A survey on approximating graph spanners

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Unweighted undirected k-spanners Peleg and Ullman 1987

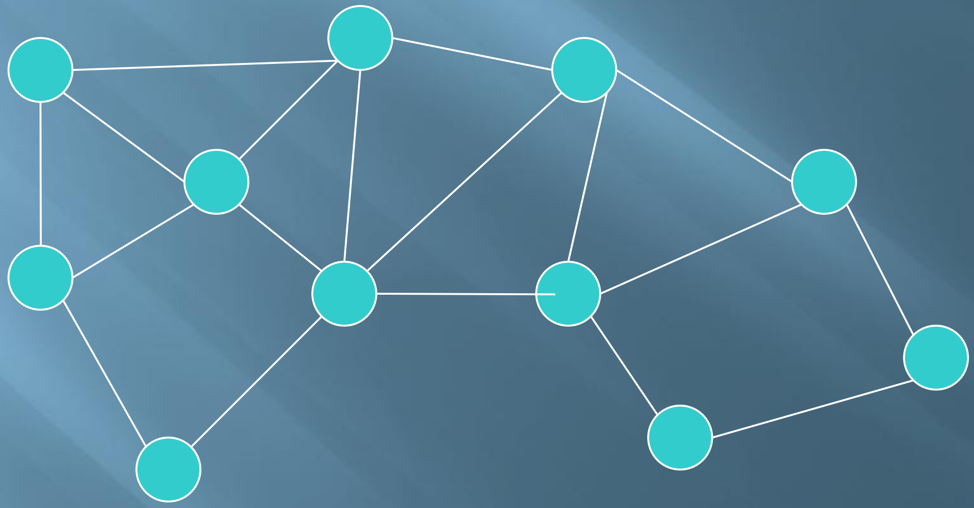
- Input: An undirected graph G(V,E) and an integer k
- **Required: a subgraph G' so that for every u and v ∈ V:**

$$Dist_{G'}(u,v)/Dist_{G}(u,v) \leq k$$

- DATA COMPRESSION

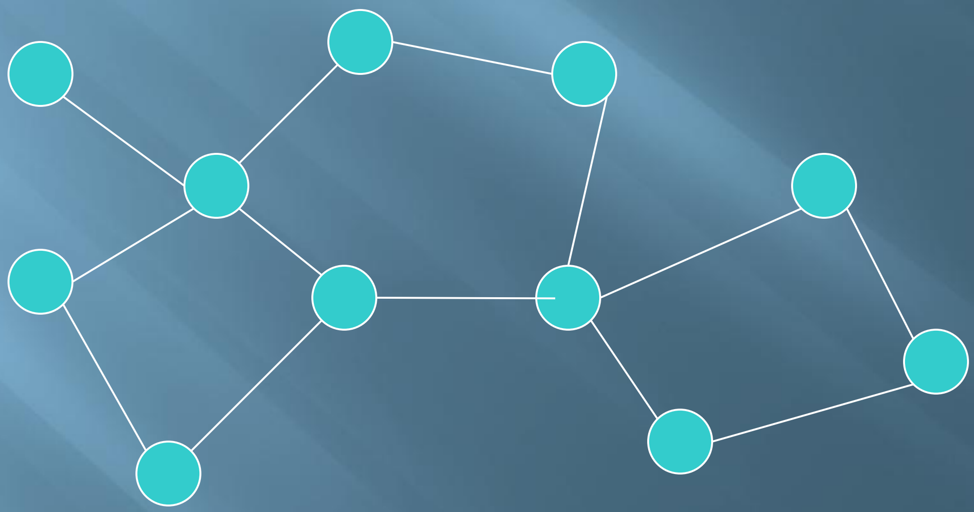
An example of a 2-spanner

- The original graph:

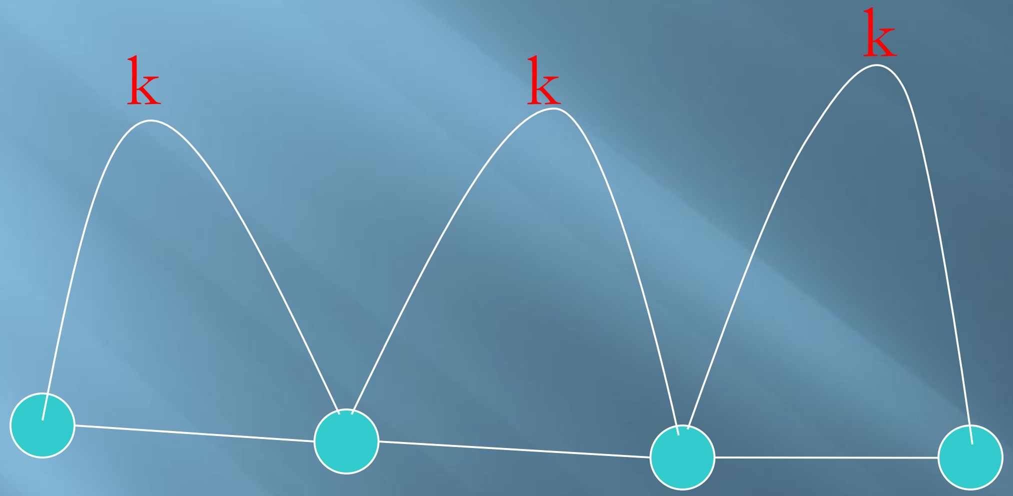


A 2-spanner

- Easy to check the new distance for every pair is at most twice the original distance.



Why dealing with edges is enough?



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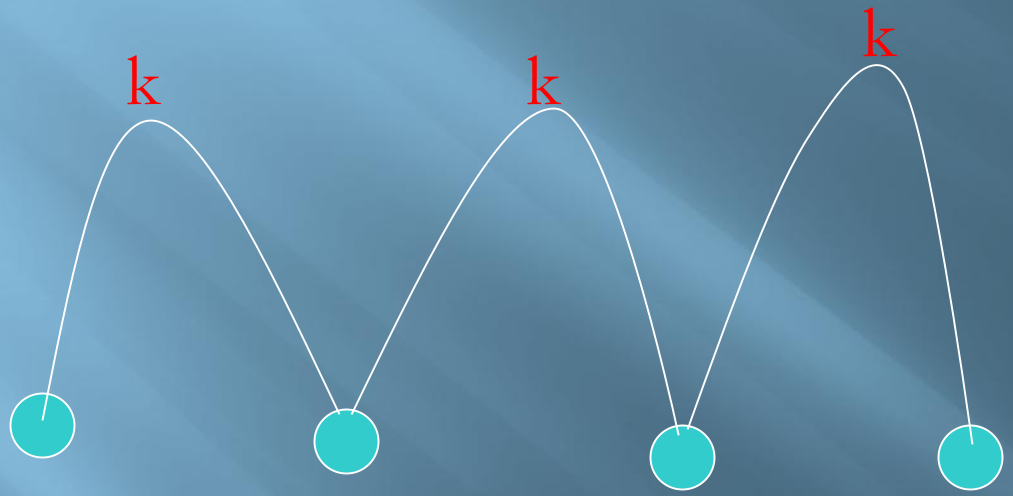


Distance 3 becomes 3k

An alternative definition

- Find a subgraph G'(V,E') so that for every edge e in E-E', adding e must close a cycle of size at most k+1.
- More general variants in which the above is not true.
- The case of general lengths over the edges.
- Then a k-spanner must be a k-spanner with respect to weighted distance.

Applications

- In geometry.
- Small routing tables: spanners have less edges. Thus smaller tables. But not much larger distance
- Synchronizers: make non synchronized distributed computation, synchronized.
- Parallel distributed and streaming algorithms.
- Distance oracles. Handle queries about distance between two vertices quick by preprocessing.
- Property testing
- Minimum time broadcast.

2-spanners

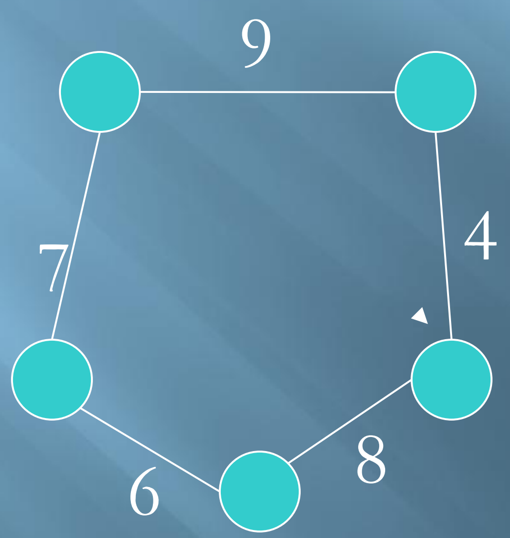
- There is a difficulty. Unlike $k \geq 3$ there are not necessarily 2 spanners with few edges.
- The only 2-spanners of a complete bipartite graph is the graph itself.
- Like in 2-SAT and 2-Coloring and other problems, 2-spanners is different than the rest.

For k at least 3 there are spanners with few edges

- As we shall see: 3-spanners with O(n*sqrt{n}) edges always exist, and the same goes for 4-spanners. And this is tight.
- The larger k is, the smaller is the upper bound on the number of edges in the best spanner.
- Remarkable fact: maximum number of edges in a graph with girth g not known.
- Maybe for 40 years the upper and lower bound are quite far!

Heaviest edge on a short cycle

For example a 4-spanner, only the edge 9 can be removed, while maintaining a 4-spanner



A generalization of the Kruskal algorithm:

- Sort the edges of the graph in increasing weights.

  $c(e_1) \leq c(e_2) \leq c(e_3) \leq \ldots\ldots\ldots\ldots \leq c(e_m)$
- Go over all edges from small cost to large.
- For the next edge e_i, if the edge does not close a cycle of length at most $k+1$ with previously added edges, add e_i to G' or else $i=i+1$
- This algorithm is due to I. Althofer, G. Das, D. Dobkin, D. Joseph, and J. Soares. 1993

The resulting graph is a k-spanner

- If an edge e is missing, then by construction, this edge is the most heavy edge in a cycle of length at most k+1.
- This is because we go over edges in non decreasing costs.
- If we reach a cycle of size k+1, then it means that previous edges were not removed.
- This implies that e is the largest edge in a cycle of length at most k+1 and it is safe to remove it.

Girth k+2

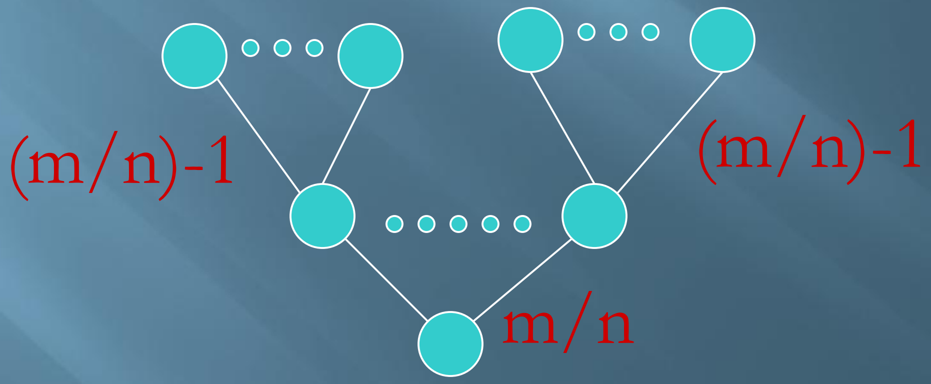
- We observe that the resulting graph has girth at least k+2
- The girth is the size of the minimum simple cycle.
- Observe that when we reach the largest edge e of a cycle with at most k+1 edges, this edge will be removed.
- Therefore, there are no k+1 size or smaller cycles.
- Graphs with large girth have "few" edges.

Example: graphs with girth 5 and 6

- We show that graphs with girth 5 and 6 have O(n*sqrt{n}) edges.
- First remove all vertices of degree strictly smaller than m/n.
- Here m is the numbers of edges and n is the number of vertices.
- Since we have removed at most n vertices and each vertex removes less than m/n edges it is clear that the resulting graph is not empty.

Two layers BFS graph

- All the vertices seen below are distinct as otherwise there is a cycle of length at most 4.

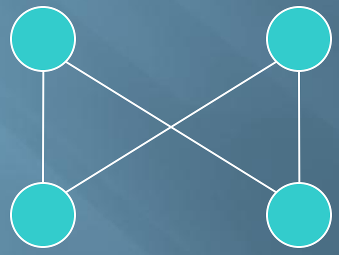


Number of edges

- This implies that $m/n(m/n-1) \leq n$, or $m^2/n^2 - m/n \leq n$
- As $m/n<n$ we get that $m^2/n^2<2n$ or $m^2<2n^3$
- Thus m=O(n sqrt{n})
- A matching lower bound. A graph of girth 6 that has Ω(n*sqrt{n}) edges.
- A projective plane for our needs is a bipartite graph with n vetices on each side and degree Θ(sqrt{n}) thus contains Θ(n*sqrt{n}) edges.
- The main property: every pair of vertices in the same side share exactly 1 neighbor.

Girth 6

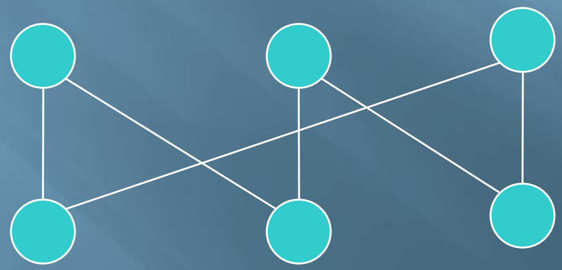
- There could not be a cycle of size 4:



A cycle of length 4 implies that two vertices on the same side share two neighbors. Contradiction

Girth 6

- There could not be a cycle of size 4:



Therefore girth 6

General bounds on the minimum number of edges for a given girth

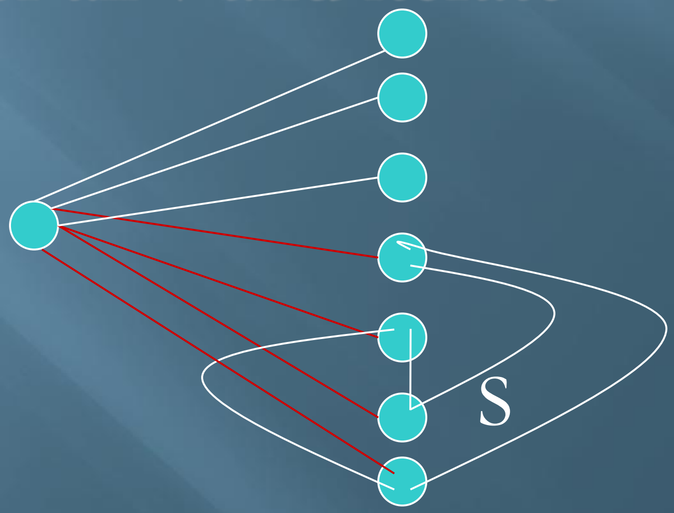
- It is known that there is always a 2k-1 spanner with $O(n^{1+1/k})$ edges.
- Using this formula: 3-spanners needs k=2. This gives the correct and tight O(n* sqrt{n}) upper bound on the number of edges in a 3-spanner.

Approximating spanners

- There are only very few approximations.
- Length 1 arbitrary costs 2-spanners.
- O(log d) approximation with d the average degree for minimum cost 2-spanners.
- As we shall see such an approximation does not exist for $k \geq 3$.

An O(log(|E|/|V|)) ratio for k=2 for arbitrary weight

- Due to K, Peleg 1992.
- For a vertex v look at the graph induced by N(v)
- Find a desnsest subgraph S(v) in N(v)
- Return the edges from v to S(v) that is the most dense set over all v and iterate



The problem we need to solve is the densest subgraph

- Let $e(S)$, $S \subset N(v)$ be the number of edges in the graph induced by S.
- This problem requires finding a subset of the vertices with maximum density $e(S)/|S|$ and can be solved exactly via flow. This implies an $O(\log d)$ ratio for d the average degree.

The problem we need to solve is the densest subgraph

- A faster algorithm, approximates the best density by 2 but gets O(n) time and not flow time. Adds 2 to the ratio (so negligible).
- Was done by K,Peleg in 1992. Also Charikar 1998.
- Very extensively cited in social networks. Almost always attribute the result to Charikar.

How hard is it to approximate spanners for $k \geq 3$?

- Strong hardness is $\exp(\log^{1-\varepsilon} n)$
- Weak hardness is $(\log n)/k$
- K. 98. First hardness. Weak hardness for $w(e)=l(e)=1$.
- Tight for $k=2$.
- Later similar methods employed for hardness for Buy at Bulk.
- Elkin Peleg: Strong hardness for:
- 1) General length
- 2) Weights=1 and general length
- 3)Unit length, arbitrary weights, $k \geq 3$
- 4) Basic but directed spanners.

Only basic spanners from now on

- From now on, edges have weights and lengths 1.
- Thus the results presented from now on are only for basic spanners.
- In fact giving a similar result for arbitrary weights already unknown for some of the problems in later slides.
- And none of the algorithms to follow work on general lengths.

A question posed in 1992

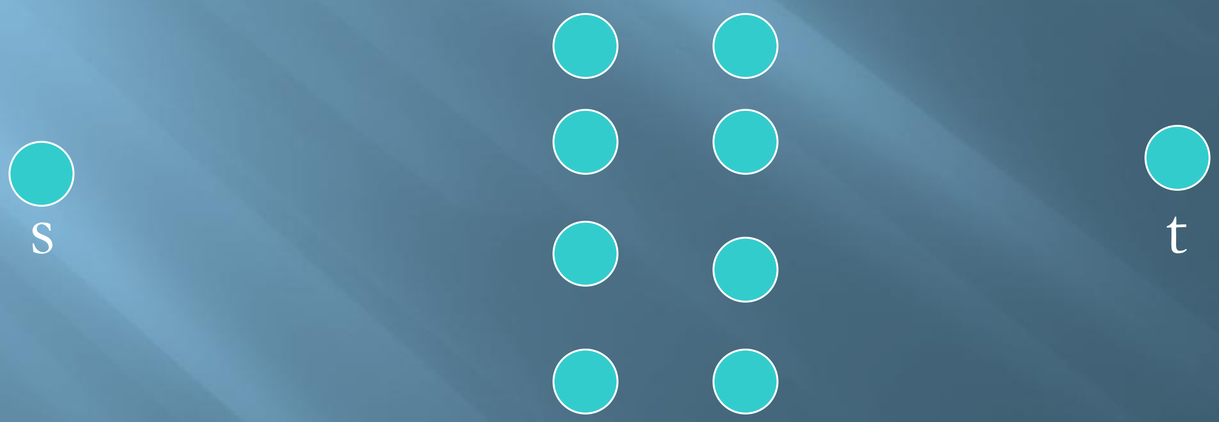
- Is undirected the basic spanner problem strongly hard?
- In ICALP 2012 Dinitz, K, Raz : $k \geq 3$ is Labelcover-Hard (means only polynomial ratio is possible).

- Second important result: Labelcover with large girth is as hard as Labelcover
- Its rare (for me) to solve a 20 years old problem.

A technique employed for approximating directed Steiner Forest

- Feldman, K. and Nutov. 2009.

The following situation:



LP flow at least ¼ between every pair s,t

At most $n^{2/5}$ vertices in every layer

An edge with large x_e

- Between every two layers there is at most $n^{4/5}$ edges.
- Let x_e be the largest capacity. Thus via every edge at most x_e flow unit pass from s to t.
- The total flow between s and t is at least ¼.
- Therefore $n^{4/5} * x_e \geq 1/4$
- Therefore there is an edge of value about $1/4n^{4/5}$
- Iterative rounding gives ratio $n^{4/5}$

Approximating directed spanners

- Krauthgamer and Dinitz 2012, employed (part of) our techniques to get an $n^{2/3}$ approximation for directed k-spanners. The techniques was (re)invented independently.
- Improvement: non iterative but randomized rounding gets about $n^{1/2}$ ratio. Very clever trick!
- Due to Berman, Bhattacharyya, Makarychev, Raskhodnikova, Yaroslavtsev. 2013.

Other results

- For $k=3$ they get ratio $n^{1/3}$ for the directed case. Note that even for undirected graphs $n^{1/2}$ is trivial but $n^{1/3}$ not.
- They also improve the result for Directed Steiner forest. The new best ratio is $n^{2/3}$.
- Can we show a better integrality gap for the natural LP?
- The answer is no.

Dinitz and Zhang 2016

- Ratio $n^{1/3}$ for k=4
- The ADDJ upper bound and the integrality gaps of the natural LP are not that far.
- Interesting proof: builds its own type of Min-Rep and uses the fact that Min-Rep is hard for large super girth several times.
- I would guess that the ratio of ADDJ will not be easily improved if at all.

Preservers

- The input contains a collection of pairs {x,y} and you want minimum edges G' so that the distance between every x,y is the same as in G.
- A paper by Chlamtac, Dinitz, K, and Laekhanukit, SODA 2017.
- Ratio $O(n^{3/5})$ approximation for preservers.
- There is a big problem. The inequality $opt \geq n-1$ does not hold.

How to overcome this

- The SODA 2017 paper introduced junction trees at the last stage.
- Junction trees are trees that connect many s,t pairs so that all paths from s,t for every pair goes via the same vertex r.
- Invented in relation to Buy at Bulk.
- Namely when the relative cost of items goes down if you buy many.

Why do the junction trees help

- Instead of bounding the cost by n-1 you bound the cost by the number of terminal pairs connected, times the maximum length.
- It has some small tricks like applying a different algorithm if the number of pairs is $\Omega(n^{4/5})$.

Approximation Steiner Forest with distance bounds

- Input: Given the pairs $\{s,t\}$ each pair has a distance bound $D(s,t)$
- Objective: find a minimum cost solution so that the distance between every pair of vertices s,t is at most $D(s,t)$.
- The same approximation ratio: $O(n^{3/5})$

Getting back to Directed Steiner Forest

- First sub-linear ratio by Feldman, Kortsarz, Nutov , 2009, $O(n^{4/5})$.
- Berman et al, 2013, improved the ratio to $O(n^{2/3})$ using their clever randomized rounding method.
- Using our additional junction tree and threshold trick we improve Berman et al to $O(n^{3/5})$ (however recall that our result is for the unweighted case). SODA 2017.

The message of this last paper

- Introducing junction tress can help approximating spanner problems. The first time junction trees ever used in spanners.
- A second message is that it seems that additive spanners are harder to approximate than usual spanners.

Additive spanners

- Aingworth, Chekuri, Indyk, Motwani 1996. For any graph, n· sqt{n} edges +2 spanners.
- Chechik. +4 spanners always exists with
  $O(n^{7/5})$ 2013.
- Baswana, Kavitha, Mehlhorn, Pettie show: Always exists +6, $O(n^{4/3})$ 2010 (before +4).
- Can we continue with this hobby for k=8, k=10 and so on?

Surprise (at least for me)

- Amir Abboud and Greg Bodwin. 2016
- The $O(n^{4/3})$ can not be not be improved.
- There are large μ, so that μ, additive spanners requires $\Omega(n^{4/3})$ edges.
- The last result for $k=6$ is best possible for much higher k.
- How do additive spanners compare to spanners for approximation? Turns out: Also harder.

The case of k=1

- We gave the first lower bound. SODA 2017.
- If we have edges of cost 0 this is easy.
- We can not show that its hard to spann edges because of the O(log n) for k=2.
- Dividing edges brings new edges that need to be spanned. Feels like catch 22.
- Overcoming that by making the new paths added the same Labelcover hard. CDLK, SODA 2017.

For $k=O(\text{polylog}(n))$

- Again Labercover hard. Harder proof.
- Additive spanners are harder to aproximate than spanners.
- Any +1 spanner is a 2-spanner but +1 spanner much harder
- Also $O(\log n)$ spanner has constant ratio but additive polylog(n) spanner is Labelcover hard.
- The +1 spanners result surprised me.

Open problems

- Transitive closure spanners. Tree spanners
- Fault tolerant spanners. Simple and nice Algorithm by Dinitz and Krauthgamer.
- Fault tolerant spanners: new version
- Preserve the distance from s to G-s under at most f edges that can fall. Parver and Peleg.
- Find a minimum H so that for any $|F| \le f$, dist(s,u,G-F)=dist(s,u,H-F). Turned to be equivalent to Set Cover. Parver and Peleg.
- Many open questions remain here.

It is not possible to predict the future. Did you know that?

- Peleg and Ulman invented spanners in 1987.
- There was nothing. Only some results from geometry.
- I would imagine Peleg and Ulman did not expect the extent of which this subject will develop back then.