On Graphs Convexities Related to Paths and Distances

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Purpose

- Parameters related to graph convexities
- Common graph convexities
- Complexity results concerning the computation of graph convexity parameters
- Bounds

Contents

- Graph Convexities:
  geodetic, monophonic, P_3
- Convexity parameters:
  hull number, interval number, convexity number
- Convexity parameters:
  Carathéodory number, Helly number, Radon number, rank
- Computing the rank:
  general graphs, special classes, relation to open packings
- Bounds

Convexity Space

A, finite set
$\mathcal{C}$ collection of subsets A

$(A, \mathcal{C})$ **Convexity space**:

- $\emptyset, A \in \mathcal{C}$
- $\mathcal{C}$ is closed under intersections

$C \in \mathcal{C}$ is called **convex**

Graph Convexity

G, graph

Convexity space $(A, \mathcal{C})$,
where $A = V(G)$, for a graph G.

Convex Hull

Convex Hull of $S \subseteq V(G)$ relative to $(V(G), \mathcal{C})$: smallest convex set $C \supseteq S$

Notation: $H(S)$

The convex hull $H(S)$ is the intersection of all convex sets containing S

Applications

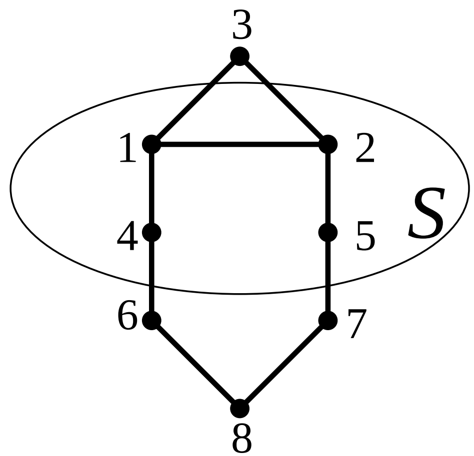
Social networks

Geodetic convexity

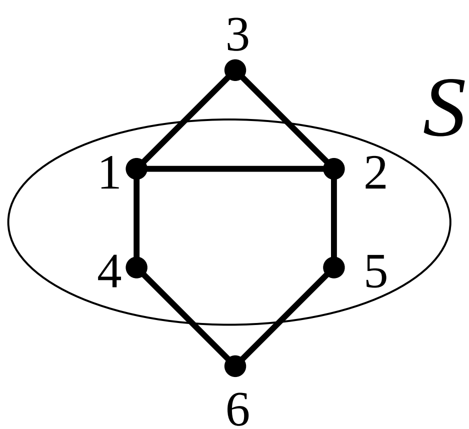
geodetic convexity:
convex sets closed under shortest paths

Van de Vel 1993
Chepói 1994
Polat 1995
Chartrand, Harary and Zhang 2002
Caceres, Marques, Oellerman and Puertas 2005

Examples



CONVEX



NOT CONVEX

Monophonic convexity

monophonic convexity:
convex sets closed under induced paths

Jamison 1982

Farber and Jamison 1985

Edelman and Jamison 1985

Duchet 1988

Caceres, Hernando, Mora, Pelayo, Puertas, Seara 2005

Dourado, Protti, Szwarcfiter 2010

P_3 convexity

P_3 **convexity**:
convex sets closed under common neighbors

Erdös, Fried, Hajnal, Milner 1972

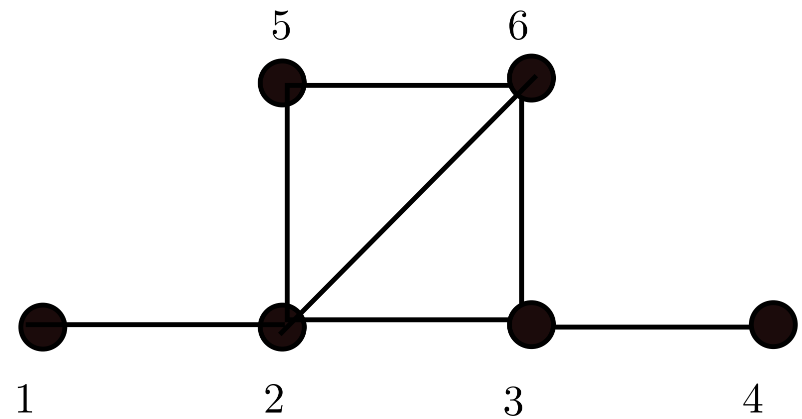
Moon 1972

Varlet 1972

Parker, Westhoff and Wolf 2009

Centeno, Dourado, Penso, Rautenbach and Szwarcfiter 2010

Example



$\{2, 3, 5, 6\}$ Convex

$\{1, 3, 5, 6\}$ Not convex

Convexity Parameters

- interval number (geodetic number)
- convexity number
- hull number

- Helly number
- Carathéodory number
- Radon number
- rank

Hull Number and Convexity Number

If $H(S) = V(G)$ then S is a **hull set.**

The least cardinality hull set of G is the **hull number** of the graph.

The largest proper convex set of G is the **convexity number** of the graph.

Interval Number

$(V(G), \mathcal{C})$ is an **interval convexity**:
$\exists$ function $I : \binom{V}{2} \rightarrow 2^V$, s.t.
$C \subseteq V(G)$ belongs to $\mathcal{C} \Leftrightarrow$
$I(x, y) \subseteq C$ for every distinct elements $x, y \in C$.

For $S \subseteq V(G)$, write $I(S) = \cup_{x,y \in S} I(x, y)$

If $I(S) = V(G)$ then S is an **interval set**

The least cardinality interval set of G is the **interval number** of the graph.

Helly number

Theorem 1 *(Helly 1923) In a d-dimensional Euclidean space, if in a finite collection of n > d convex sets any d+1 sets have a point in common, then there is a point common to all sets of the collection.*

Helly number

The smallest k, such that every k-intersecting subfamily of convex sets has a non-empty intersection.

Helly-Independence

For $S \subseteq V(G)$, the set

$$\cap_{v \in S} H(S \setminus \{v\})$$

is the **Helly-core** of S.

S is *Helly-independent* if it has a non-empty Helly-core, and *Helly-dependent* otherwise.

$h(G)$ = Helly number

the maximum cardinality of a Helly-independent set.

Carathéodory number

Theorem 2 *(Carathéodory 1911) Every point* u, *in the convex hull of a set* $S \subset \mathbb{R}^d$ *lies in the convex hull of a subset* F *of* S, *of size at most* $d+1$.

Carathéodory number

$c(G)$ = Carathéodory number,
the smallest k, s.t.
for all $S \subseteq V(G)$, and all $u \in H(S)$,
there is $F \subseteq S$, $|F| \leq k$,
satisfying $u \in H(F)$.

Carathéodory-Independence

For $S \subseteq V(G)$, let

$$\partial S = \cup_{v \in S} H(S \setminus \{v\})$$

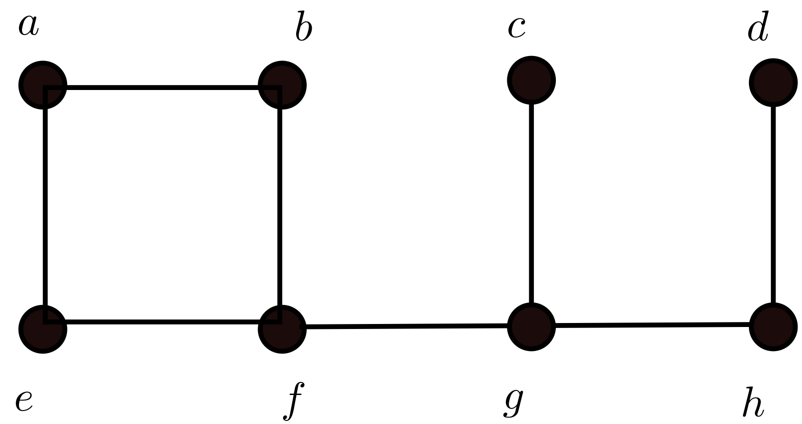
.

S is **Carathéodory-independent** (or **irredundant**) if $H(S) \neq \partial S$, and **Carathéodory-dependent** (or **redundant** otherwise.

$c(G)$ = Carathéodory number

maximum cardinality of a Carathéodory-independent set.

Example



P_3 convexity:
$\{e, b, c, d\}$, largest Carathéodory-independent set
$\Rightarrow c(G) = 4$

Radon Number

Theorem 3 *(Radon 1921):*
Every set of $d+2$ *points in* $\mathbb{R}^d$ *can be partitioned into two sets, whose convex hulls intersect.*

Radon number

Let $R \subseteq V(G)$ and $R = R_1 \cup R_2$
$R = R_1 \cup R_2$ is a **Radon partition**:
$H(R_1) \cap H(R_2) \neq \emptyset$

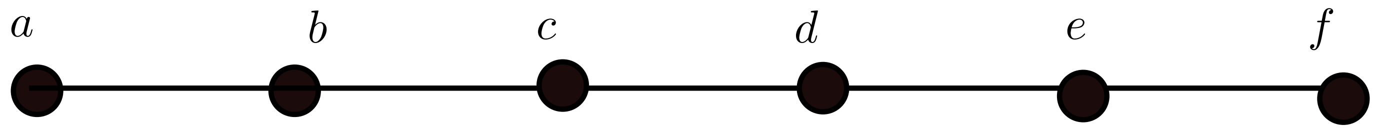
R is a **Radon set** if it admits a Radon partition,
$R(G) =$ **Radon number,**
least k, s.t. all sets of size $\leq k$ admit a Radon partition

Radon-Independence

A set $R \subset V(G)$ admitting no Radon partition is called **Radon-independent** (or **anti-Radon**, or **simploid** c.f. Nesetril and Strausz 2006).

$r(G) = 1+$ maximum cardinality of an anti-Radon set of G.

Example



P_3 convexity:

$\{a, b, d, e\}$, largest Radon-independent set

$\Rightarrow r(G) = 5$

Convex Rank

A set $S \subseteq V(G)$ is **convex-independent** if

$$s \notin H(S \setminus \{s\}),$$

for every $s \in S$, and **convex-dependent**, otherwise.

$rank(G) =$ maximum cardinality of a convex-independent set

Notation: $rk(G)$

Heredity

Helly-independence,
Radon-independence,
convex-independence:

are hereditary

Carathéodory-independence:

not necessarily

Implications

Radon-independence $\Rightarrow$ Helly-independence $\Rightarrow$ convex-independence

Carathéodory-independence $\Rightarrow$

convex-independence

Relationships

- $h + 1 \le r$ (Levi 1951)
- $r \le ch + 1$ (Kay and Womble 1971)

Basic problems - geodetic convexity

Given $S \subseteq V(G)$:

- Compute $I(S)$ - Poly
- Decide if S is convex - Poly
- Decide if S is an interval set - Poly
- Compute $H(S)$ - Poly
- Decide if S is a hull set - Poly

Basic problems - P_3 convexity

Given $S \subseteq V(G)$:

- Compute $I(S)$ - Poly
- Decide if S is convex - Poly
- Decide if S is an interval set - Poly
- Compute $H(S)$ - Poly
- Decide if S is a hull set - Poly

Basic problems - monophonic convexity

Given $S \subseteq V(G)$:

- Compute $I(S)$ - NPH
- Decide if S is convex - Poly
- Decide if S is an interval set - NPH
- Compute $H(S)$ - Poly
- Decide if S is a hull set - Poly

Complexity - Geodetic Convexity

Parameter	Status	Reference
interval number	NPC	Atici 2002
hull number	NPC	Dourado, Gimbel, Kratochvil, Protti, Szwarcfiter 2009
convexity number	NPC	Gimbel 2003
Helly number	Co-NPC	Polat 1995
Carathéodory number	NPC	Dourado, Rautenbach, Santos, Schäfer, Szwarcfiter 2013
Radon number	NPH	Dourado, Szwarcfiter, Toman 2012
rank	NPC	Kanté, Sampaio, Santos, Szwarcfiter 2016

Complexity - P_3 Convexity

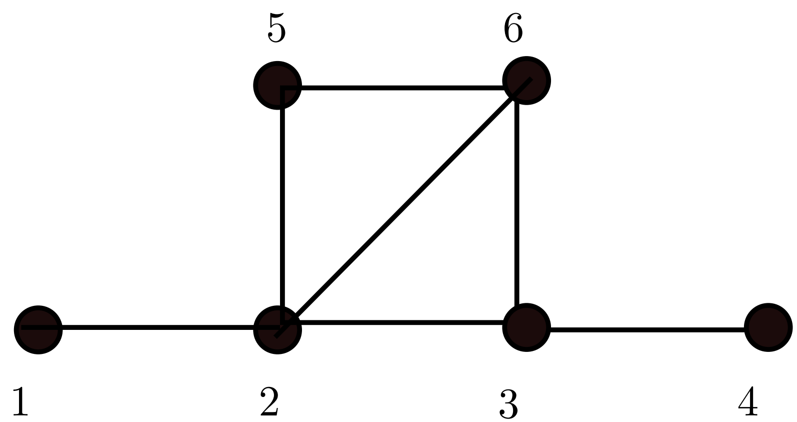
Parameter	**Status**	**Reference**
interval no.	NPC	Chang, Nemhauser 1984
hull no.	NPC	Centeno, Dourado, Penso, Rautenbach, Szwarcfiter 2011
convexity no.	NPC	Centeno, Dourado, Szwarcfiter 2009
Helly no.	Co-NPC	
Carathéodory no.	NPC	Barbosa, Coelho, Dourado, Rautenbach, Szwarcfiter 2012
Radon no.	NPH	Dourado, Rautenbach, Santos, Schäfer, Szwarcfiter, Toman 2013
rank	NPC	Ramos, Santos, Szwarcfiter 2014

Complexity - Monophonic Convexity

Parameter	Status	Reference
interval number	NPC	Dourado, Protti, Szwarcfiter 2010
hull number	Poly	Dourado, Protti, Szwarcfiter 2010
convexity number	NPC	Dourado, Protti, Szwarcfiter 2010
Helly number	NPH	Duchet 1988
Carathéodory number	Poly	Duchet 1988
Radon number	NPH	Duchet 1988
rank	NPC	Ramos, Santos, Szwarcfiter 2014

Convex independence

Example (for P_3 convexity)



$\{1, 4, 5\}$ is convexly-independet

$\{1, 3, 5\}$ is convexly-dependent

Problem Statement

MAXIMUM CONVEXLY INDEPENDENT SET
INPUT: Graph G, integer k
QUESTION: Does G contain a convexly independent set of size $\geq k$?

A related problem

An **open packing** of G is a subset $S \subseteq V(G)$ whose open neighborhoods are pairwise disjoint.

Henning and Slater (1999)

A related problem

MAXIMUM OPEN PACKING
INPÙT: Graph G, integer k
QUESTION: Does G contain an open packing of size $\geq k$?

Notation: $\rho(G) =$ maximum open packing of the graph

Relation: $\rho(G) \leq rk(G)$

Open packing - Hardness

Theorem 4 *(Henning and Slater 1999) The maximum open packing problem is NP-complete, even for chordal graphs.*

Split graphs and Convexly indep sets

Lemma 1 *: Let* C *be any clique of some graph* G*, and* $v_1, v_2 \in C$*. Then* $H(\{v_1, v_2\}) \subseteq C$*.*

Lemma 2 *: Let* G *be a split graph with bipartition* $C \cup I = V(G)$*, minimum degreee* ≥ 2*, and* S *a convexly indep set of size* > 2*. Then* $S \subseteq I$*.*

Sketch

(i) $|S \cap C| \geq 2 \;\Rrightarrow H(S) = V(G)$, contradiction

(ii) $|S \cap C| = 1$: Let $v_1 \in S \cap C$ and $v_2 \in S \cap I$. Then there is $v_3 \in C$ adjacent to v_1. Consequently, $v_3 \in H(\{v_1, v_2\})$, implying $H(S) = V(G)$, again a contradiction

Lemma

Lemma 3 *Let G be a split graph with bipartition $C \cup I = V(G)$, minimum degree ≥ 2, and S, $|S| > 2$ a proper subset of $V(G)$. Then S is convexly indep iff $H(S) = S$.*

Sketch: Let S be convexly indep. By the previous lemma, $S \subseteq I$. By contradiction, suppose $H(S) \neq S$. Then $\exists w \in C \cap H(S)$ such that w is adjacent to $v_1, v_2 \in S$. Since $\delta(G) \geq 2$, $\exists v_3 \in C$, $v_3 \neq w$, such that v_1, v_3 are adjacent. Consequently, $H(S) = V(G)$, implying that S is not convexly indep. The converse is similar.

Hardness - Rank

Theorem 5 *The maximum convexly indep set problem is NP-complete, even for split graphs of minimum degree* ≥ 2.

Reduction: Set packing

Hardness - Open packing

Corollary 1 *The maximum open packing problem is NP-complete, even for split graphs of minimum degree* ≥ 2*.*

Note: Improves the NP-completess for chordal graphs, by Henning and Slater.

More hardness

Theorem 6 *The maximum convexly indep set problem is NP-complete for bipartite graphs having diameter* ≤ 3

Reduction: From the NP-completeness of maximum convexly indep set for split graphs.

More hardness - Monophonic

Theorem 7 *In the monophonic convexity, the maximum convexly indep set problem is NP-complete for graphs having no clique cutsets.*

Reduction: From maximum clique problem

Polynomial time

- Threshold graphs
- Biconnected interval graphs
- trees

Threshold graphs

Theorem 8 *Let G be a threshold graph, $|V(G)| \geq 3$, and D the subset of minimum degree vertices of G. Then*

- *(i): G is a star $\Rrightarrow rk(G) = |V(G)| - 1$. Otherwise*
- *(ii): $\delta(G) = 1 \Rrightarrow rk(G) = |D| + 1$. Otherwise*
- $rk(G) = 2$

Threshold graphs

Sketch:

- (i): No leaf v of a graph belongs to the hull set of any set not containing v.
- (iii): Any two vertices of G form a maximal convexly indep set.
- (ii) All degree one vertices have a common neighbor. Then |D| is convexly indep. However we can still add an additional vertex $u \neq v$ to the set and maintain it as convexly independent.

Biconnected interval graphs

Lemma 4 *Let* G *be a biconnected chordal graph, and* u, v *a pair of distinct vertices of* G*, at distance* ≤ 2*. Then* $H(u, v) = V(G)$*.*

Biconnected interval graphs

Let G be an interval graph, and $\mathcal{I}$ the family of intervals representing G.
Greedy Algorithm:

1. Define $S := \emptyset$, and sort $\mathcal{I}$ in non-decreasing ordering of the endpoints of the intervals.
2. while $\mathcal{I} \neq \emptyset$, choose the vertex v having the least endpoint in $\mathcal{I}$, add v to S, and remove from $\mathcal{I}$ the intervals of v and all vertices lying at distance ≤ 2 from v in G.
3. Terminate the algorithm: S is a maximum convexly indep set of G.

Trees

T tree, rooted at $r \in V(T)$.
Let u, v be adjacent vertices of T, and S a subset of $V(T)$ containing both u, v. Then u **sends a unit of load** to v if

$$u \in H_{T-v}(S - v)$$

(u does not depend on v to be inside $H(S - v)$

Notation:

$ch(v)$ = total load that v received by v, considering all its neighbors in $H_{T-v}(S - v)$.

Trees

Lemma 5 *Let* $S \subseteq V(T)$ *be a convexly indep set, and* $v \in V(T)$. *Then* $v \in H(S - v)$ *iff* $ch(v) \geq 2$.

Corollary 2 $S \subseteq V(T)$ *is convexly indep iff exists no* $v \in S$, s.t. $ch(v) \geq 2$.

Trees

$P_v(i,j,k)$, the *contribution* of v = size of max convexly indep set using only vertices from the subtree rooted in v in the state defined by i, j and k.

If $P_v(i,j,k)$ is not defined then v's contribution is $-\infty$.

- $i = 1$ means that v receives 1 unity of charge from its parent, while $i = 0$ means it does not.
- $j = 1$ means that v is part of the convexly independent set, while $j = 0$ means the opposite.
- k is the amount of charge that v receives from its children.

Trees

Notation: p_v = parent of v; $N'(v) = N(v) \setminus \{p_v\}$.

Define the functions:

$$f(v,i) = \max\{P_v(i,0,0), P_v(i,0,1)\} \tag{1}$$

$$h(v,i) = \max\{\max_{2 \le k < d(v)} \{P(i,0,k)\}, \max_{0 \le k \le d(v)} P_v(i,1,k)\} \tag{2}$$

$$g(v,i_1,i_2) = h(v,i_1) - f(v,i_2) \tag{3}$$

$$P_v(0,0,0) = \sum_{u \in N'(v)} f(u,0); \tag{4}$$

$$P_v(0,0,1) = \begin{cases} -\infty, & \text{if } v \text{ has no child,} \\ \sum_{u \in N'(v)} f(u,0) + \max_{u \in N'(v)} g(u,0,0), & \text{otherwise;} \end{cases} \tag{5}$$

$$P_v(0,0,2) = \begin{cases} -\infty, & \text{if } v \text{ has less than } 2 \text{ children,} \\ \sum_{u \in N'(v)} f(u,1) + \max_{\substack{\forall X \subseteq N'(v) \\ |X|=2}} \sum_{u \in X} g(u,0,1), & \text{otherwise;} \end{cases} \tag{6}$$

$$\underset{k \geq 3}{P_v(0,0,k)} = \begin{cases} -\infty, & \text{if } v \text{ has less than } k \text{ children,} \\ \sum_{u \in N'(v)} f(u,1) + \max_{\substack{\forall X \subseteq N'(v) \\ |X|=k}} \sum_{u \in X} g(u,1,1), & \text{otherwise;} \end{cases} \tag{7}$$

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$$P_v(0,1,0) = \sum_{u \in N'(v)} f(u,1) + 1; \tag{8}$$

$$\sum P_v(0,1,1) = \begin{cases} -\infty, & \text{if } v \text{ has no child,} \\ \sum_{u \in N'(v)} f(u,1) + \max_{u \in N'(v)} g(u,1,1) + 1, & \text{otherwise;} \end{cases}$$

$$\underset{k \geq 2}{P_v(0,1,k)} = -\infty; \tag{9}$$

$$P_v(1,0,0) = \begin{cases} -\infty, & \text{if } v = r, \\ \sum_{u \in N'(v)} f(u,0), & \text{otherwise;} \end{cases} \tag{10}$$

$$P_v(1,0,1) = \begin{cases} -\infty, & \text{if } v \text{ has no child or } v = r, \\ \sum_{u \in N'(v)} f(u,1) + \max_{u \in N'(v)} g(u,0,1), & \text{otherwise;} \end{cases}$$

(11)

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$$P_v(1,0,k)_{k\geq 2} = \begin{cases} -\infty, & \text{if } v \text{ has less than } k \text{ children or } v = r, \\ \sum_{u\in N'(v)} f(u,1) + \max_{\substack{\forall S\subseteq N'(v)\\ |S|=k}} \sum_{u\in S} g(u,1,1), & \text{otherwise;} \end{cases} \tag{12}$$

$$P_v(1,1,0) = \begin{cases} -\infty, & \text{if } v = r, \\ \sum_{u\in N'(v)} f(u,1) + 1, & \text{otherwise;} \end{cases} \tag{13}$$

$$P_v(1,1,k)_{k\geq 1} = -\infty. \tag{14}$$

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Trees - geodetic and monophonic

Theorem 9 *The set of leaves of a tree* T *is the maximum convexly indep set of* T*, in both the geodetic and monophonic convexities.*

Bounds

Theorem 10 *Let G be a graph with minimum degree $\delta(G)$. Then*

$$rk(G) \leq \frac{2n}{\delta(G)+1}$$

Moreover, this bound is tight.

A similar expression has been obtained by Henning, Rautenbach and Schafer (2013), for bounding the Radon number.

Note that the rank of a graph can be used as a tighter bound for the Radon number, since

$$rd(G) - 1 \leq rk(G) \leq \frac{2n}{\delta(G)+1}$$

Further problems

This was essentially the first computational study of this parameter. There are many open problems, as the study of the rank of a graph in the geodetic convexity.

THANK YOU FOR THE ATTENTION