalent to the Chu Shen seller competition mechanism in the Chu Shen model of double-sided auction with pair-wise transaction costs. When the X set of our GTR algorithm includes all and only the buyers it performs essentially equivalent to the Chu Shen buyer competition mechanism in the Chu Shen model of double-sided auction with pair wise transaction costs. This note points out again, as was pointed out earlier in the paper, that the choice of an optimal X set is a very important open question.

The following corollary is a result of the above analysis.

COROLLARY 5.3. *The GTR-2 for double-sided auctions with pair wise transaction costs (problem B.2) reduces at most as much as the Chu Shen mechanisms ([9]) do and there exists many cases in which the Chu Shen mechanisms reduce a trade while GTR-2 does not reduce any trade.*

The third domain to be examined is the spatially distributed markets without strategic edges [3].

5.3 Spatially Distributed Markets without Strategic Edges

In this model a global market for a single good is constructed from a set of k markets $M_1, ..., M_k$, each in a different location. These markets are the nodes of a simple directed graph representing the possible commercial relationships between the markets. If a good can be shipped from M_i to M_j then there is a directed edge (M_i, M_j) in the graph. For each edge (M_i, M_j) there is an internal cost of $c_{i,j} \geq 0$, which is the cost of shipment of one unit of the good from market M_i to market M_j along the edge (M_i, M_j).

Each buyer (seller) is a single parameter agent, which means that she wants to buy (sell) a single unit of the good in one particular market, and has one parameter that represents the value (cost) that she gets from trading. A buyer (seller) trades if she buys (sells) a unit of the good in her market. Excess demand (supply) in a given market will be matched to a surplus of supply (demand) in other markets by shipment of goods from one market to the other. x denotes the shipment vector, where $x_{i,j} \geq 0$ is the number of units shipped from market M_i to market M_j along the edge (M_i, M_j). There is a trade between market M_i and market M_j if there is a shipment of goods from M_i to $M_j (x_{i,j} > 0)$.

Similar to other trade reduction procedures, [3] uses the efficient allocation as a building block for the trade reduction.

The efficient allocation of the spatially distributed market graph problem is solved by reducing the problem to a minimum cost flow problem and finding the efficient solution of the minimum cost flow graph. The minimum cost flow graph is constructed in the following way:

- Include all notes and edges of the spatially distributed market graph described above.
- Add another sink node
- For each buyer b with value $v_b \geq 0$ in market M_j, add an edge with capacity one and cost of $-v_b$ from the market to the sink node.
- For each seller c_s with cost $c_s \geq 0$ in market M_i, add an edge with capacity one and cost of c_s from the sink node to market M_i.
- The capacity of each edge (M_i, M_j) is $u_{i,j} = \infty$.

The VCG pricing scheme is defined by the residual graph of the minimum cost flow and constructed in the following manner:

- Replace each edge $(i, j) \in E$ by two edges, (i, j) and (j, i).
- The edge (i, j) has cost $c_{i,j}$ and residual capacity $u_{i,j} - x_{i,j}$, and the edge (j, i) has a cost $c_{j,i} = -c_{i,j}$ and residual capacity $x_{i,j}$.
- The residual graph consists only of edges with positive residual capacity.

The trade reduction mechanism of [3] requires the following definitions:

Definition 5.1. *Markets M_i,M_j are in a commercial relationship (CR) if there is a trade between M_i and M_j or between M_j and M_i in the efficient allocation.*

Definition 5.2. *The Commercial Relationship Component (CRC) of a market M_i is the transitive closure of the commercial relationship property.*

In essence the CRC of market M_i contains all of the markets with which M_i has an direct or indirect commercial relationship.

The reduced residual graph that is used to derive the trade reduction allocation is constructed in the following way:

- Consists of all notes of the residual minimum cost flow graph.
- Add the following edges from the residual minimum cost flow graph and use the cost of the added edge as length:
  - For each edge (M_i, M_j) such that there is flow on the edge, add the edge with its cost and its reversed residual edge with the negative cost.
  - For each market M_i add the residual edges corresponding to the trading buyer with the minimal valuation (if such buyer exists) and the trading seller with the maximal cost (if such seller exists).

Note that all edges between CRCs as well as edges corresponding to non trading agents are not in the reduced residual graph.

Finally the trade reduction allocation in [3] is computed in the following way:

- Calculate the efficient allocation using the minimum cost flow graph and find the residual minimum cost flow graph and the reduced residual graph.
- For each CRC calculate the minimal positive cycle in the reduced residual graph and remove it from the allocation.

Before we can move on and compare the performance difference in the spatially distributed markets model between the [3] and our GTR algorithm we first start by explaining the meaning of the minimal positive cycle reduction from every CRC in [3]'s work.

As defined above, the [3] mechanism reduces a single shortest cycle from the reduced residual graph. Recall that in the reduced residual graph for each market M_i there is an edge from the sink to the minimum buyer where the length of the edge is the value of that buyer and there is an edge from the maximum seller to the sink where the length of the edge is the cost of that seller.

It is fairly intuitive to see, and is proved in [3], that the shortest positive cycle goes into the sink once and leaves the sink once. Otherwise the cycle could be split into two shorter cycles. As the shortest cycle goes into the sink once there is only one seller on it and as the shortest cycle leaves the sink once there is only one buyer on it. Therefore by removing a single shortest cycle from every CRC in the reduced residual graph only a single trade is reduced.

Now we can continue and compare the performance difference in the spatially distributed markets model between [3] and the GTR algorithm. Recall that the reduced residual graph is defined to add edge (M_i, M_j) if there is flow on the edge in the residual graph of the minimum cost flow problem. Such flow can occur only if there is trade between M_i and M_j. Also recall that in [3]'s specially distributed market model only the excess demand/supply of a market M_i is traded outside the market. So consider a specially distributed market network where all markets M_i have no excess supply or demand. In this case every market M_i is a CRC by itself and [3]'s mechanism will have k CRC and will reduce k trades, one from each market.

Following the above scenario consider the GTR algorithm in a specially distributed market network where all markets M_i have no excess supply or demand. When looking for the conditions sufficient for competition, the GTR algorithm will consider competition for players in market M_i, also in other markets M_j where $j \neq i$. When players in markets M_j can provide external/internal competition for the players in market M_i then no trade reduction will occur in market M_i. The last observation is a key observation as in the GTR algorithm the competition between markets defines the CRCs and therefore much larger CRCs can potentially be created. If larger CRCs are created then there will be fewer CRCs. Thus potentially the GTR algorithm can reduce much less trades than [3]'s mechanism. But in no case will the GTR algorithm reduce more trade than [3]'s mechanism.

The following corollaries are a result of theorem 4.2 and the above analysis.

Corollary 5.4. *The GTR-2 algorithm for spatially distributed markets without strategic edges (problem B.3) reduces* at most *one cycle per connected component*

Corollary 5.5. *The GTR-2 for spatially distributed markets without strategic edges (problem B.3) reduces at most as much as the Babaioff-Nisan-Pavlov mechanism ([3]) does and* *there exists many cases in which the Babaioff-Nisan-Pavlov mechanism reduces many more trades then the GTR-2 does.*

The fourth domain to be examined is the supply chain problems in [2, 4].

5.4 Supply Chains with Unique Manufacturing Technology

We will start by examining the supply chain problem in [2]. The model of supply chain discussed in [2] is a chain graph. Meaning the supply chain model is constructed from a collection of markets that are trading in a chain format, i.e. the buyers of the first market are the exact same sellers of the second market and the buyers of the second market are the exact same sellers of the third market etc. Every buyer in a market is interested in a single good and can produce a single good. All buyers of one market produce the same type of good. In other words the [2] supply chain model can be described as a chain of double-sided auctions and as such [2] adopts McAfee [13]'s trade reduction technique.

Denote the n markets in the linear chain at [2] by M^t $1 \leq t \leq n$. Two different procedures are given for achieving budget balance with a trade reduction; the symmetric protocol and the pivot protocol. The symmetric protocol first propagates the supply/demand in all the markets and then all markets perform a reduction in parallel on a single trade in every market. The pivot protocol also propagates the supply/demand in all the markets and then performs a trade reduction in the last market i.e., M^n if necessary. If a trade reduction occurred in market M^n then the trade reduction propagates to the adjacent market, meaning market M^{n-1} and so on until all markets have a single trade removed. The reason that a trade has to be removed in all markets once market M^n has reduced a trade is that [2] requires a material balance, meaning that every market produces the same number of items as it consumes.

If one views the [2] model as a graph of nodes and each node is a connected component in the graph then one can see another way to describe the trade reduction procedures in [2], similar to the approach in [3], by demanding a single reduction in every connected component. Similar to our improved performance from [3] also in this case the GTR algorithm approaches the whole chain of markets as a single connected component in the graph. When looking for external competition for the sellers in the M^t market the GTR algorithm considers the sellers of the M^{t-1} market, as the buyers of the M^{t-1} market are the sellers of the M^t market. Also when looking for external competition for the buyers in the M^t market the GTR algorithm considers the buyers of the M^{t+1} market as the sellers of the M^{t+1} market, are the buyers of the M^t market. Nevertheless the procedure described above of identifying external competition does not cover the internal competition of all the different equivalent classes. Recall that a procurement class in [2]'s supply chain model is a different type of seller in every market (or a different type of buyer in every market) as sellers (buyers) of every market introduce a new type of players. Therefore the number of equivalent classes in this model is the number of markets and thus for the GTR algorithm to cover all internal competition n players should be removed in the worst case.

It turns out that only in the worst case does the GTR algorithm reduce as much as [2]. Consider a market chain of three markets where the markets M^3 and M^1 have more buyers than sellers. Then the M^3 extra buyers are providing external competition to the sellers in M^3 but also to the buyers in M^2 (as they are the same) and the M^1 extra buyers are providing internal competition to the sellers in M^2. In this case the GTR algorithm will not reduce any trade in the whole chain of markets while [2]'s mechanisms, symmetric or pivot, will reduce 3 trades out of the 3 chain markets.

It is also interesting to note that although [2] describes many methods of trade reduction they all essentially reduce to the McAfee method in which a single trade is reduced from the double-sided auction. The GTR algorithm improvement over [2]'s reduction is therefore applied to all methods presented at [2].

The following corollaries are a result of theorem 4.2 and the above analysis and are similar to corollary 5.4.

Corollary 5.6. *The GTR-2 algorithm for supply chain (problem B.5.1)* at most *reduces one cycle per connected component, i.e., a player of each equivalent class.*

Corollary 5.7. *The GTR-2 for supply chain (problem B.5.1) reduces at most as much as the Babaioff-Nisan mechanism does [2] and there exists many cases in with the Babaioff-Nisan mechanism reduces many more trades then the GTR-2 does.*

We continue by examining the supply chain problem in [4]. The model of supply chain discussed in [4] is a tree graph. Every node in the tree is a double-sided auction market. The buyers in every market are interested in a single bundle of goods and a buyer of a market is interested in the same single bundle as all the other buyers in his market. The buyers can produce a single product out of the single bundle of goods they buy, meaning they are buying a bundle but selling a single good. [4]'s model assumes a unique manufacturing technology that essentially means that a given good type can be produced by a single node in the tree graph. The tree structure imposes the situation that there may be multiple goods of the same type produce in a single market and therefore can be sold to multiple markets. In other words the above means that the buyers of a single market can become the sellers of different markets, unlike the chain model where all the buyers of one market are all the sellers of the next market in the chain.

The trade reduction is conducted in [4] in the following way:

- In every consumers' market meaning leafs of the supply chain tree, a single trade is reduced or at least a single player is reduced which is the player that his value is used as a critical price value for all other trading players in the market.
- After a trade was reduced in every market leaf the trade in the nodes that are connected to the leafs is reduced. If a node market is connected to two leaf markets then two trades will be reduced out of the node market etc.
- In general the number of trades that will be reduced in every node is the sum of the number of trades that were reduced in all his children's nodes. That general scheme also apply for multiple units requested by a node market from his child-node market.

As [4]'s model allows for buyers to request bundles of goods, the problem of every market in this supply chain is a single-minded combinatorial auction and therefore is NP-hard to solve in polynomial time. [4] goes about that problem by assuming a fixed number of leafs in the supply chain tree, i.e., markets that are not producing any good, and computing the allocation in polynomial time. The GTR algorithm does assume a fixed number of any problem parameters even in the single-minded combinatorial case as the GTR algorithm accepts an IC IR problem with an allocation that was already found.

The above observation leads to a difference in performance between GTR algorithm and the [4] mechanism. Like GTR [4]'s mechanism for the known single-minded combinatorial auction case is proven to be IC in dominant strategies. Unlike GTR, [4] can not prove that the unknown single-minded combinatorial auction case is IC in dominant strategy and show the unknown single-minded case only in Nash equilibrium. The reason that [4] can not prove the unknown single-minded case in dominant strategies is that in the unknown bundle case there may be a scenario where a player can lie and report an increased bundle. By doing so the player may increase potential competition and therefore decrease his probability or the likelihood of being reduced in the trade reduction. Following the above observation that the GTR algorithm accepts an IC (in dominant strategies) problem it is clear that such problem can not occur in our model.

In order to continue and compare the performance of the GTR algorithm and the mechanism in [4] we first need to understand how theorem 4.2 applies to GTR on a supply chain problem, as supply chain problems in general are not procurement-class based domains and not class based domains. The reason that the GTR algorithm applies to the supply chain model of [4] (and also [2]) is that by limiting their supply chain mechanism to a unique manufacturing technology, [4] essentially limits their solution to be a procurement class based domain. Consider a supply chain model as in [4] but without unique manufacturing technology. Also consider buyer i in market M^t. In one division of D the dividing function i can be matched in a procurement set s_j with seller z who produced his good type g in technique a and in another division of D the dividing function i can be matched in a procurement set s'_j with seller z' who produced his good type g (same type of good as z produced) in technique b. Clearly there does not exist a bijection $f : s_j \to s'_j$ such that $f(z)$ is a replacement set for z in s'_j as $f(z) = z'$ in this case and as z and z' produce their good in a different procedure and may require different goods to be bought in the markets where they are buyers.

It is interesting to note that [4] claims that the unique manufacturing technology is necessary to ensure BB and their efficiency competitive ratio. In our paper's words [4] is claiming that they do not know how to bound the welfare loss by the trade reduction procedure without limiting their supply chain model to be a procurement-class based domain. Unlike [4] we can also bound the loss of welfare by GTR for class based domains and not just for procurement class based domains as theorem 4.2 implies. Though for class based domains we can not guaranty a polynomial time algorithm as the diving function may not be computed in polynomial time.

We continue by comparing the GTR algorithm trade reduction's performance and the [4] mechanism. Recall the [4] mechanism reduces a single trade in every leaf market in the supply chain tree. Similar to the above comparison between the GTR algorithm and [2]'s mechanism where the supply chain graph is a simple chain of nodes, the GTR algorithm may not reduce a trade in a leaf market (consumer market). Consider a leaf market M^t with n sellers selling good type a and a node market $M^{t'}$ where the n sellers of M^t are buyers of $M^{t'}$. Also consider the case where $M^{t'}$ has another n buyers that are sellers in market $M^{t''}$. If the n sellers of the leaf market M^t have internal competition as buyers in market $M^{t'}$ by the other buyers in $M^{t'}$ (the ones that are sellers in M'') then the buyers in M^t have external competition and the sellers of M^t have internal competition and the GTR algorithm will not remove any trade in the leaf market M^t. If no leaf market has a trade reduced from it then no trade will be reduced of any market node in the supply chain tree, while [4] might reduce the number of trades that is equal to the number of nodes in the tree in this case.

The following corollaries are a result of theorem 4.2 and the above analysis and are similar to corollary 5.4.

COROLLARY 5.8. *The GTR-2 algorithm for supply chain (problem B.5.2)* at most *reduces one cycle per connected component, i.e., a player of each equivalent class.*

COROLLARY 5.9. *The GTR-2 for supply chain (problem B.5.2) reduces at most as much as the Babaioff-Walsh mechanism does [4] and there exists many cases in with the Babaioff-Walsh mechanism reduces many more trades then the GTR-2 does.*

5.5 Closing Open Problems

In this subsection we analyze the implication of the GTR algorithm on problems for achieving BB in special domains that were left as open problems in the literature.

We start by analyzing the open problem left in [3]. The problem left open in [3] is how to achieve an IC, IR, and BB mechanism in the extended model where the routs between markets are provided by strategic transportation companies. More specifically the problem is a natural generalization of the model in [3] which assumes that carriers bid for the privilege of performing a shipment of a unit from one market to the other and report a cost of the shipment to the auction before it runs. According to [3] the minimum cost flow problem can be computed as in the non-strategic case after replacing the edge from market M_i to market M_j by multiple edges of capacity one and the costs reported by the shipment agents.

The problem of distributed markets with strategic edges is not a class-based domain as replacing sets may be of size larger than one. As distributed markets with strategic edges is not a class-based domain it is also not a procurement-class based domain. Therefore theorem 4.2 does not necessarily apply for this problem. Nevertheless for this problem although it is not class based domain the theorem applies. If we remove a representative of every single equivalent class then we will cover the internal competition of all the trading procurement sets. As there is only one representative of every equivalence class in a procurement set in this problem removing a representative of every single equivalent class also provides the sufficient external competition to all trading players. The number of different equivalent classes in the distributed markets with strategic edges is the number of edges plus two (Two-The buyers class and the sellers class). The number of edges in a graph of n nodes (markets) is n^2 and therefore removing n^2+2 procurement sets will suffice.

The following corollary solves the open problem at [3].

COROLLARY 5.10. *For distributed markets on n nodes with strategic transportation agents(problem B.6) it suffices to remove at most n^2 procurements sets.*

If we limit our problem of distributed markets with strategic edges to a problem with bounded paths of length K, then reducing only $K * n$ procurement sets is sufficient. The corollary follows:

The following corollary improves on corollary 5.10.

Corollary 5.11. *For distributed markets on n nodes with strategic transportation agents and paths of bounded length K (problem B.6) it suffices to remove at most $K*n$ procurements sets.*

Proof. Every path has to start in a market. If every market trades using the longest path the allocation will include at most n paths of length K. If every path includes different edges of transportation, by removing $K*n$ procurement sets we can cover sufficient competition to all equivalent classes presented in the solution. □

We continue by analyzing the open problem of double sided CA with single value players (problem B.8). Though the supply chain problem in [4] dealt with a combinatorial auction setting they only dealt with single-minded combinatorial auction which means that every buyer is only interested in a single bundle. [4] left open the question of a combinatorial auction where buyers are multi minded and are allocated a single bundle out of number of different desired bundles. Though the GTR algorithm does not answer the general question of a multi minded combinatorial auction trade reduction, it does allow for a generalization of the single minded combinatorial auction and a step towards the multi minded case. The case the GTR algorithm can solve in combinatorial auctions is that of single value players. In this case buyers may bid for multiple bundles like the general multi minded case but they do assign the same value for all bundles they desire.

We provide results for two special cases of double sided combinatorial auction with single value players (problem B.8). The first is a single value combinatorial auction where there are at most M different kinds of procurement sets and the second is a single value combinatorial auction where there are K types of goods and each procurement set consists of at most one of each type.

The problems above are not necessarily class based domain problems as replacing sets may be of size larger than one. Consider buyer i who desires two different bundles. If one bundle desired by buyer i includes a single good of a single seller j and the other desired bundle includes a set of goods belonging to sellers t where for all t, $t \neq j$ then the set of sellers t can replace seller j and thus the replacing set $|R_j| \neq 1$.

The problems above are not a procurement-class based domains as they are not class domains. Therefore theorem 4.2 does not necessarily apply for these problems. Nevertheless the sufficient amount of competition can be achieved by reducing less procurement sets then theorem 4.2 requires. In the case of M different kinds of procurement sets it is possible to cover the sufficient competition by reducing a representative of every kind of procurement set. Even if multiple representatives of the same equivalent class are in the same procurement set, by removing a representative of that kind of procurement set we are removing the sufficient amount of the equivalent class representatives. In the case of K types of goods and each procurement set consists of at most one of each type if we remove $K + 1$ procurement sets that do not include identical types of goods, then similar to theorem 4.2 we remove at least one representative of every equivalent class.

The corollaries follow:

Corollary 5.12. *If there are at most M different kinds of procurement sets it suffices to remove M procurement sets, one of each kind.*

COROLLARY 5.13. *If there are K types of goods and each procurement set consists of at most one of each type it suffices to remove at most $K + 1$ procurement sets.*

6 Conclusions and Future Work

In this paper we presented a general solution procedure called the *Generalized Trade Reduction (GTR)*. GTR accepts an IR and IC mechanism as an input and outputs mechanisms that are IR, IC, and BB. The output mechanisms achieve welfare that is close to optimal for a wide range of domains.

The GTR procedure improves on existing results such as homogeneous double-sided auctions, distributed markets, and supply chains, and solves several open problems such as distributed markets with strategic transportation edges and bounded paths, multi-minded single value combinatorial double-sided auctions with bounded size procurements sets, and multi-minded single-value combinatorial double-sided auctions with a bounded number of procurement sets.

The question of the quality of welfare approximation in general domains that are not procurement class based domains or class based domains is an important and interesting open question. We also leave open the question of upper bound for the quality of approximation of welfare. Although we know that it is impossible to have IR, IC, and BB in an efficient mechanism it would be interesting to have an upper bound on the approximation of the welfare achievable in an IR, IC, and BB mechanism.

The GTR procedure outputs a mechanism that depends on a set $X \subset N$. Another interesting question is what the quality of approximation is when X is chosen randomly from N before valuations are declared.

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A Calculating the Optimal Dividing Function in Procurement Class Domains in Polynomial Time

In this section we show how to calculate the optimal dividing function for procurement class based domains in polynomial time. We first define a special dividing function D_0' that is easy to calculate:

We define the dividing function D_0' recursively as follows: At stage j, D_0' divides the trading players into two sets A_j and A_j' s.t.

- A_j is a procurement set
- A_j' can be divided into a disjoint union of procurement sets.
- A_j has minimal value from all possible such partitions.

Define $\mathbf{s}_j = A_j$ and recursively invoke D_0' and A_j' until $A_j' = \varnothing$.

We now prove that D_0' is the required dividing function.

Lemma A.1. *For procurement class domains $D_0 = D_0'$.*

Proof. Since the domain is a procurement-class based domain, for every reduced procurement set the set of players who achieve competition (either internal or external) is fixed. Therefore, the number of procurement sets that are reduced is independent of the dividing function D. Since the goal is to optimize welfare by reducing procurement sets with the least value we can optimize welfare. This is achieved by D_0. □

B Problems and Examples

For completeness we present the formal definitions of the problems that we use to illustrate our mechanism.

The first problem that we define is the *double-sided auction with homogeneous goods.*

Problem B.1. **Double-sided auction with homogeneous goods:** *There are m sellers each of which have a single good (all goods are identical) and n buyers each of which are interested in receiving a good. We denote the set of sellers by S and the set of buyers by B. Every player $i \in S \cup B$ (both buyers and sellers) has a value v_i for the good. In this model a procurement set consists of a single buyer and a single seller, i.e., $|\mathbf{s}| = 2$. The value of a procurement set is $W(\mathbf{s}) = v_j - v_i$ where $j \in B$ and $i \in S$, i.e., the gain from trade.*

A trade reduction solution for problem B.1 was presented by [13].

A related model is the *pair related costs* [9] model.

Problem B.2. **The pair related costs:** *A double-sided auction B.1 in which every pair of players $i \in S$ and $j \in B$ has a related cost $F(i, j) \geq 0$ in order to trade. $F(i, j)$ is a friction cost that should be minimized in order to maximize welfare.*

[9] defines two budget-balanced mechanisms for this case. One of [9]'s mechanisms has the set of buyers B as the X set for the X-external mechanism and the other has the set of sellers S as the X set for the X-external mechanism.

A similar model is the *spatially distributed markets (SDM)* model [3] in which there is a graph imposing relationships on the costs.

PROBLEM B.3. **Spatially distributed markets:** *there is a graph $G = (V, E)$ such that each $v \in V$ has a set of sellers S^v and a set of buyers B^v, all trading identical goods and every node in the graph is a market. Each edge $e \in E$ has an associated cost which is the cost to transport a single unit of good along the edge. The edges are non strategic but all players are strategic. The excess demand/suply of every market is traded with other markets. If every node in the graph consists of a single buyer or a single seller then problem B.3 converges to problem B.2.*

Another graph model is the model defined in [6].

PROBLEM B.4. **Trading Networks:** *Given a graph and buyers and sellers who are situated on nodes of the graph. All trade must pass through a* trader*. In this case procurement sets are of the form $(buyer, seller, trader)$ where the possible sets of this form are defined by a graph.*

The supply chain model [2, 4] can be seen as a generalization of [6] in which procurement sets consist of the form $(producer, consumer, trader_1, \ldots, trader_k)$.

PROBLEM B.5. **Supply Chain:** *A graph $G = (V, E)$. The graph nodes are markets M^t, $1 \leq t \leq n$ and the edges $e \in E$ define trading relations between the markets. There are producer markets where the buyers in those markets produce new goods out of the goods they bought and sell them in other markets. Some producer markets are assumed to produce goods out of nothing (called first producer). There are also consumer markets that do not produce any goods. All players have value for goods they buy or sell. An entire chain of interim trades is necessary to create a viable procurement set that includes a first producer and a consumer.*

[2, 4] consider a *unique manufacturing technology* in which the graph defining possible relationships is a tree. [2] considers a simple chain structure and [4] considers a more general tree structure.

SUB-PROBLEM B.5.1. **Supply Chain: Simple Chain** *In [2] every node in the supply chain is a simple double-sided auction B.1, together with the unique manufacturing technology assumption the structure of the supply chain is a simple production line.*

SUB-PROBLEM B.5.2. **Supply Chain: Tree** *In [4] every node is a known single-minded double combinatorial auction (see problem B.8). A buyer can produce a single good out of the single bundle of goods he buys and not all buyers in market M^t sell their good to the same market M^{t+j} as in sub-problem B.5.1. Together with the unique manufacturing technology assumption the above supply chain creates a tree structure.*

[2, 4] left open the question of budget balanced mechanisms for supply chains where there is no unique manufacturing technology. Section 5 explains why removing the unique manufacturing technology assumption results in a problem that is not a procurement class based domain.

All the problems presented so far in this section are procurement-class based domains. We also consider several problems that are not procurement class based domains and generally the questions of budget balance have been left as open problems.

An open problem raised in [3] is the specially distributed markets model in which edges are strategic.

PROBLEM B.6. **Spatially distributed markets with strategic edges:** *there is a graph $G = (V, E)$ such that each $v \in V$ has a set of sellers S^v and a set of buyers B^v. Each edge $e \in E$ has an associated cost which is the cost to transport a single unit of good along the edge. Each buyer,seller and edge has a value for the trade, i.e., all entities are strategic.*

Another interesting problem is *transport networks*.

PROBLEM B.7. **Transport networks:** *A graph $G = (V, E)$ where the edges are strategic players with costs and the goal is to find a minimum cost transportation route between a pair of privileged nodes $Source, Target \in V$.*

It was shown in [1] that the efficient allocation can have a budget deficit that is linear in the number of players. Similar to problem B.6, this problem is not a procurement class based domain and [1] left the question of a budget balanced mechanism open.

Another non procurement-class based domain mechanism is the *double-sided combinatorial auction (CA) with single-value players.*

PROBLEM B.8. **Double-sided combinatorial auction (CA) with single value players:** *There exists a set S of sellers each selling a single good. There also exists a set B of buyers each interested in* bundles *of 2^S[14].*

There are two variants of this problem. In the single minded *case each buyer has a positive value for only a single subset whereas in the* multi minded *case each buyer can have multiple bundles with positive valuation but all of the values are the same. In both cases we assume* free disposal *so that all bundles containing the desired bundle have the same value for the buyer.*

We also consider problems that are non class domains.

PROBLEM B.9. **Double-sided combinatorial auction (CA) with general multi-minded players:** *same as B.8 but each buyer can have multiple bundles with positive valuation that are not necessarily the same.*

[14] We abuse notation and identify the seller with the good.

C Comparing Different Choices of X

The choice of X can have a large impact on the welfare (and revenue) of the reduced mechanism and therefore the question arises of how one should choose the set X.

As the X-external mechanism is required to maintain IC, clearly the choice of X can not depend on the value of the players as otherwise the reduced mechanism will not be truthful.

In this section we motivate the choice of small X sets for procurement class based domains and give intuition that it may also be the case for some other domains.

We start by illustrating the effect of the set X over the welfare and revenue in the double-sided auction with homogeneous goods problem B.1. Similar examples can be constructed for the other problems defined is B.

The following example shows an effect on the welfare.

EXAMPLE C.1. *There are two buyers and two sellers and two non intersecting (incomparable) sets $X = \{buyers\}$ and $Y = \{sellers\}$. If the values of the buyers are $101, 100$ and the sellers are $150, 1$ then the X-external mechanism will yield a gain from trade of 0 and the Y-external mechanism will yield a gain from trade of 100.*

Conversely, if the buyers values are $100, 1$ and the sellers are $2, 3$ the X-external mechanism will yield a gain from trade of 98 and and the Y-external mechanism will yield a gain from trade of zero.

The example clearly shows that the difference between the X-external and the Y-external mechanism is unbounded although as shown above the fraction each of them reduces can be bounded and therefore the multiplicative ratio between them can be bounded (as a function of the number of trades).

On the revenue side we can not even bound the ratio as seen in the following example:

EXAMPLE C.2. *Consider k buyers with value 100 and $k+1$ sellers with value 1.*

If $X = \{buyers\}$ then there is no need to reduce any trade and all of the buyer receive the good and pay 1. $k+1$ of the sellers sell and each of them receive 1. This yields a net revenue of zero.

If $Y = \{sellers\}$ then one must reduce a trade! This means that all of the buyers pay 100 while all of the sellers still receive 1. the revenue is then $99k$.

Similarly, an example can be constructed that yields much higher revenue for the X-external mechanism as compared to the Y-external mechanism.

The above examples refer to sets X and Y which do not intersect and are incomparable. The following theorem compares the X-external and Y-external mechanisms for procurement class based domains where X is a subset of Y.

THEOREM C.1. *For procurement class based domains, if $X \subset Y$ and for any $\mathbf{s} \in \mathbf{S}$, $\mathbf{s} \cap X \cap Y \neq \varnothing$ then:*

1. *The efficiency of the X external mechanism in GTR-1 (and hence GTR-2) is at least that of the Y-external mechanism.*

2. *Any winning player that wins in both the X-external and Y-external mechanisms pays no less in the Y-external than in the X-external and therefore the ratio of budget to welfare is no worse in the Y external then the X-external.*

Proof. 1. For any dividing function D there are two possible reasons why a procurement set $\mathbf{s}_j$ would be reduced in the X-external mechanism:

   (a) $\mathbf{s}_j$ lacks external competition in the X-external mechanism. In this case $\mathbf{s}_j$ lacks external competition in the internal mechanism.

   (b) $\mathbf{s}_j$ has all required external competitions in X-external. In this case $\mathbf{s}_j$ has all required internal competitions in Y-external by lemma 3.1 but might lack some external competition for $\mathbf{s}_j \cup \{Y \setminus X\}$ and be reduced.

2. This follows from the fact that for any ordering D any procurement set $\mathbf{s}$ that is reduced in the X-external mechanism is also reduced in the Y-external mechanism. Therefore the critical value is no less in the Y-external mechanism than the X-external mechanism.

□

REMARK C.1. *For any two sets X, Y it is easy to build an example in which the X-external and Y-external mechanisms reduce the same procurement sets so the inequality is weak.*

Theorem C.1 shows an inequality in welfare as well as for payments but it is easy to construct an example in which the revenue can increase for X as compared to Y, as well as the opposite. This suggests that in general we want X to be as small as possible although in some domains it is not possible to compare different X's.