Universal Linkage and the Uniqueness of EDM Completions

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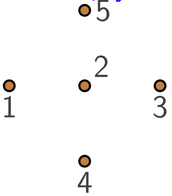
Introduction

Every configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^n$ defines EDM $D = (d_{ij} = ||p^i - p^j||^2)$. For example,



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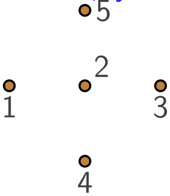

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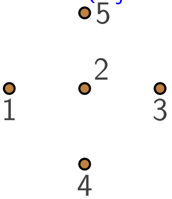

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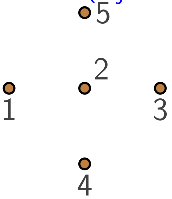

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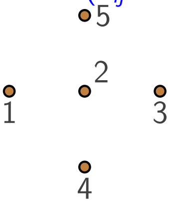

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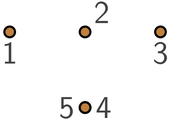

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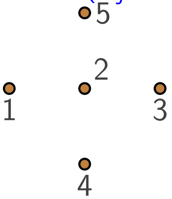

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$$D = \begin{bmatrix} 0 & 1 & 4 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 0 & x \\ 2 & 1 & 2 & x & 0 \end{bmatrix}.$$

for all $0 \leq x \leq 4$.

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- CCS is a spectrahedron, i.e., intersection of psd cone with an affine space.

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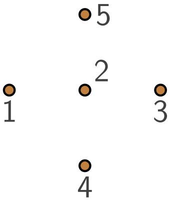
- emb dim of D for $d_{45} = 0$ or 4 is 2, while it is 3 for $0 < d_{45} < 4$.

Bar-and-Joint Frameworks



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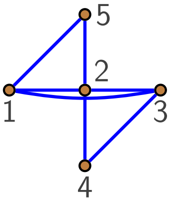

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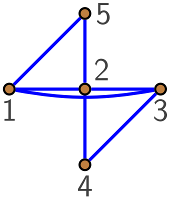

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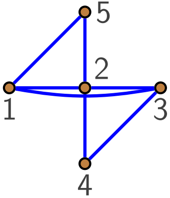

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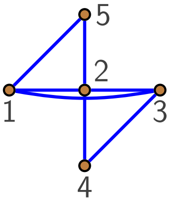

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- $\{k, l\}$ is universally linked iff its CCS is contained in the hyperplane $y_{kl} = 0$ in $\mathbb{R}^{\bar{m}}$, $\bar{m} =$ num. of missing edges of G.

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- (G, p) has an affine motion if $\exists$ (G, q): (i) $H \circ D_p = H \circ D_q$, (ii) $D_p \neq D_q$ and (iii) $q^i = Ap^i + b$ for $i = 1, \ldots, n$.

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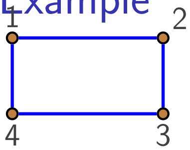
- Theorem [A 2005] (G, p) is universally rigid iff it is both dimensionally rigid and has no affine motions.

Example



$$D = \begin{bmatrix} 0 & 4 & 5 + y_{13} & 1 \\ 4 & 0 & 1 & 5 + y_{24} \\ 5 + y_{13} & 1 & 0 & 4 \\ 1 & 5 + y_{24} & 4 & 0 \end{bmatrix}$$

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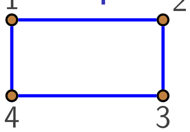

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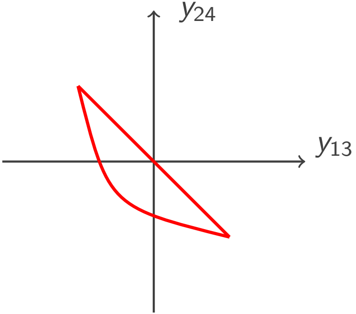

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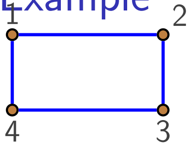

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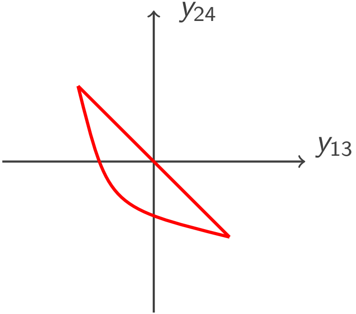

Obviously (G, p) is not dimensionally rigid. It has an affine motion, and neither $\{1, 3\}$ nor $\{2, 4\}$ is universally linked.

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- If (G, p) is r-dimensional, then rank $\Omega \leq n - 1 - r$.
- Ω is optimal dual variable in a certain Semidefinite programming problem.

- Theorem[A. '05, Connelly '82]: Let Ω be a stress matrix of r-dimensional framework (G, p), $r \leq n-2$. If Ω is psd and of rank $n-r-1$, then (G, p) is dimensionally rigid.

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- Theorem[A and Nguyen '13]: Let Ω be a stress matrix of r-dimensional framework (G, p). $r \leq n-2$. If rank $\Omega = n-r-1$ and if for each vertex i, the set $\{p^i\} \cup \{p^j : \{i, j\} \in E(G)\}$ is in general position, then (G, p) has no affine motion.

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- e is the vector of all 1s.
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$$\mathcal{T}(D) = -\frac{1}{2}(I - \frac{ee^T}{n})D(I - \frac{ee^T}{n}) \succeq 0.$$

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Moreover, the embedding dimension of D is equal to rank $\mathcal{T}(D)$.
- $B = \mathcal{T}(D)$ is the Gram matrix of the generating points of D.
- B is not invariant under translations. Thus impose $Be = 0$.

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- Let V be $n \times (n-1)$ matrix: $V^T e = 0$ and $V^T V = I$.
- Let $X = V^T B V = -VDV^T/2$ or $B = VXV^T$. Thus X is called the projected Gram matrix of D.

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- Let $X = V^T B V = -VDV^T/2$ or $B = VXV^T$. Thus X is called the projected Gram matrix of D.
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- Thus there is a one-to-one correspondence between $n \times n$ EDMs D and psd matrices of order $n-1$.
- The CCS of (G, p) is given by

$$\{y = (y_{ij}) : X + \sum_{ij:\{i,j\} \notin E(G)} y_{ij} M^{ij} \succeq 0\},$$

where X is the projected Gram matrix of (G, p) and M^{ij}s are universal matrices.

Facial Structure of CCS

- Let $\mathcal{X}(y) = X + \sum_{ij:\{i,j\}\notin E(G)} y_{ij} M^{ij}$. Thus CCS is given by

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- Theorem: Let U be the matrix whose columns form an orthonormal basis of null $(\mathcal{X}(y))$. Let Ω be a non-zero psd stress matrix of (G, p). Then

$$\begin{aligned} \text{minface}(y) &= \{x \in \mathcal{F} : \text{null}(\mathcal{X}(y)) \subseteq \text{null}(\mathcal{X}(x))\} \\ \text{relint(minface)(y)} &= \{x \in \mathcal{F} : \text{null}(\mathcal{X}(y)) = \text{null}(\mathcal{X}(x))\} \\ \text{aff(minface)(y)} &= \{x \in \mathbb{R}^{\bar{m}} : \mathcal{X}(x)U = 0\} \\ \Omega V \mathcal{X}(x) V^T &= 0 \text{ for all } x \in \mathcal{F}. \end{aligned}$$

Strong Arnold Property (SAP)

- Given graph G, let A be an $n \times n$ symmetric matrix A such that $A_{ij} = 0$ for all $\{i, j\} \in E(\overline{G})$. Then A satisfies SAP if $Y = 0$ is the only symmetric matrix satisfying: (i) $Y_{ij} = 0$ if $i = j$ or $\{i, j\} \in E(G)$ and (ii) $AY = 0$.

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- Thus our sufficient condition for universal rigidity is equivalent to the assertion that stress matrix Ω satisfies SAP.

Transversal Intersections

- Given graph G, let rank $\Omega = k$ and let $\mathcal{S}_k = \{A \text{ is symm} : \text{rank } A = k\}$. Further, let $\mathcal{T}_\Omega$ be the tangent space to $\mathcal{S}_k$ at Ω.

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SDP Non-degeneracy (Alizadeh et al '97)

- Consider the pair of dual SDPs:

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\begin{array}{lll}
\text{(P)} & \max_y & 0^T y \\
 & \text{subject to} & \mathcal{X}(y) = X + \sum_{ij:} y_{ij} M^{ij} \succeq 0 \\
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\text{(D)} & \min_Y & \operatorname{trace}(XY) \\
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- Theorem[Alizadeh et al '97]: If (D) has an optimal non-degenerate Y, then y in (P) is unique.

Thank You