The Triangle Algorithm: An Algorithmic Separation Theorem and its Applications

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Given a subset $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$, and $p \in \mathbb{R}^m$, either give a certificate that proves $p \in conv(S)$, or one that proves $p \notin conv(S)$.

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When $p \notin conv(S)$ a certificate is a separating hyperplane.

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- Historically speaking, the first two polynomial-time LP algorithm happened to be (implicitly) designed for solve H-CHMP:

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- Khachiyan-K. matrix scaling algorithm (1992): Given an $n \times n$ symmetric psd matrix A, test the solvability of the following nonlinear dual to H-CHMP ($0 \in conv(A)$?):

$$DADe = e, \quad D = \mathrm{diag}(d_1, \cdots, d_n), \quad d_i > 0, \quad e = (1, \ldots, 1)^T.$$

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Moreover, algorithmic applications of these for semidefinite programming and self-concordant programming have been analyzed, e.g. "Semidefinite programming and matrix scaling over the semidefinite cone," *Linear Algebra and its Applications*, 2003, B.K.

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- **Step 1.** Given *iterate* $p' = \sum_{i=1}^{n} \alpha_i v_i \in conv(S)$, check if there exists a *pivot* : $v_j \in S$ s.t. $d(p', v_j) \geq d(p, v_j)$.

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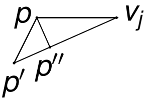

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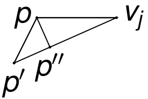

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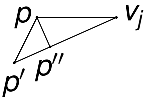

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- **Step 2.** Otherwise, compute $p'' = nearest(p; p'v)$:

$$p'' = (1-\alpha)p' + \alpha v_j = \sum_{i=1}^{n} \alpha_i' v_i, \quad \alpha = \frac{(p - p')^T (v_j - p')}{d^2(v_j, p')},$$

$$\alpha_j' = (1-\alpha)\alpha_j + \alpha, \quad \alpha_i = (1-\alpha)\alpha_i, \quad \forall i \neq j.$$

  Replace p' with p'' and Go to Step 1.

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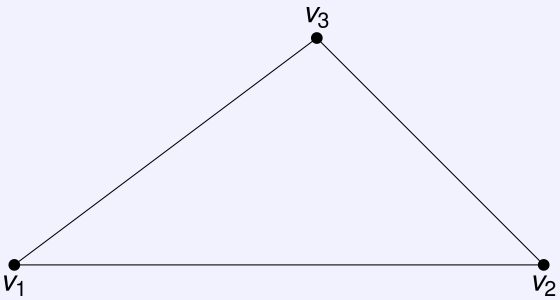

Figure: Triangle Algorithm for testing if $p \in conv(\{v_1, v_2, v_3\})$.

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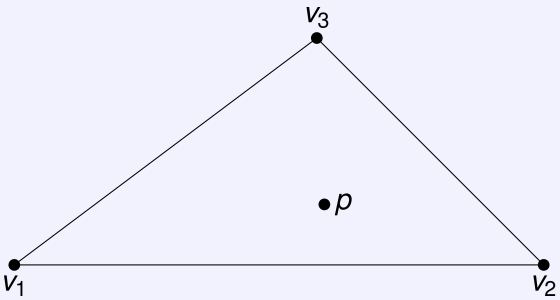

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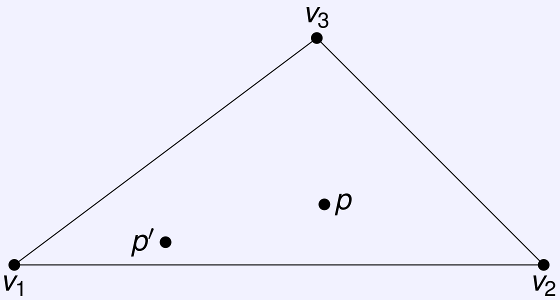

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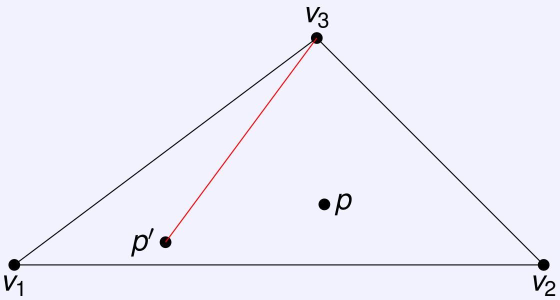

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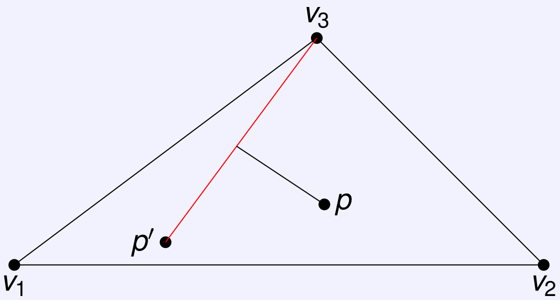

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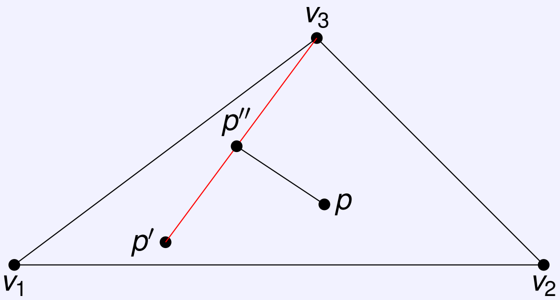

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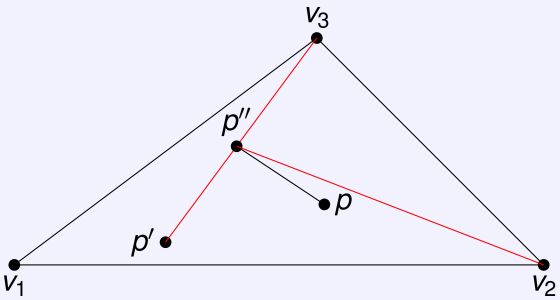

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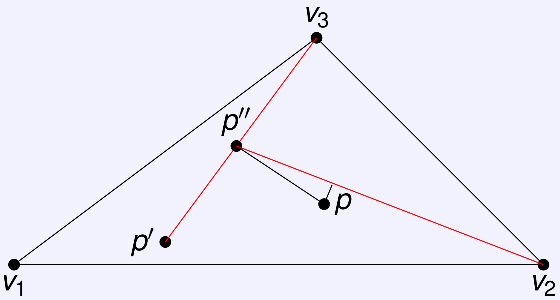

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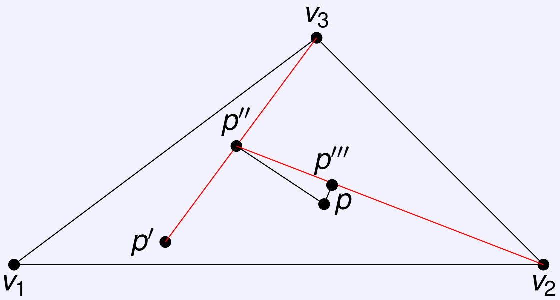

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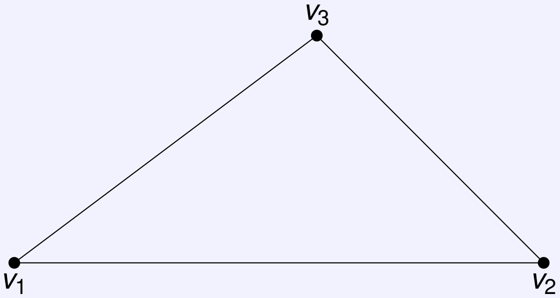

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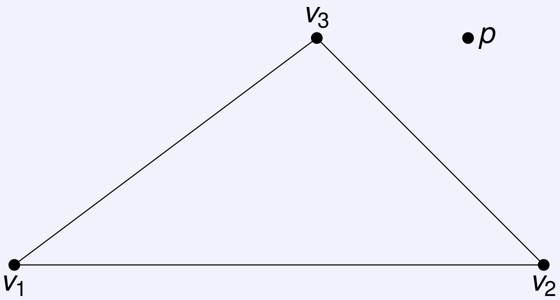

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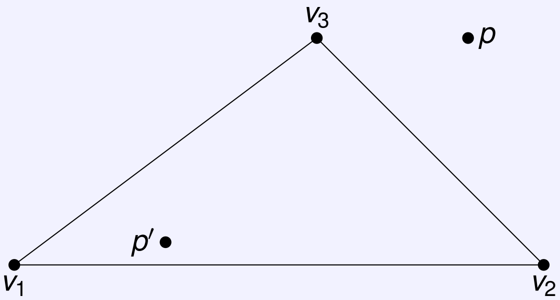

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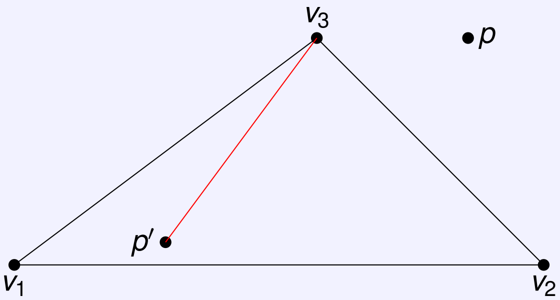

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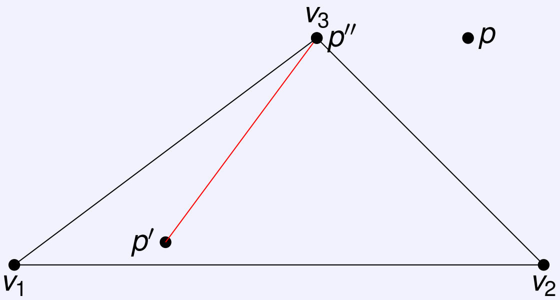

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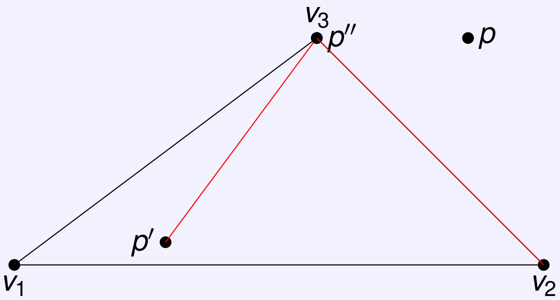

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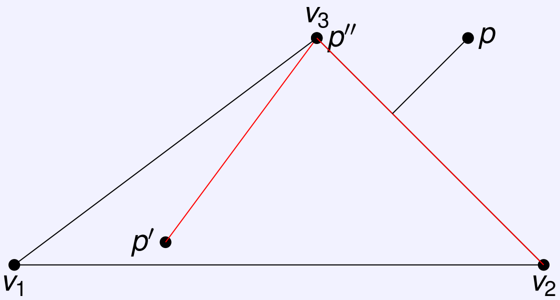

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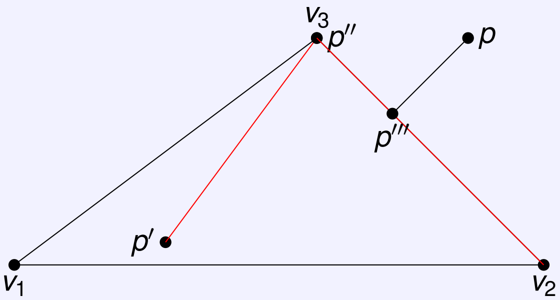

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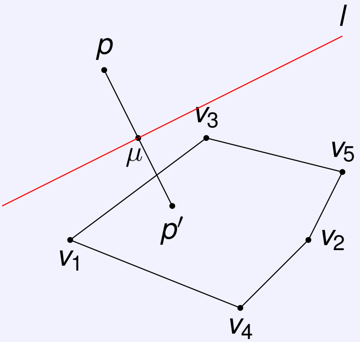

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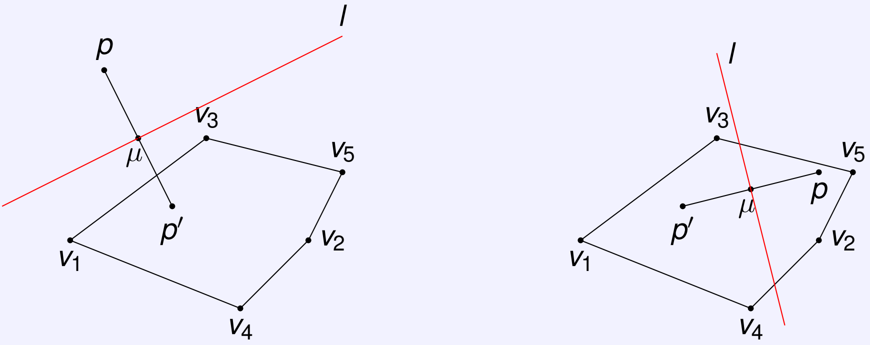

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Remark

H.W. Kuhn (1967), proves this in the Euclidean plane making use of several results, including Ville's Lemma. Some generalizations of the theorem over normed spaces is given by Durier and Michelot (1986).

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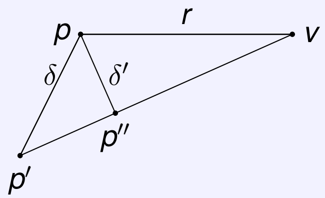

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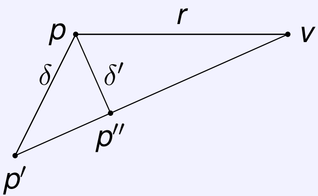

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Given two consecutive iterates p', p'', corresponding to the triangle $\triangle pp'v$ with v a pivot, let $\delta = d(p', p)$, $\delta' = d(p'', p)$, and $r = d(p, v)$.

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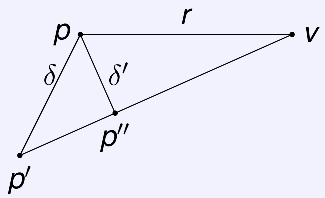

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$$\delta' \leq \delta \sqrt{1 - \frac{\delta^2}{4r^2}}.$$

Complexity of Triangle Algorithm: First Bound

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(ii) Suppose $p \notin conv(S)$. The number of iterations to compute a witness p' in $conv(S)$ is

$$O\left(\frac{R^2}{\Delta^2}\right), \quad \Delta = \min\left\{d(x, p) : \quad x \in conv(S)\right\}.$$

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To find pivot Triangle Algorithm does not require taking square-roots:

$$d(p', v) \geq d(p, v) \iff \|p'\|^2 - \|p\|^2 \geq 2v^T(p' - p).$$

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- *Worst-case complexity of each iteration of Triangle Algorithm is $O(mn)$. However, even without preprocessing, often, each iteration requires only $O(m+n)$.*
- *Triangle Algorithm could outperform these due to distance duality, simplicity and degrees of freedom it offers.*

Experimental Results with Triangle Algorithm

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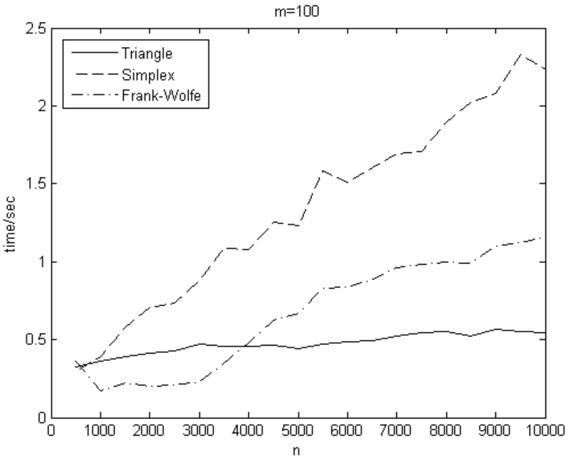

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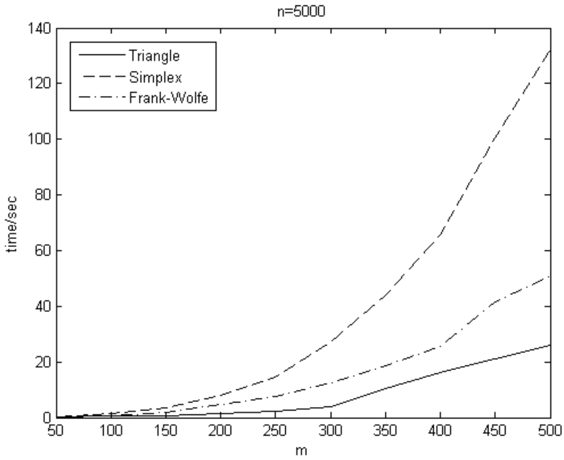

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n	# of points visited per iteration	iterations
500	185	459.6
1000	228.26	479.6
3000	240.37	540.4
5000	242.22	541.6
10000	254.84	535.4

Table: The performance of Triangle algorithm when m=100

Properties and Characterizations of Witnesses: Separation Property

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If $p' \in W_p$ the orthogonal bisecting hyperplane of the line segment pp' separates p from $conv(S)$.

Properties and Characterizations of Witnesses: Approximation of Distance to Convex Hull

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$$\frac{1}{2} d(p,p') \leq \Delta \leq d(p,p').$$

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Corollary

Given $S = \{v_1, \dots, v_n\}$ and p all in $\mathbb{R}^m$, consider the set of open balls B_i balls centered at v_i with radius $d(p, v_i)$, $i = 1, \dots, n$.
Then $p \in conv(S)$ if and only if $(\cap_{i=1}^n B_i) \cap conv(S) = \emptyset$.
Equivalently, $p \in conv(S)$ if and only if $(\cap_{i=1}^n \overline{B}_i) \cap conv(S) = \emptyset$.

A Case with No Witness: $p \in conv(S)$

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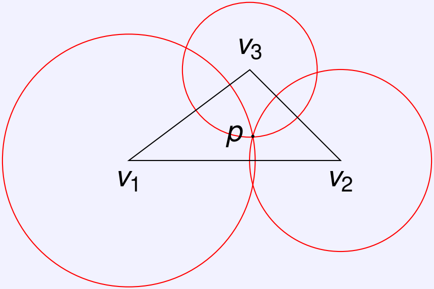

Figure: No witnesses: $p \in conv(S)$. The three discs intersect only at p.

Some Cases with Witnesses: $p \notin conv(S)$

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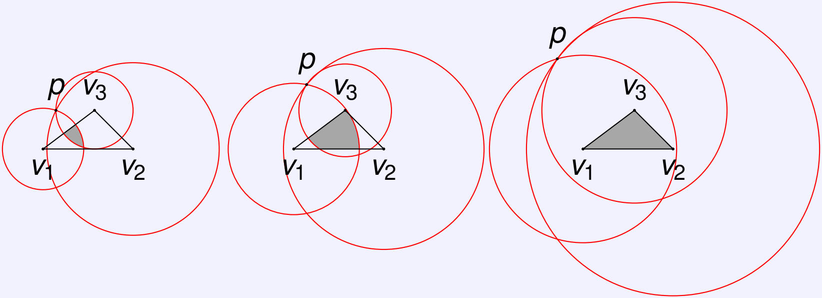

Figure: Examples with $W_p \neq \emptyset$, $p \notin conv(S)$. W_p is interior of gray areas: For any $p' \in W_p$ the bisector of pp' separates p from $conv(S)$.

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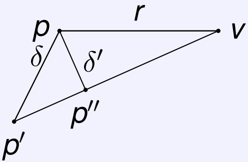

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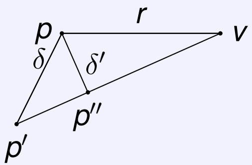

Theorem

(Strict Distance Duality) Assume $p \notin S$. Then $p \in conv(S)$ if and only if for each $p' \in conv(S)$ there exists a strict pivot.

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Assume p lies at the center of a ball of radius ρ in the relative interior of $conv(S)$, and Triangle Algorithm uses strict pivot in each iteration. The number of iterations to compute $p_\varepsilon \in conv(S)$ such that

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$$O\left(\left(\frac{R}{\rho}\right)^2 \log\frac{1}{\varepsilon}\right).$$

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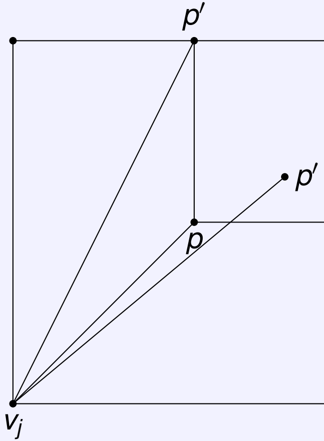
$$O\left(\frac{1}{c} \ln \frac{1}{\varepsilon}\right), \quad (1)$$

where c is the visibility factor, a constant satisfying the inequalities

$$\sin(pp'v') \leq \frac{1}{\sqrt{1+c}}, \quad c \geq \varepsilon^2, \quad (2)$$

over all the iterates p' *having corresponding pivot* v'*.*

Example



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Proposition

We have

$$\widehat{W}_p = \Big\{ x \in conv(S) : (x-p)^T(v_i - p) > 0, i = 1, \dots, n \Big\}.$$

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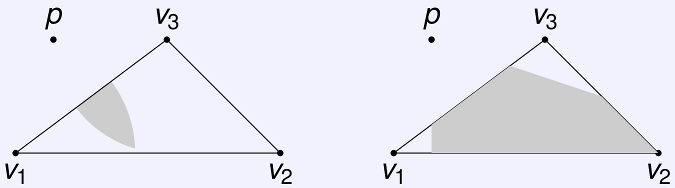

Figure: Witness set W_p (left) and Strict Witness set $\widehat{W}_p$ (right).

Solving Strict LP Feasibility Via Triangle Algorithm

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In other words, triangle algorithm gives complete answer when testing the feasibility of $Ax < b$, not just a yes answer.

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Numerical Experiments for Solving $Ax = b$

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In several experiments performed by DIMACS REU student, MS students, a Postdoc: generating different systems, including those from finite difference discretization, Incremental Triangle Algorithm has outperformed Jaobi, Gauss-Seidel, SOR, and AOR, taking much fewer iterations than these methods.

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In some examples triangle algorithm used only one iteration to compute solutions to absolute accuracy 10^{-10}. In particular, in an example (from Stanford) where the dimension of A was approximately $300,000$. (Rutgers MS thesis of Hao Shen (2014-2015) includes details.)

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Separation of Convex Sets

Definition

Given two compact convex subsets K, K' of $\mathbb{R}^m$, we say $H = \{x : h^T x = a\}$ is a separating hyperplane if

$$h^T x < a, \quad \forall x \in K, \quad h^T x < a, \quad \forall x \in K'.$$

Definition

$$\delta_* = d(K, K') = \min\{d(p, p') : p \in K, p' \in K'\} = d(p_*, p'_*).$$

Fact

Then $\delta_* = 0$ if and only if $K \cap K' \neq \emptyset$.

Four Problems Associated to A Pair of Convex Sets

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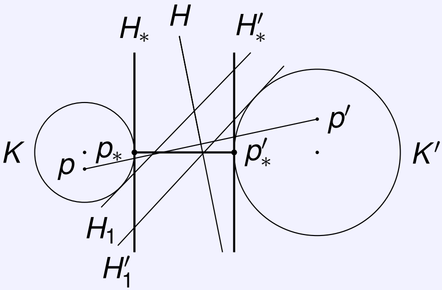

Figure: (p_*, p'_*) optimal pair, (H_*, H'_*) optimal support; (p, p') a pair whose orthogonal bisector separator H; (H_1, H'_1) a supporting pair.

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Definition

Given $(p, p') \in K \times K'$, we say it is a *witness pair* if the orthogonal bisecting hyperplane of the line segment pp' separates K and K'.

Triangle Algorithm I (Testing if K and K' intersect)

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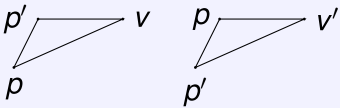

Figure: v is p'-pivot for p (left); v' is p-pivot for p' (right).

Voronoi Diagrams

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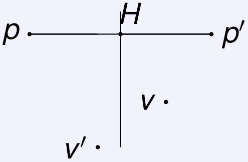

Figure: In the Figure, the point v and v' are pivots for p' and p, respectively.

A New Separating Hyperplane Theorem

A New Separating Hyperplane Theorem

Theorem

(Krein-Milman) *Let K be a compact convex subset of $\mathbb{R}^m$. Then K is the convex hull of its extreme points. In notation, $K = conv(\mathrm{ex}(K))$.*

A New Separating Hyperplane Theorem

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Theorem

(Distance Duality) Let K, K' *be compact convex subsets in* $\mathbb{R}^m$*, with* $\mathrm{ex}(K)$ *and* $\mathrm{ex}(K')$ *as their corresponding set of extreme points. Let* S *be a subset of* K *containing* $\mathrm{ex}(K)$*, and* S' *a subset of* K' *containing* $\mathrm{ex}(K')$*. Then,* $K \cap K' \neq \emptyset$ *if and only if for each* $(p, p') \in K \times K'$*, either there exists* $v \in S$ *such that* $d(p', v) \geq d(p, v)$*, or there exists* $v' \in S'$ *such that* $d(p, v') \geq d(p', v')$*.*

A New Separating Hyperplane Theorem

Theorem

(Distance Duality) Let K, K' be compact convex subsets in $\mathbb{R}^m$, with $\mathrm{ex}(K)$ and $\mathrm{ex}(K')$ as their corresponding set of extreme points. Let S be a subset of K containing $\mathrm{ex}(K)$, and S' a subset of K' containing $\mathrm{ex}(K')$. Then, $K \cap K' \neq \emptyset$ if and only if for each $(p, p') \in K \times K'$, either there exists $v \in S$ such that $d(p', v) \geq d(p, v)$, or there exists $v' \in S'$ such that $d(p, v') \geq d(p', v')$.

An alternative description of the Distance Duality is the following.

Theorem

(Distance Duality) Let K, K' be compact convex subsets in $\mathbb{R}^m$, with $\mathrm{ex}(K)$ and $\mathrm{ex}(K')$ as their corresponding set of extreme points. Then, $K \cap K' = \emptyset$ if and only if there exists $(p, p') \in K \times K'$ such that $d(p, v) < d(p', v)$ for all $v \in \mathrm{ex}(K)$ and $d(p', v') < d(p, v')$ for all $v' \in \mathrm{ex}(K')$. (Such pair is necessarily a witness pair)

Iterative Step in Triangle Algorithm I

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Each iteration of Triangle Algorithm I computes for given pair $(p, p') \in K \times K'$, either $v \in K$ that is a p'-pivot for p; or $v' \in K'$, a p-pivot for p'.

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Thus the worst-case complexity in each iteration is

$$T = \max\{T_K, T_{K'}\}.$$

Triangle Algorithm I

Triangle Algorithm I ($(p_0, p'_0) \in K \times K'$, $\varepsilon \in (0, 1)$)

- **Step 0.** Set $p = v = p_0$, $p' = v' = p'_0$.
- **Step 1.** If $d(p, p') \le \varepsilon d(p, v)$, or $d(p, p') \le \varepsilon d(p', v')$, stop.
- **Step 2.** Test if there exists $v \in K$ that is a p'-pivot for p, i.e.

  $$2v^T(p' - p) \ge \|p'\|^2 - \|p\|^2$$

  (e.g. by solving $\max\{(p' - p)^T v : v \in K\}$). If such pivot exists, set $p \leftarrow nearest(p'; pv)$, and go to Step 1.
- **Step 3.** Test if there exists $v' \in K'$ that is a p-pivot for p', i.e.

  $$2v'^T(p - p') \ge \|p\|^2 - \|p'\|^2.$$

  (e.g. by solving $\max\{(p - p')^T v' : v' \in K'\}$). If such pivot exists, set $p' \leftarrow nearest(p; p'v')$, and go to Step 1.
- **Step 4.** Output (p, p') as a witness pair, stop ($K \cap K' = \emptyset$).

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When $\delta_* > 0$, the number of iterations of Triangle Algorithm I to compute a witness pair $(p, p') \in K \times K'$ is

$$O\left(\left(\frac{\rho_*}{\delta_*}\right)^2\right),$$

$$\rho_* = \max\{\Delta_0, \Delta_0'\}, \quad \Delta_0 = \mathrm{diam}(K), \quad \Delta_0' = \mathrm{diam}(K').$$

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Suppose $\delta_* > 0$. We say a pair of hyperplanes (H, H') supports (K, K'), if they are parallel, H supports K and and H' supports K'.

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Suppose $\delta_* > 0$. We say a witness pair $(p, p') \in K \times K'$ gives an ε-approximate supporting hyperplane, if it is an ε-approximate solution and there exists a pair or supporting hyperplane (H, H'), parallel to the orthogonal bisecting hyperplane of (p, p'), satisfying

$$\delta_* - d(H, H') \leq \varepsilon d(p, v), \quad \text{for some} \quad v \in K,$$

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Triangle Algorithm II (Start With a Witness Pair)

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Given a witness pair $(p, p') \in K \times K'$, it computes an ε-approximate solution, i.e. such that $d(p, p')$ approximates $\delta_* = d(K, K')$.

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However, if $d(p, p')$ does not sufficiently approximate δ_*, we will make use of *weak-pivot*, to defined.

Algorithm for Approximation of Distance

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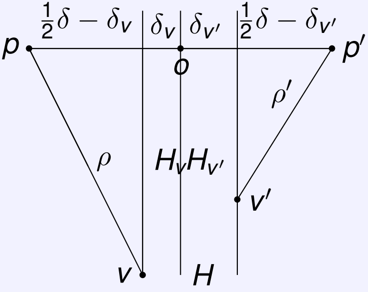

Figure: Depiction of the orthogonal bisector hyperplane H to pp' and parallel supporting hyperplanes H_v and $H_{v'}$ that separate K and K'.

$$\delta_v + \delta_{v'} = d(H_v, H_{v'}) < \delta_* < d(p, p').$$

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Theorem

Suppose $(p, p') \in K \times K'$ *is a witness pair. Let the orthogonal bisecting hyperplane to the line* pp' *be* $H = \{x : h^T x = (p - p')^T x = a\}$*. Let*
$v = \mathrm{argmin}\{h^T x : x \in K\}, \quad v' = \mathrm{argmax}\{h^T x : x \in K'\},$

$H_v = \{x : h^T x = h^T v\}, \quad H_{v'} = \{x : h^T x = h^T v'\}.$

Then H_v *and* $H_{v'}$ *are supporting hyperplane to* K *and* K'*, respectively.*

Also, if $\delta_v = d(v, H), \quad \delta_{v'} = d(v', H), \quad \underline{\delta} = \delta_v + \delta_{v'}$*, we have*

$$d(H_v, H_{v'}) = \underline{\delta} = \frac{h^T v - h^T v'}{\|h\|},$$

$$\underline{\delta} \leq \delta_* \leq \delta = d(p, p').$$

Triangle Algorithm II

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Definition

Given a witness pair $(p, p') \in K \times K'$, let H be the orthogonal bisecting hyperplane of pp'. We shall say $v \in K$ is a *weak p'-pivot* for p if

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Similarly, we shall say $v' \in K'$ is a *weak p-pivot* for p' if

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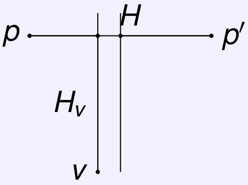

Triangle Algorithm II

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Theorem

Let

$$\Delta_0 = \mathrm{diam}(K), \quad \Delta'_0 = \mathrm{diam}(K'),$$

$$\rho_* = \max\{\Delta_0, \Delta'_0\}.$$

The total arithmetic complexity of Triangle Algorithm II is

$$O\left(T\left(\frac{\rho_*}{\delta_* \varepsilon}\right)^2 \ln \frac{\rho_*}{\delta_*}\right).$$

In particular, when K or K' is a singleton we have

$$O\left(T\left(\frac{\rho_*}{\delta_* \varepsilon}\right)^2\right).$$

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Thus, the algorithm requires searching for a weak-pivot or a pivot to reduce the gap $\delta_k = d(p_k, p'_k)$ until the desired approximation is attained.

Summary of Triangle Algorithms I and II

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Let T be the worst-case complexity of computing a pivot for a point in K, or K'. The total number of arithmetic operations in Triangle Algorithm I to get an ε-approximate solution when $\delta_* = 0$, or a witness pair is

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Special Applications and Extensions

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- When $K = conv(V)$, $V = \{v_1, \ldots, v_n\}$, $K' = conv(V')$, $V' = \{v'_1, \ldots, v'_{n'}\}$). In particular, when one set is a single point. This includes applications such as SVM. In this case

$$T = O(m(n+n')), \quad \text{with preprocessing} \quad T = O(m+\max\{n,n'\}).$$

CS Masters Thesis, Mayank Gupta, 2015-1016, extensive computation and comparison with *sequential minimal optimization* (SMO). The results are very good! Article to be released in near future.

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- Applications in non-convex optimization.
- Applications in combinatorial and graph problems.
- Applications in conic programming.

Approximation of An NP-Complete Problem

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Let $Z = \{x : x^T Q x = 0 : x^T x \leq 1\}$.

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This problem is NP-complete.

Let $Z = \{x : x^T Qx = 0 : x^T x \leq 1\}$.

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In particular, in the later case when Z is convex, the algorithm gives an approximate zero of Q in S.

Related Articles and Forthcoming Work

- A characterization theorem and an algorithm for a convex hull problem, *Annals of operations Research*, Volume 226, Issue 1, pp 301-349, 2014.
- “Finding a Lost Treasure in Convex Hull of Points From Known Distances", Proceedings of the 24th Canadian Conference on Computational Geometry (2012).
- “Three Convex Hull Theorems On Triangles and Circles,” with Jong Youll Park, Honam Mathematical Journal (Korean Journal, in English), 2014.
- “An Algorithmic Separating Hyperplane Theorem and Its Application,” 2014, arxiv.org/pdf/1412.0356v1.pdf.
- Solving Linear System of Equations Via A Convex Hull Algorithm, http://arxiv.org/abs/1210.7858
- On the Triangle Algorithm for Convex Hull Membership, 23rd Fall Workshop on Computational Geometry, City College of New York, Oct 25, 2013, with Michael Saks (2-page abstract).
- Experiments with the Triangle Algorithm for Linear Systems, 23rd Fall Workshop on Computational Geometry, City College of New York, Oct 25, 2013, with Thomas Gibson (2-page abstract).
- Experimental Study of the Convex Hull Decision Problem via a New Geometric Algorithm, 23rd Fall Workshop on Computational Geometry, City College of New York, Oct 25, 2013, with Meng Li. (2-page abstract).
- “Randomized triangle algorithms for convex hull membership, 2-page Extended Abstract in 24nd Annual Fall Workshop on Computational Geometry, Connecticut, 2014.
- A Geometric Polynomial-Time Algorithm for Bipartite Perfect Matching Problem, forthcoming.
- An Approximation to an NP-Complete Problem via The Triangle Algorithm, forthcoming.

- Finally, there remain many open problems.