Generic global rigidity of graphs

Tibor Jordán

Department of Operations Research and the Egerváry Research Group on Combinatorial Optimization, Eötvös University, Budapest

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Bar-and-joint frameworks

A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

Bar-and-joint frameworks II.

We say that (G, p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework which is equivalent to (G, p) is congruent to (G, p).

The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p).

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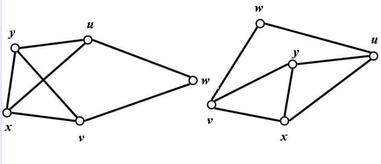
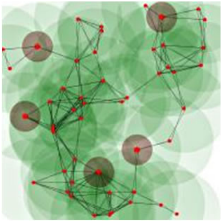
A planar framework



A rigid but not globally rigid two-dimensional framework.

Global rigidity: applications



A subset of pairwise distances may be enough to uniquely determine the configuration and hence the location of each sensor (provided we have some anchor nodes whose location is known).

Bar-and-joint frameworks: generic realizations

Testing rigidity is NP-hard for $d \geq 2$ (T.G. Abbot, 2008). Testing global rigidity is NP-hard for $d \geq 1$ (J.B. Saxe, 1979).

The framework is *generic* if there are no algebraic dependencies between the coordinates of the vertices.

The rigidity (resp. global rigidity) of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity (resp. global rigidity) of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. (Asimow and B. Roth 1979; R. Connelly 2005, S. Gortler, A. Healy and D. Thurston (2010).) We say that the graph G is *rigid* (*globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid.

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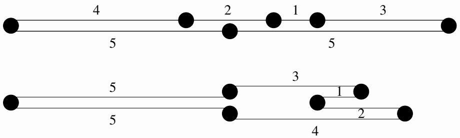
Combinatorial (global) rigidity

- Characterize the rigid graphs in $\mathbb{R}^d$,
- Characterize the globally rigid graphs in $\mathbb{R}^d$,
- Find an efficient deterministic algorithm for testing these properties,
- Obtain further structural results (maximal rigid subgraphs, maximal globally rigid clusters, globally linked pairs of vertices, etc.)
- Solve the related optimization problems (e.g. make the graph rigid or globally rigid by pinning a smallest vertex set or adding a smallest edge set)

Frameworks on the line

Lemma

A one-dimensional framework (G, p) is rigid if and only if G is connected.



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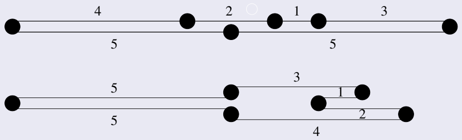

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Matrices and matroids

The *rigidity matrix* of framework (G,p) is a matrix of size $|E| \times d|V|$ in which the row corresponding to edge uv contains $p(u)-p(v)$ in the d-tuple of columns of u, $p(v)-p(u)$ in the d-tuple of columns of v, and the remaining entries are zeros. For example, the graph G with $V(G)=\{u,v,x,y\}$ and $E(G)=\{uv,vx,ux,xy\}$ has the following rigidity matrix:

$$\begin{array}{cc} & \begin{array}{cccc} \quad u \quad & \quad v \quad & \quad x \quad & \quad y \quad \end{array} \\ \begin{array}{c} uv \\ vx \\ ux \\ xy \end{array} & \begin{pmatrix} p(u)-p(v) & p(v)-p(u) & 0 & 0 \\ 0 & p(v)-p(x) & p(x)-p(v) & 0 \\ p(u)-p(x) & 0 & p(x)-p(u) & 0 \\ 0 & 0 & p(x)-p(y) & p(y)-p(x) \end{pmatrix} \end{array}.$$

Graph G is rigid if and only if the generic rank of its rigidity matrix equals $d|V|-\binom{d+1}{2}$.

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Equilibrium stresses

The function $\omega : e \in E \mapsto \omega_e \in \mathbb{R}$ is an *equilibrium stress* on framework (G, p) if for each vertex u we have

$$\sum_{v \in N(u)} \omega_{uv}(p(v) - p(u)) = 0. \qquad (1)$$

The *stress matrix* Ω of ω is a symmetric matrix of size $|V| \times |V|$ in which all row (and column) sums are zero and

$$\Omega[u, v] = -\omega_{uv}. \qquad (2)$$

The generic framework (G, p) is globally rigid in $\mathbb{R}^d$ if and only if there exists an equilibrium stress whose stress matrix has rank $|V| - (d + 1)$.

Globally rigid graphs - necessary conditions

We say that G is *redundantly rigid in* $\mathbb{R}^d$ if removing any edge of G results in a rigid graph.

Theorem (B. Hendrickson, 1992)

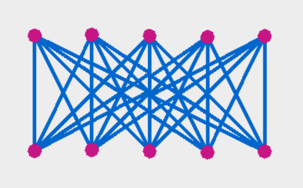
Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G is
(i) $(d+1)$-connected, and
(ii) redundantly rigid in $\mathbb{R}^d$.

H-graphs

We say that a graph G is an *H-graph* in $\mathbb{R}^d$ if it satisfies Hendrickson's necessary conditions in $\mathbb{R}^d$ (i.e. $(d+1)$-vertex-connectivity and redundant rigidity) but it is not globally rigid in $\mathbb{R}^d$.

Theorem (B. Connelly, 1991)

The complete bipartite graph $K_{5,5}$ is an H-graph in $\mathbb{R}^3$. Furthermore, there exist H-graphs for all $d \geq 4$ as well (complete bipartite graphs on $\binom{d+2}{2}$ vertices).



H-graphs

Theorem (S. Frank and J. Jiang, 2011)

There exist two more (bipartite) H-graphs in $\mathbb{R}^4$ and infinite families of H-graphs in $\mathbb{R}^d$ for $d \geq 5$.

Theorem (T.J, C. Király, and S. Tanigawa, 2016)

There exist infinitely many H-graphs in $\mathbb{R}^d$ for all $d \geq 3$.

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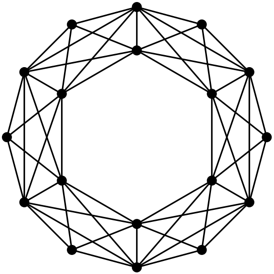
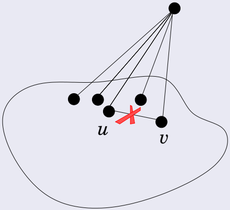
A non-bipartite H-graph in $\mathbb{R}^3$.



Global rigidity - sufficient conditions



The d-dimensional *extension* operation.

Theorem (B. Connelly, 1989, 2005)

Suppose that G can be obtained from K_{d+2} by extensions and edge additions. Then G is globally rigid in $\mathbb{R}^d$.

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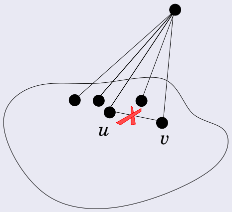

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Suppose that G can be obtained from K_{d+2} by extensions and edge additions. Then G is globally rigid in $\mathbb{R}^d$.

Global rigidity on the line and in the plane

Lemma

Graph G is globally rigid in $\mathbb{R}^1$ if and only if G is a complete graph on at most two vertices or G is 2-connected.

Theorem (B. Jackson, T. J., 2005)

Let G be a 3-connected and redundantly rigid graph in $\mathbb{R}^2$ on at least four vertices. Then G can be obtained from K_4 by extensions and edge-additions.

Theorem (B. Jackson, T. J., 2005)

Graph G is globally rigid in $\mathbb{R}^2$ if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.

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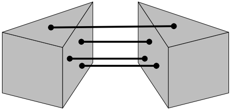
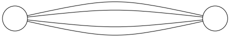
Body-bar frameworks

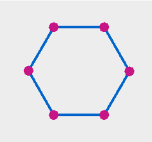


A d-dimensional *body-bar framework* is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by bars. The bars are pairwise disjoint. Two bodies may be connected by several bars. In the underlying multigraph of the framework the vertices correspond to the bodies and the edges correspond to the bars.



Tree-connectivity.

Let $M = (V, E)$ be a multigraph. We say that M is *k-tree-connected* if M contains k edge-disjoint spanning trees. If M contains k edge-disjoint spanning trees for all $e \in E$ then M is called *highly k-tree-connected*.

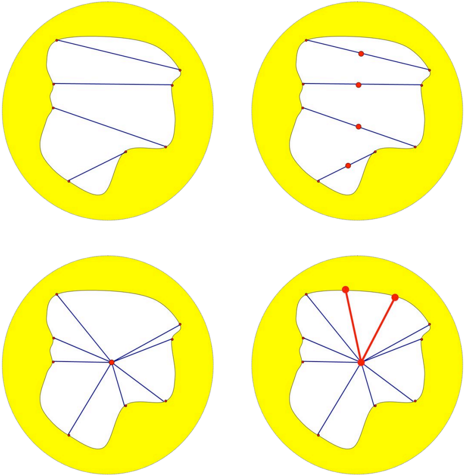


Rigid body-bar frameworks

Theorem (T-S. Tay, 1989, W. Whiteley, 1988)

A generic body-bar framework with underlying multigraph $H = (V, E)$ is rigid in $\mathbb{R}^d$ if and only if H is $\binom{d+1}{2}$-tree-connected.

Pinching edges



The pinching operation ($m = 6, k = 4$).

Another constructive characterization

Theorem (A. Frank, L. Szegő 2003)

A multigraph H is highly m-tree-connected if and only if H can be obtained from a vertex by repeated applications of the following operations:

- (i) adding an edge (possibly a loop),
- (ii) pinching k edges $(1 \leq k \leq m-1)$ with a new vertex z and adding $m-k$ new edges connecting z with existing vertices.

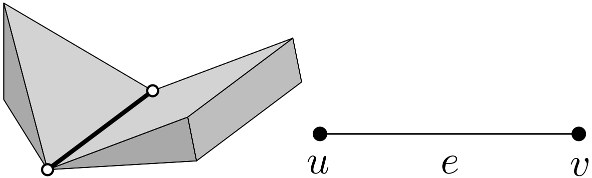
Globally rigid body-bar graphs

Theorem (B. Connelly, T.J., W. Whiteley, 2013)

A generic body-bar framework with underlying multigraph $H = (V, E)$ is globally rigid in $\mathbb{R}^d$ if and only if H is highly $\binom{d+1}{2}$-tree connected.

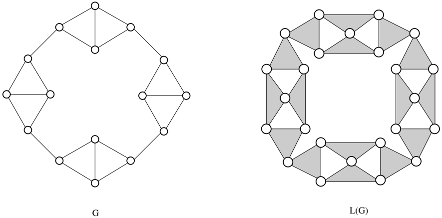
Body-hinge frameworks

A d-dimensional *body-hinge framework* is a structure consisting of rigid bodies in d-space in which some pairs of bodies are connected by a hinge, restricting the relative position of the corresponding bodies. Each hinge corresponds to a $(d-2)$-dimensional affine subspace. In the underlying multigraph of the framework the vertices correspond to the bodies and the edges correspond to the hinges.



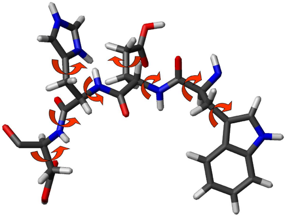
Bodies connected by a hinge in 3-space and the corresponding edge of the underlying multigraph.

Body-hinge frameworks II.



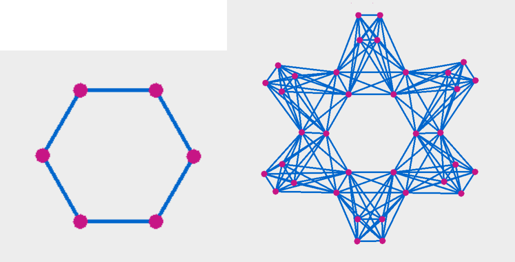
A 2-dimensional body-hinge structure (right) and its multigraph (left).

Body-hinge frameworks: applications



Fixed distances and angles give rise to a body-hinge structure in 3-space, with concurrent hinges at each body.

Bar-and-joint models of body-hinge structures



The 3-dimensional body-hinge graph induced by a six-cycle.

Rigid body-hinge frameworks

For a multigraph H and integer k we use kH to denote the multigraph obtained from H by replacing each edge e of H by k parallel copies of e.

Theorem (T-S. Tay, 1989, W. Whiteley, 1988)

A generic body-hinge framework with underlying multigraph $H = (V, E)$ is rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H$ is $\binom{d+1}{2}$-tree-connected.

Globally rigid body-hinge graphs in $\mathbb{R}^d$

Theorem (T.J., C. Király, S. Tanigawa, 2014)

Let $H = (V, E)$ be a multigraph and $d \geq 3$. Then the body-hinge graph G_H is globally rigid in $\mathbb{R}^d$ if and only if $(\binom{d+1}{2} - 1)H$ is highly $\binom{d+1}{2}$-tree-connected.

Sufficiency was conjectured in (B. Connelly, T.J., W. Whiteley, 2013).

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Sufficient conditions - rigidity

We say that G is *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V(G)$.

Theorem (S. Tanigawa, 2013)

If G is vertex-redundantly rigid in $\mathbb{R}^d$ then it is globally rigid in $\mathbb{R}^d$.

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If G is rigid in $\mathbb{R}^{d+1}$ then it is globally rigid in $\mathbb{R}^d$.

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Every rigid graph in $\mathbb{R}^d$ on $|V|$ vertices can be made globally rigid in $\mathbb{R}^d$ by adding at most $|V| - d - 1$ edges. This bound is tight for all $d \geq 1$.

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Sufficient conditions - connectivity

Theorem (B. Jackson, T.J., 2005)

If a graph G is 6-vertex-connected, then G is globally rigid in $\mathbb{R}^2$.

Conjecture (L. Lovász and Y. Yemini, 1982, B. Connelly, T.J., W. Whiteley, 2013)

Sufficiently highly vertex-connected graphs are rigid (globally rigid) in $\mathbb{R}^d$.

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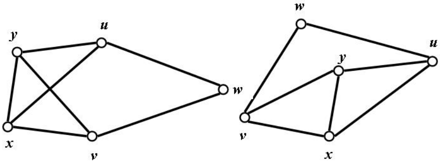
Globally linked pairs

We say that a pair $\{u, v\}$ of vertices of G is *globally linked* in G in $\mathbb{R}^d$ if for all generic d-dimensional realizations (G, p) we have that the distance between $q(u)$ and $q(v)$ is the same in all realizations (G, q) equivalent with (G, p).



Globally linked pairs in minimally rigid graphs

Theorem (B. Jackson, T.J., Z. Szabadka (2014)

Let $G = (V, E)$ be a minimally rigid graph in $\mathbb{R}^2$ and $u, v \in V$. Then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$ if and only if $uv \in E$.

Molecular graphs

The *square* G^2 of graph G is obtained from G by adding the edges uv for all non-adjacent vertex pairs u, v with a common neighbour in G.

Conjecture

Let G be a graph with no cycles of length at most four. Then G^2 is globally rigid in $\mathbb{R}^3$ if and only if G^2 is 4-connected and the multigraph $5G$ is highly 6-tree connected.

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Thank you.