# Distance Problems on Points and Lines

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## 1 Abstract

A number of high dimensional problems can be solved or approximated by reducing them to lower dimensional problems. For example, approximating a set of labeled data points in $\mathcal{R}^d$, for some integer $d \geq 3$, by a minimum complexity poly-line such that each point is no more than a specified distance from the poly-line can be solved by reduction to lower dimensional problems. We will discuss a number of optimal distance problems involving points and lines in the $\mathcal{R}^2$ plane. For some of the problems previous results are known (and we just summarize them), for some of the problems we propose results that are either the first known or improve over previously best known results, and we state some other problems as open problems.

## 2 Problems

**Problem P1**: Preprocess a set $S$ of $n$ labeled points $\{p_1, p_2, \ldots, p_n\}$ to answer the following queries: given a pair $(p_i, l_i)$, where $p_i \in S$ and $l_i$ is a line through $p_i$, find the farthest point $p_k$ from $p_i$ such that $p_k \in S$, $k < i$ (or $k > i$), and $p_k$ is to the left (or right) of the line $l_i$, assuming a direction is assigned to $l_i$. The problem appears in polygonal chain approximation under $L_2$ distance metric. To see this, consider a line $l(p_i, p_j)$ that goes through $p_i$ and $p_j$, $i < j$, and stabs all $\epsilon$-radius disks with centers at points $p_k$, $i < k < j$. Then the pair $(p_i, p_j)$ corresponds to a valid approximating shortcut under infinite beam metric [1,2]. To decide if $(p_i, p_j)$ is valid under the more restrictive tolerance zone metric [1,2] we need to perform the following operation with respect to $p_i$ (and similarly for $p_j$): find the farthest point $p_k$ from $p_i$, $i < k < j$, such that $p_k$ and $p_j$ are on opposite sides of the line $l_i$ passing through $p_i$ and orthogonal on $l(p_i, p_j)$.
**Problem P1'**: Preprocess a set $S$ of $n$ points $\{p_1, p_2, \ldots, p_n\}$ to answer the following queries: given a pair $(p, l_p)$, where $p$ is a point and $l_p$ is a line through $p$, find the farthest point $p_j$ from $p$ such that $p_j \in S$ and $p_j$ is to the left (or right) of the line $l_p$.

The following two problems are related to the problems above:
**Problem P2**: Preprocess a planar convex polygon $C$ with $n$ vertices to answer the following queries: given a pair $(p, l_p)$, where $p$ is a point and $l_p$ is a line through $p$, find the farthest point $q$ from $p$ such that $q \in C$ and $q$ is to the left (or right) of the line $l_p$. Clearly, $q$ is a vertex of the polygon $C$.
**Problem P3**: Preprocess a convex polygon $C$ with $n$ vertices (or a set of $n$ points) to answer the following queries: given a point $p$, find the farthest vertex of $C$ from $p$.
**Problem P4**: *Farthest lines from points.* Given a set $S$ of $n$ points, for each point in $S$ find farthest line defined by two points in $S$. Note that the points defining the farthest line from a point may not be in $CH(S)$, the convex hull of $S$.

**Problem P4'**: *Closest lines from points.* Given a set $S$ of $n$ points, for each point in $S$ find closest line defined by two other points in $S$.

**Problem P4"**: *Closest and farthest line from point.* Given a set $S$ of $n$ points, and a point $p \notin S$, find the closest (or farthest) line to $p$ defined by two points in $S$.

**Problem P5**: *Closest and farthest points from lines.* Given a set $S$ of $n$ points, for each line $l$ defined by two points in $S$ find the closest (or farthest) point $p \in S$, $p \notin l$.

**Problem P5'**: *Closest and farthest point from line.* Given a set $S$ of $n$ points and a line $l$, find smallest (or largest) distance from a point $p \in S$ to the line.

**Problem P6**: Find the convex hull of the vertices of an arrangement of $n$ lines in the plane. Note that an $O(n^2 \log n)$ algorithm is possible by first computing the vertices of the arrangement and then their convex hull.

**Problem P7**: Find the minimum area triangle in a set $s$ of $n$ points when a vertex of the triangle is fixed. Note: same complexity as minimum area triangle problem?

**Problem P8**: Given a set $S$ of $n$ points, find the $k$-th smallest distance from a point $p \in S$ to the lines defined by two other points of $S$.

**Problem P8'**: Given a set $S$ of $n$ points and a line $l$ defined by two points of $S$ (or not), find the $k$-th smallest distance from $l$ to a point of $S$.

**Problem P9**: Given a set $S$ of $n$ (possible intersecting) line segments, and a set $P$ of $m$ points, for each line segment in $S$ find the closest point in $P$.

**Problem P9'**: Given a set $S$ of $n$ (possible intersecting) line segments, and a set $P$ of $m$ points, for each point in $P$ find the closest line segment in $S$.

# References


[1] H. Imai and M. Iri, "Polygonal approximations of a curve-Formulations and algorithms," *Computational Morphology*, pg. 71-86, 1988.

[2] D.Z. Chen and O. Daescu, "Space-efficient algorithms for approximating polygonal curves in two dimensional space," *Proc. of the 4th Annual International Computing and Combinatorics Conference*, pg. 55-64, 1998.