Dimension reduction for semidefinite programming

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Joint work with **Pablo Parrilo** (MIT)

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Semidefinite programs (SDPs)

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \mathbb{S}^n_+ \end{array}$$

Formulated over vector space $\mathbb{S}^n$ of $n \times n$ symmetric matrices.

- variable $X \in \mathbb{S}^n$
- $\mathcal{A} \subseteq \mathbb{S}^n$ an affine subspace, $C \in \mathbb{S}^n$ cost matrix
- $\mathbb{S}^n_+$ cone of psd matrices

Efficiently solvable in theory; in practice, solving some instances impossible unless special structure is exploited.

Dimension reduction

Reformulate problem over subspace $\mathcal{S} \subseteq \mathbb{S}^n$ intersecting set of optimal solutions

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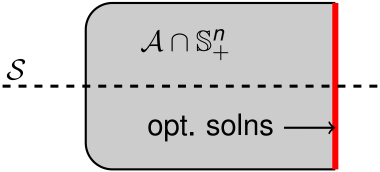

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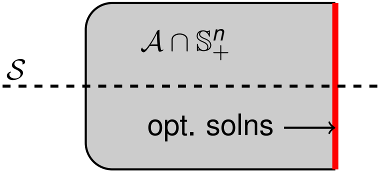

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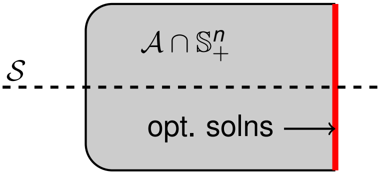

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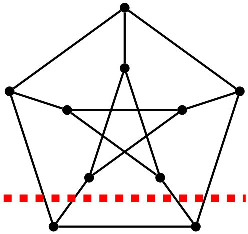
Reduction methods: *symmetry reduction* and *facial reduction*

Symmetry reduction (MAXCUT relaxation example)



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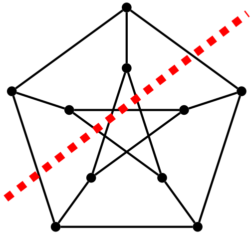

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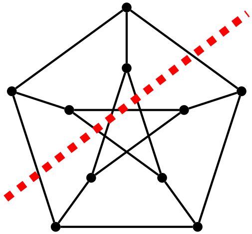

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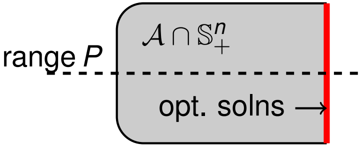

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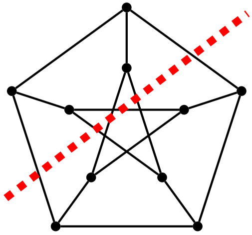

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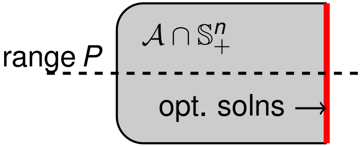

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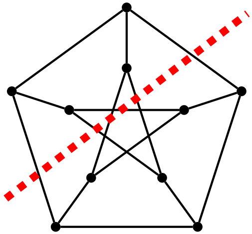

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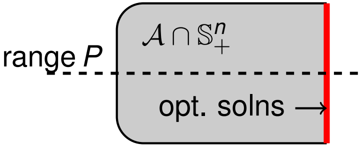

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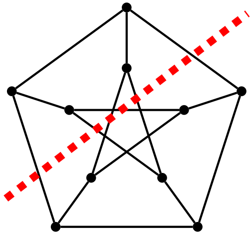

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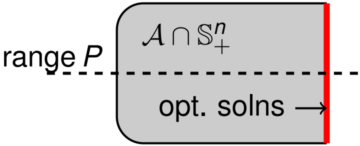

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(e.g., Schrijver '79; Gatermann, Parrilo. '03)

Our approach

This talk: a reduction method subsuming symmetry reduction

- Notion of ‘optimal’ reduction and algorithm for finding it
- Jordan algebra interpretation: immediate extension to symmetric cone optimization (e.g., LP, SOCP).

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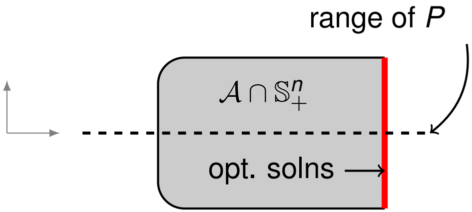
- Borwein-Wolkowicz ’81: minimal face/facial reduction algorithm
- Alizadeh-Schmieta ’00: extension of interior-point methods

How does symmetry reduction work?

Given SDP $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \operatorname{Tr} CX$, method finds special orthogonal projection $P : \mathbb{S}^n \to \mathbb{S}^n$



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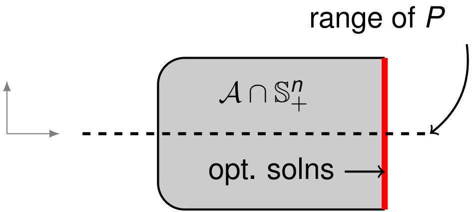

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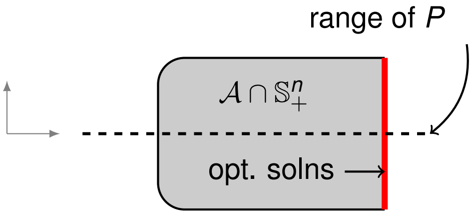

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- Hence, if X feasible then $P(X)$ feasible with equal cost:

$$\operatorname{Tr} CX = \operatorname{Tr} P(C)X = \operatorname{Tr} CP(X).$$

Our approach: optimize over projections

Given SDP $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \langle C, X \rangle$, find map P that solves

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Main properties:

- Can be solved in polynomial time.
- Range of P structured: a *Jordan subalgebra* of $\mathbb{S}^n$.
- $\mathbb{S}^n_+ \cap \operatorname{range} P$ equals a product of symmetric cones.

The optimal subspace of $\min_{X \in \mathcal{A} \cap \mathbb{S}^n_+} \langle C, X \rangle$

Theorem (Parrilo-P.)

Orthogonal projection $P : \mathbb{S}^n \to \mathbb{S}^n$ *solves*

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where affine set $\mathcal{A} = X_{\mathcal{L}^\perp} + \mathcal{L}$ *and* $P_{\mathcal{L}}$ *is proj. map onto* $\mathcal{L}$.

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Properties of minimization problem:

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- A unique solution.

Decomposition via Jordan algebras

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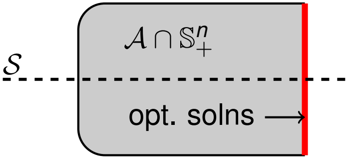

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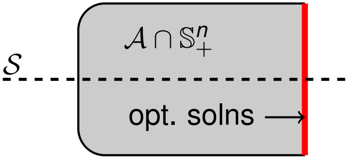

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$\mathcal{S}_i$ are *simple Jordan algebras*

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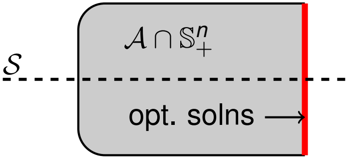

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$\mathcal{S}_i$ are *simple Jordan algebras*

- Number of distinct eigenvalues of generic element equals *rank* of $\mathcal{S}_i$—a complexity measure.

Minimizing dimension optimizes decomposition

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All feasible subspaces have decomp. $\mathcal{S} = \oplus_{i=1}^{d_\mathcal{S}} \mathcal{S}_i$. In what sense does solution $\mathcal{S}^*$ optimize the ranks of each $\mathcal{S}_i$?

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- $\mathcal{S}^*$ minimizes $\sum_i \operatorname{rank} \mathcal{S}_i$ and $\max_i \operatorname{rank} \mathcal{S}_i$
- *Majorization* inequalities hold, i.e., for each $m \geq 1$

$$\sum_{i=1}^{m} \operatorname{rank} \mathcal{S}_i^* \leq \sum_{i=1}^{m} \operatorname{rank} \mathcal{S}_i$$

(ranks sorted in decreasing order)

Majorization example

Subspaces (parametrized by u_i and v_i) and their rank vectors

$$\begin{pmatrix} u_1 & u_2 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_4 & 0 & 0 \\ 0 & 0 & 0 & u_5 & u_6 \\ 0 & 0 & 0 & u_6 & u_7 \end{pmatrix} \qquad \begin{pmatrix} v_1 & v_2 & 0 & 0 & 0 \\ v_2 & v_3 & 0 & 0 & 0 \\ 0 & 0 & v_4 & v_5 & v_6 \\ 0 & 0 & v_5 & v_7 & v_8 \\ 0 & 0 & v_6 & v_8 & v_9 \end{pmatrix}$$

$$r_u = (2, 1, 2) \qquad r_v = (2, 3)$$

Majorization example

Subspaces (parametrized by u_i and v_i) and their rank vectors

$$\begin{pmatrix} u_1 & u_2 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_4 & 0 & 0 \\ 0 & 0 & 0 & u_5 & u_6 \\ 0 & 0 & 0 & u_6 & u_7 \end{pmatrix} \qquad \begin{pmatrix} v_1 & v_2 & 0 & 0 & 0 \\ v_2 & v_3 & 0 & 0 & 0 \\ 0 & 0 & v_4 & v_5 & v_6 \\ 0 & 0 & v_5 & v_7 & v_8 \\ 0 & 0 & v_6 & v_8 & v_9 \end{pmatrix}$$

$$r_u = (2, 1, 2) \qquad\qquad r_v = (2, 3)$$

Vector $r'_u = (2, 2, 1)$ majorized by $r'_v = (3, 2, 0)$:

$$2 \le 3, \qquad 2 + 2 \le 3 + 2, \qquad 2 + 2 + 1 \le 3 + 2 + 0$$

Structure theorem of Jordan-von Neumann-Wigner

If $\{X^2 : X \in \mathcal{S}\} \subseteq \mathcal{S}$, then $\mathcal{S} = \oplus_{i=1}^{m} \mathcal{S}_i$ with $\mathcal{S}_i$ isomorphic to

- Algebra of Hermitian matrices with real, complex or quaternion entries
- A spin-factor algebra

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Allows us to reformulate over $\mathcal{K}_1 \times \cdots \times \mathcal{K}_m$:

$$\begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \mathbb{S}_+^n \end{array} \qquad \begin{array}{ll} \text{minimize} & \operatorname{Tr} CX \\ \text{subject to} & X \in \mathcal{A} \cap \underbrace{T(\mathcal{K}_1 \times \cdots \times \mathcal{K}_m)}_{\mathcal{S} \cap \mathbb{S}_+^n} \end{array}$$

Computational results

Instances from DIMACS SDP problem library:

instance	$\dim \mathcal{S}^*$	$\dim \mathbb{S}^n$
hamming_7_5_6	5	8256
hamming_8_3_4	5	32896
hamming_9_5_6	6	131328
hamming_9_8	6	131328
hamming_10_2	7	524800

$\mathcal{S} \cap \mathbb{S}^n_+$ isomorphic to non-negative orthant—problem converted into linear program.

Results: SOSOPT (Seiler '13) Demo scripts

Script Name	n (before)	n (after)
sosoptdemo2	13, 3	$3, 2 \times 3, 1 \times 7$
sosoptdemo4	35	$5 \times 5, 1 \times 10$
gsosoptdemo1	9, 5	$6, 3 \times 2, 2$
IOGainDemo_3	15, 8	$10, 5 \times 2, 3$
Chesi(1|2)_IterationWithVlin	9, 5	$6, 3 \times 2, 2$
Chesi3_GlobalStability	14, 5	8, 6, 3, 2
Chesi(3|4)_IterationWithVlin	9, 5	$6, 3 \times 2, 2$
Chesi(5|6)_Bootstrap	19, 9	$13, 6 \times 2, 3$
Chesi(5|6)_IterationWithVlin	19, 9	$13, 6 \times 2, 3$
Coutinho3_IterationWithVlin	9, 5	$6, 3 \times 2, 2$
HachichoTibken_Bootstrap	19, 9	12, 7, 6, 3
HachichoTibken_IterationWithVlin	19, 9	12, 7, 6, 3
Hahn_IterationWithVlin	9, 5	6, 3, 3, 2
KuChen_IterationWithVlin	19, 9	$13, 6 \times 2, 3$
Parrilo1_GlobalStabilityWithVec	3, 2	$2, 1 \times 3$
Parrilo2_GlobalStabilityWithMat	3, 2	$2, 1 \times 3$
VDP_IterationWithVball	5, 4	$3 \times 2, 2, 1$
VDP_IterationWithVlin	9, 5	$6, 3 \times 2, 2$
VDP_LinearizedLyap	9, 5	$6, 3 \times 2, 2$
VannelliVidyasagar2_Bootstrap	19, 9	$13, 6 \times 2, 3$
VannelliVidyasagar2_IterationWithVlin	19, 9	$13, 6 \times 2, 3$
VincentGrantham_IterationWithVlin	9, 5	$6, 3 \times 2, 2$
WTBenchmark_IterationWithVlin	19, 9	$13, 6 \times 2, 3$

Conclusions

New reduction method for SDP.

- Generalizes symmetry reduction and *-algebra-methods
- Fully algorithmic, don’t need to compute automorphisms!
- Yields optimal ‘block-diagonalization’ (majorization)
- Through Jordan algebra theory, extends to LP/SOCP/...

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Thanks for your attention!