A Survey of Computational Assumptions on Bilinear and Multilinear Maps

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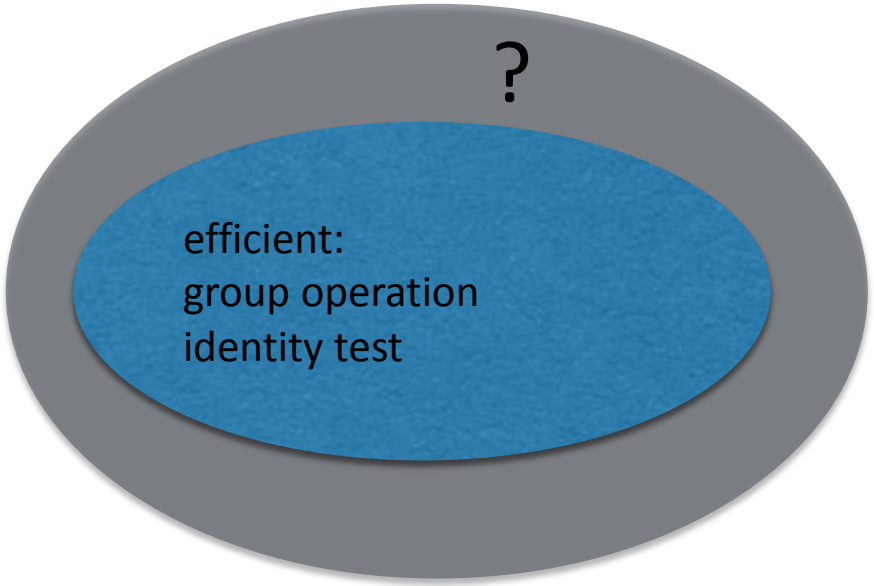
Group Basics

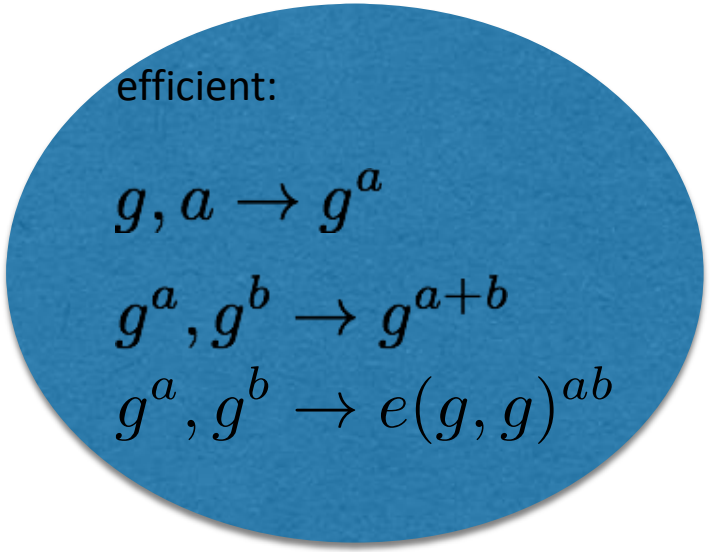
"There are two kinds of people in this world. Those who like additive group notation, and those who like multiplicative group notation."

$$g, g^a, g^b, g^{a+b}$$

inefficient:
discrete log



Bilinear Groups



When Faced with a New Group Assumption:

Kinds of Assumptions

- Generic group models
- q-type assumptions
- static assumptions

Variants:
Symmetric/Asymetric
Composite Order/ Prime Order
Linear/ Bilinear/ MultiLinear

Kinds of Proof Techniques

- Brute Force
- Cancelation
- Encoding
- Dual System
- Deja Q
- …

basic generic group arguments
BB IBE
G06, W'05 …
W09, LW10, LOSTW10,…
CM14, W16

Billinear Diffie-Hellman Assumption

Symmetric group:

$$g \in G, e : G \times G \rightarrow G_T$$

Given:

$$g, g^x, g^y, g^z$$

Distinguish:

$$e(g,g)^{xyz} \text{ from } e(g,g)^r$$

SXDH Assumption

Asymmetric group:

$$g \in G, h \in H, e : G \times H \rightarrow G_T$$

Given:

$$g, h, g^a, g^b$$

Distinguish:

$$g^{ab} \text{ from } g^r$$

A Basic q-type Assumption

Symmetric group:

$$g \in G, e : G \times G \rightarrow G_T$$

Given:

$$g, g^s, g^a, g^{a^2}, g^{a^3}, \ldots, g^{a^q}$$

Distinguish:

$$e(g,g)^{sa^{q+1}} \text{ from } e(g,g)^r$$

A Driving Example: IBE

$$PP : g, g^a, g^b, e(g,g)^\alpha$$

$$CT : Me(g,g)^{\alpha s} \qquad g^s \qquad g^{s(aID+b)}$$

$$SK : \qquad g^{\alpha + r(aID+b)} \qquad g^r$$

Decryption:

$$e(g,g)^{\alpha s + sr(aID+b) - sr(aID+b)}$$

$$= e(g,g)^{\alpha s}$$

Arguing Generic Security

Look at exponents you are given in G:

Look at the blinding factor:

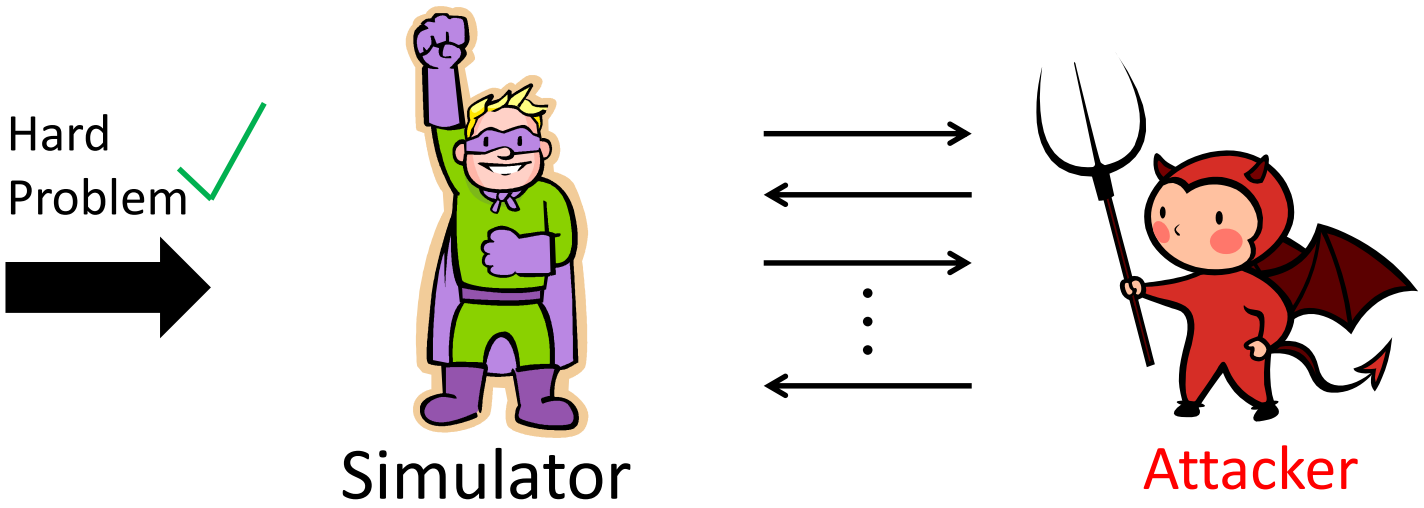
$$\alpha s$$

$$a,\ b,\ s,\ s(aID^* + b),$$
$$\alpha + r_1(aID_1 + b), r_1, \ldots, \alpha + r_q(aID_q + b), r_q$$

All you can do is take linear combinations of degree at most 2

$$\text{No way to get } \alpha s \text{ alone when } ID^* \neq ID_i \forall i$$

Proof Challenges Beyond Generic Security



Simulator must balance two competing goals:

answer
attacker
queries

leverage
attacker
success

Arguing Selective Security

- Embed the challenge as a function of known ID*

$$\alpha = xy,\ s = z$$

$$a = x,\ b = -xID^* + w$$

Given:

g, g^x, g^y, g^z

$$PP : g, g^a = g^x, g^b = (g^x)^{-ID^*} g^w, e(g,g)^\alpha = e(g^x, g^y)$$

$$CT : g^s = g^z, g^{s(aID^*+b)} = (g^z)^w$$

Distinguish:

$e(g,g)^{xyz}$ from $e(g,g)^r$

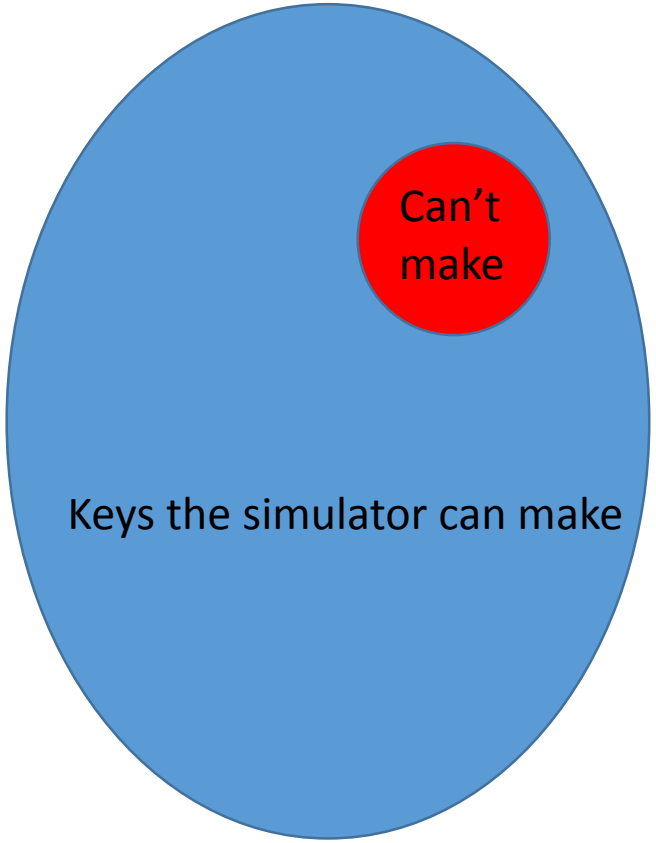
Choose $r = -\frac{y}{ID - ID^*} + r'$

then $\alpha + r(aID + b) = xy - xy + rw = rw$

Simulator can produce key for any ID not equal to ID!*

How to Leverage a q-Type Assumption [example from W05]

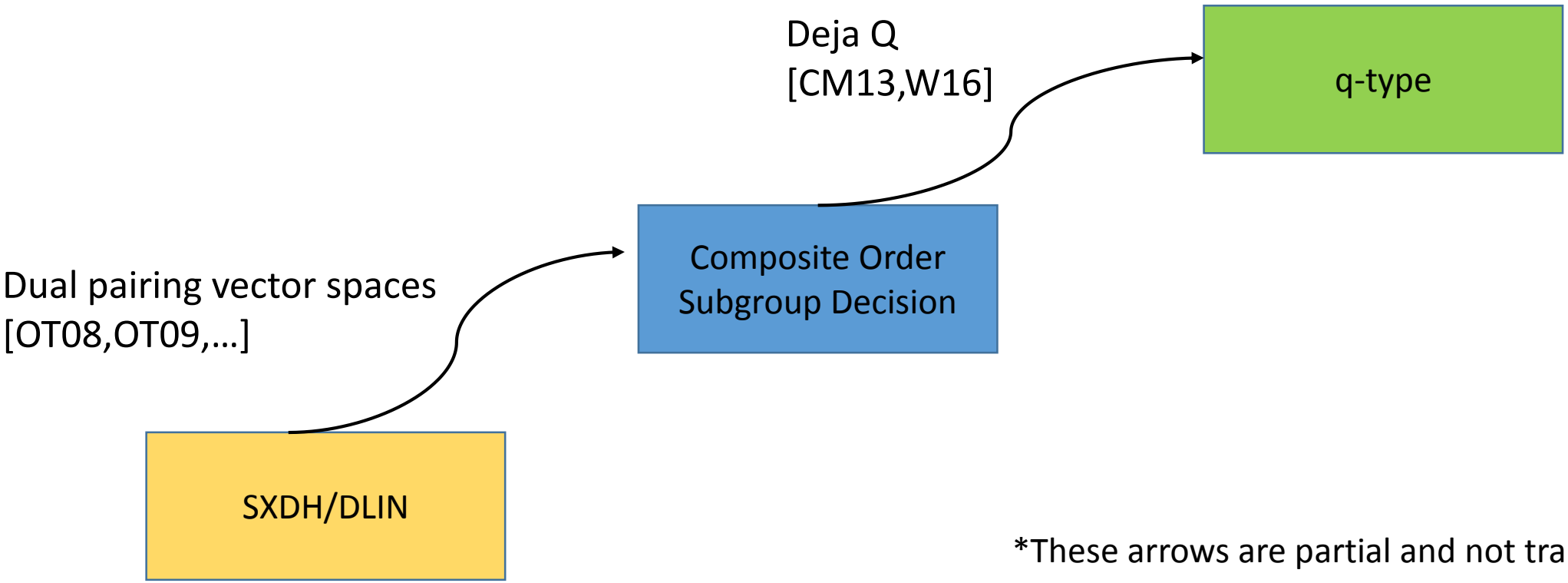
What if we don't want to fix ID* ahead of time?



To partition small PP with parameter q:

Use a q-size assumption!

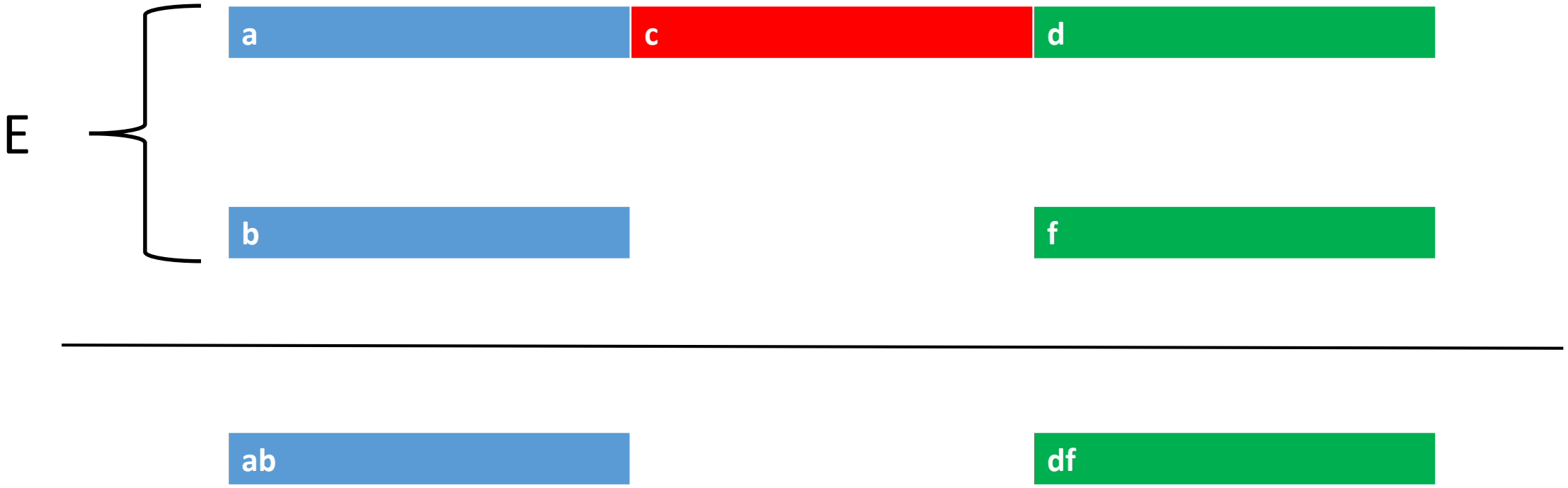
Simulation Techniques



*These arrows are partial and not transitive!

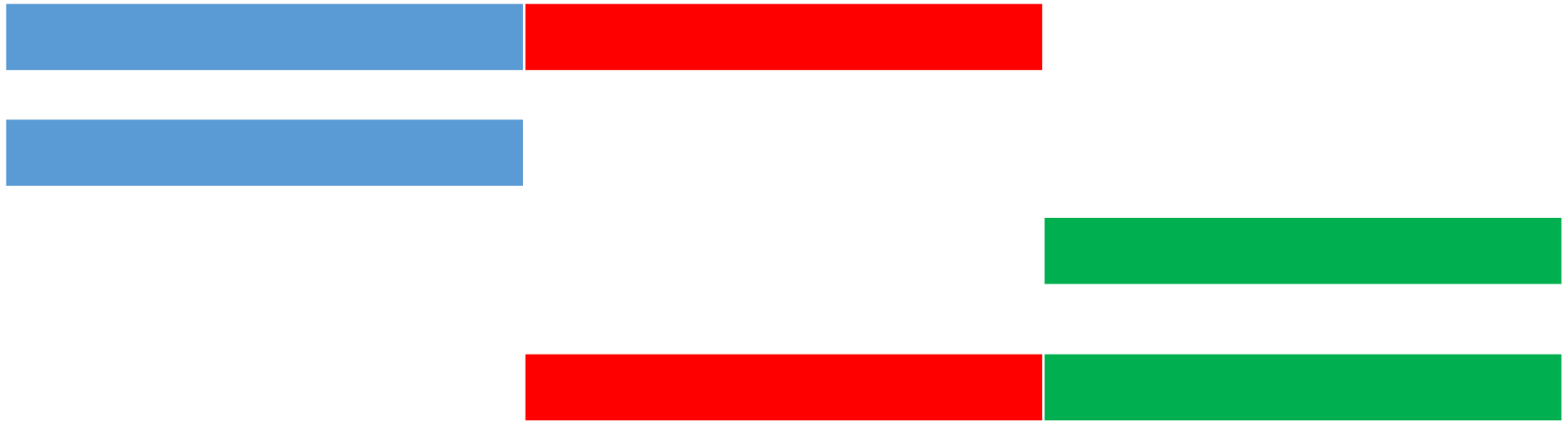
Composite Order Bilinear Groups

How the pairing operates:



Subgroup Decision Assumptions in Composite Order Bilinear Groups

Example: Given



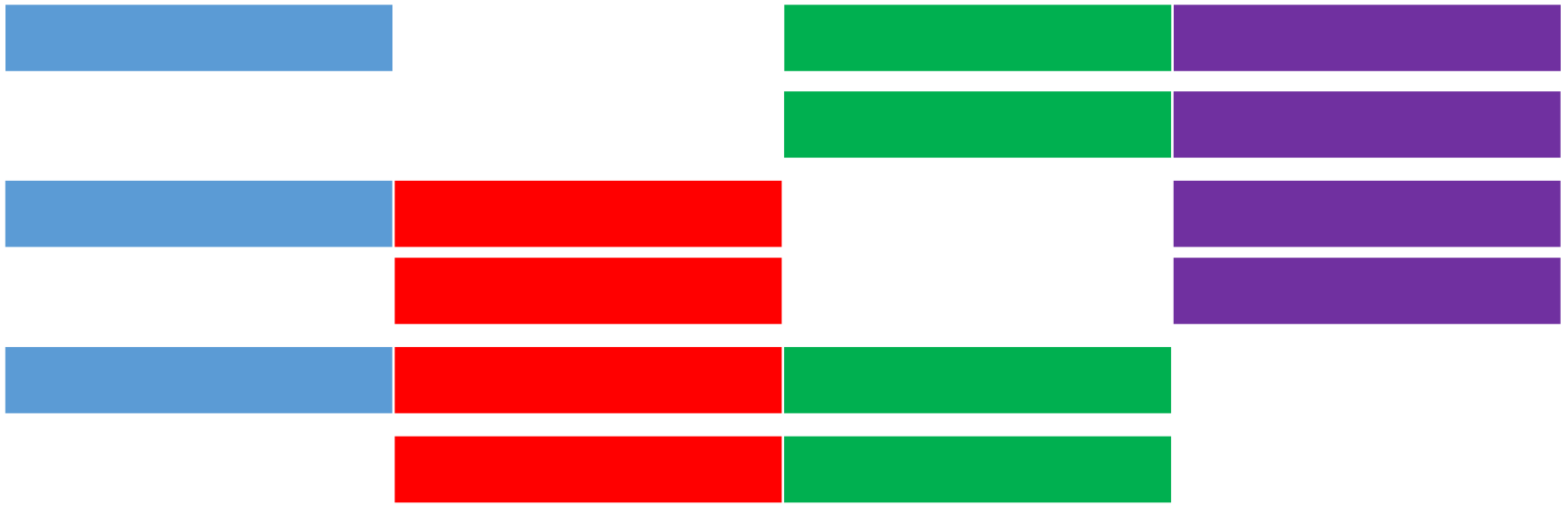
Distinguish

from

Subgroup Decision in a Multilinear Group?

Here's what it might look like in a 3-linear group:

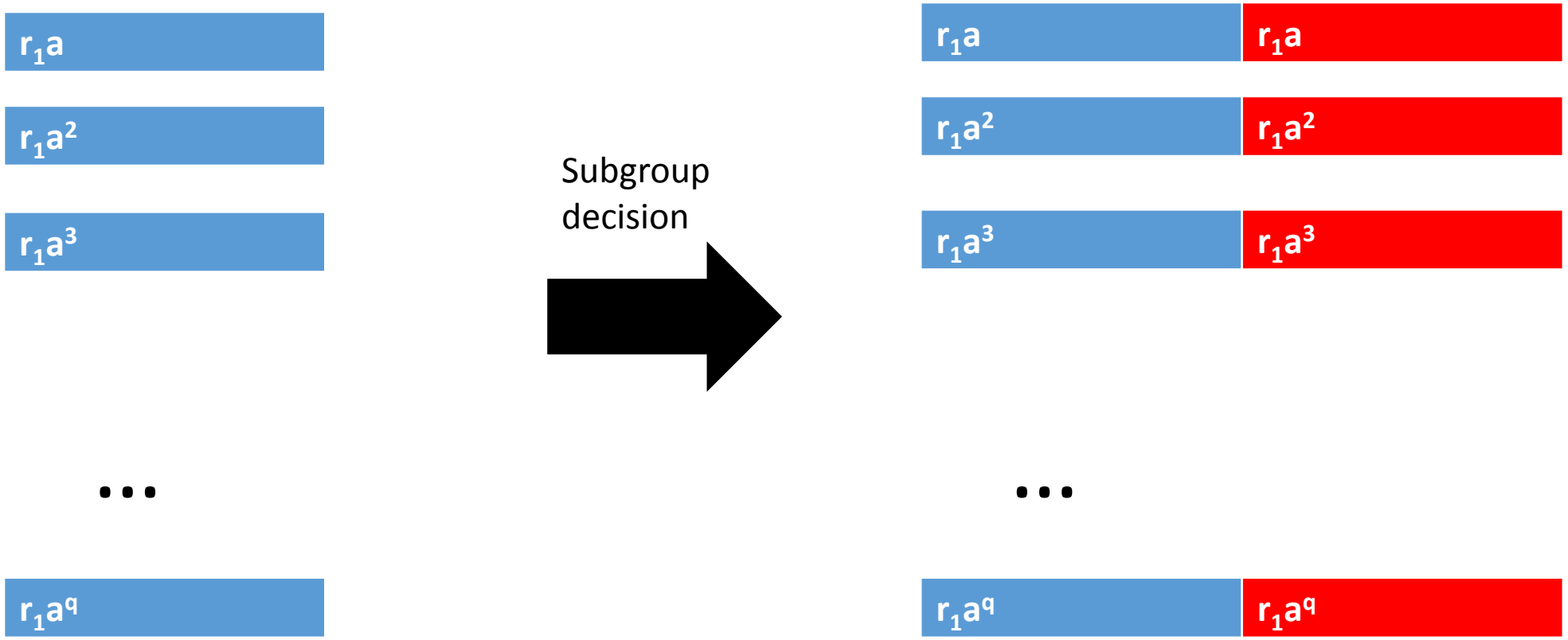
Given



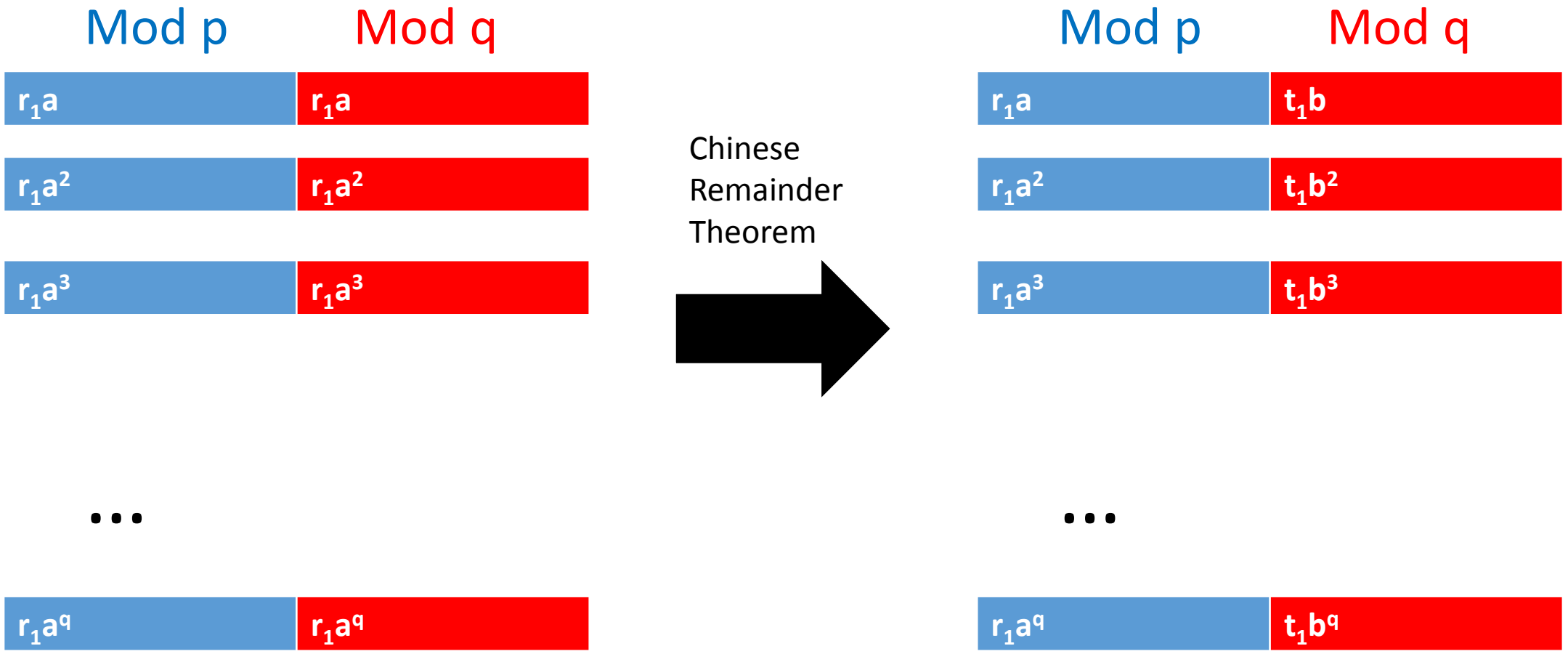
Distinguish

from

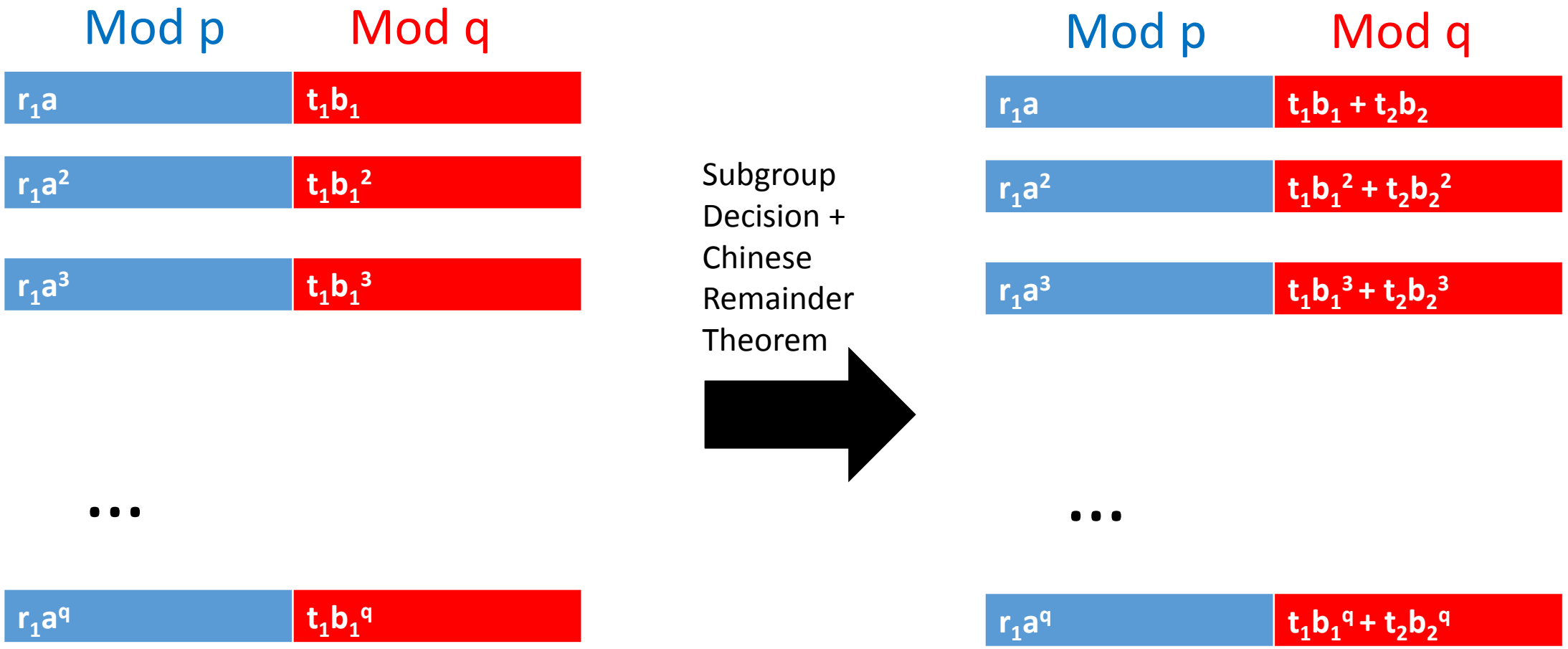
Deja Q – Basic Example



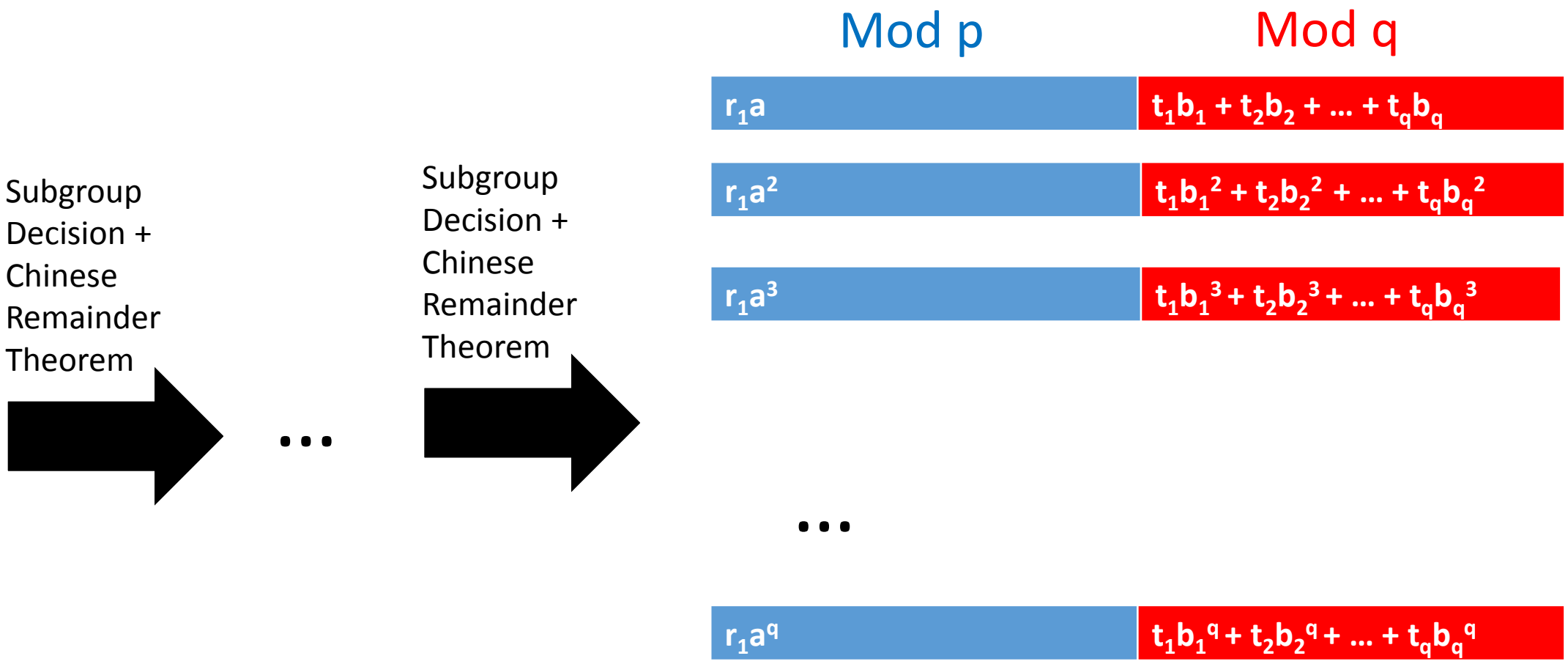
Deja Q – Basic Example



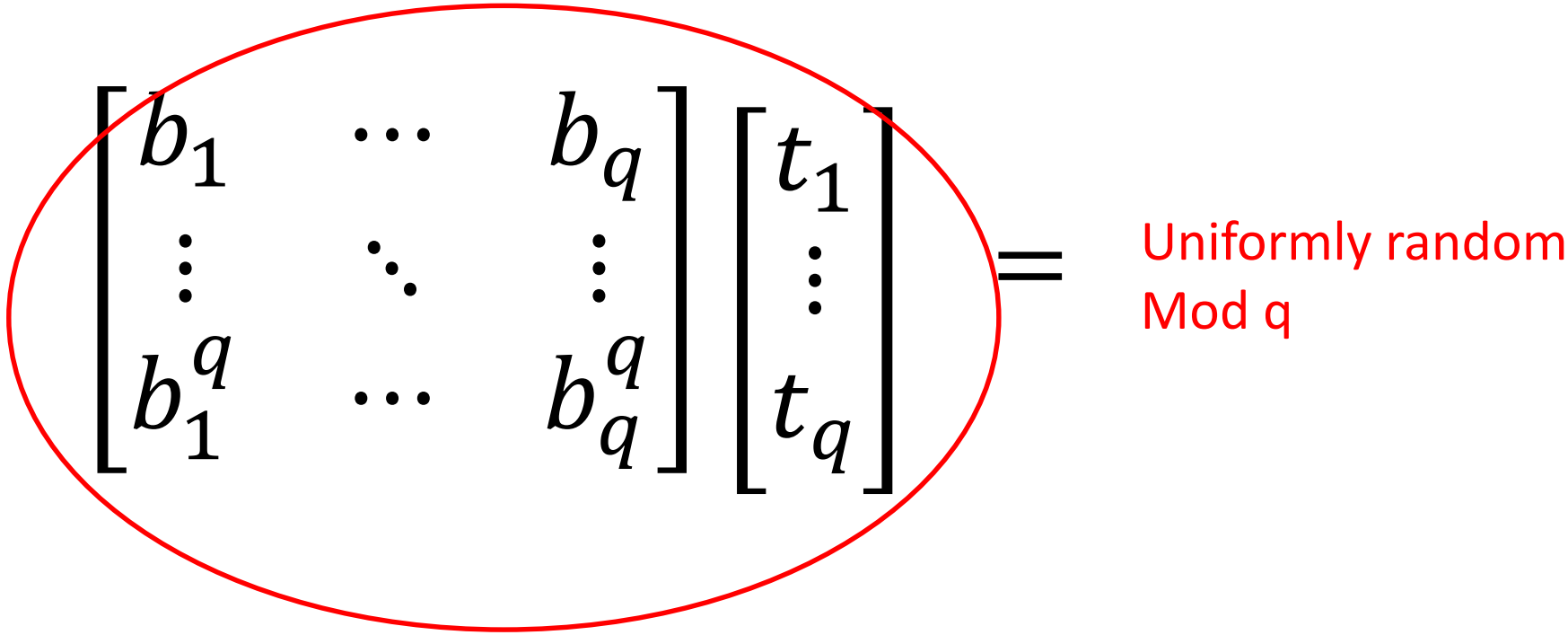
Deja Q – Basic Example



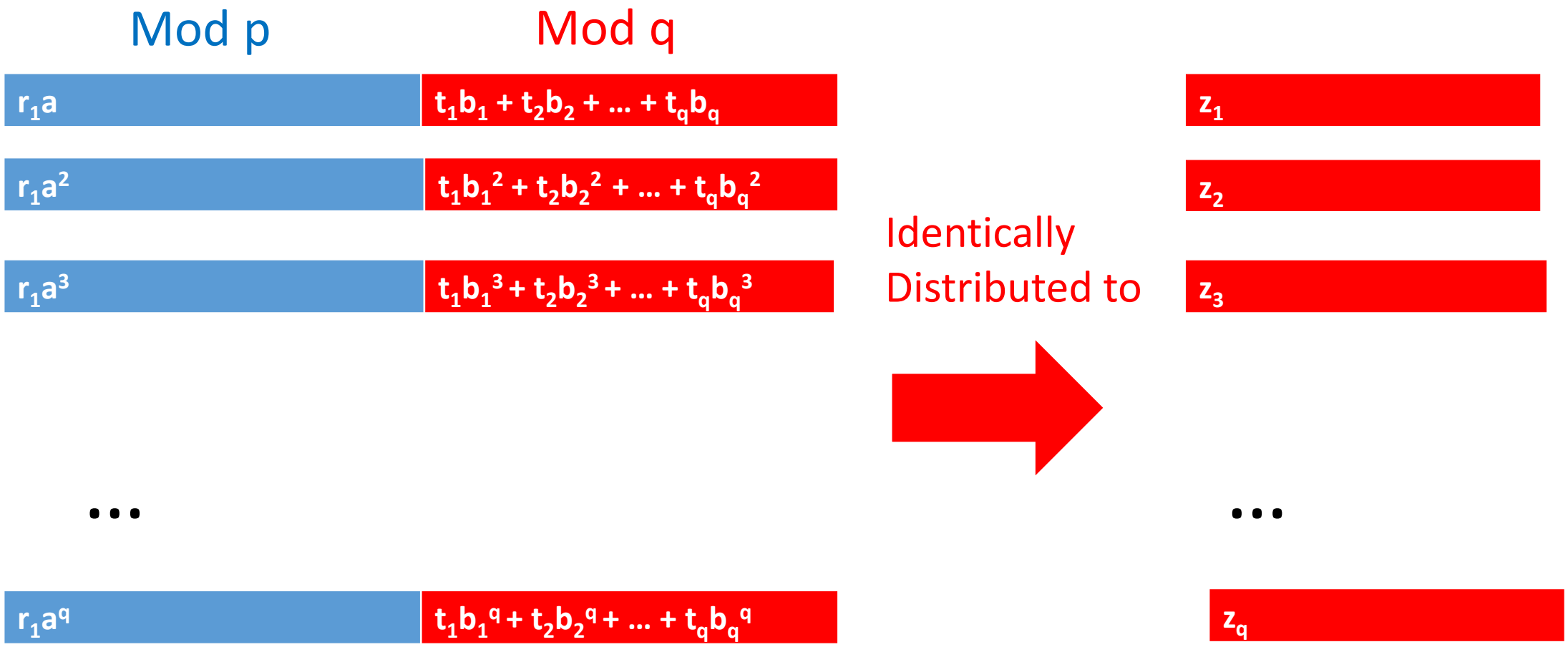
Deja Q – Basic Example



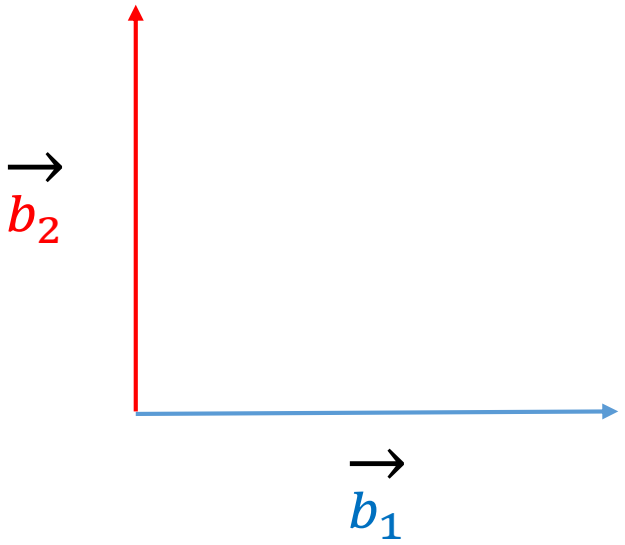
Deja Q – Basic Example



Deja Q – Basic Example



Dual Pairing Vector Spaces



$$b_1, b_2 \leftarrow \mathbb{Z}_p^d$$

$$b_1^*, b_2^* \text{ s.t. } b_i \cdot b_j^* = 0 \text{ if } i \neq j,$$

$$b_i \cdot b_i^* = 1$$

Emulates some features of composite order, asymmetric group:

E

$$\frac{g^{rb_1+tb_2} \quad g^{sb_1^*+zb_2^*}}{e(g,g)^{rs+tz}}$$

Emulating Subgroup Decision using SXDH

Asymmetric group: $g \in G, h \in H, e : G \times H \to G_T$

Given:

g, h, g^a, g^b

Distinguish:

g^{ab} from g^r

$$b_1 := ax_1 + x_2 \quad b_2 := x_2$$

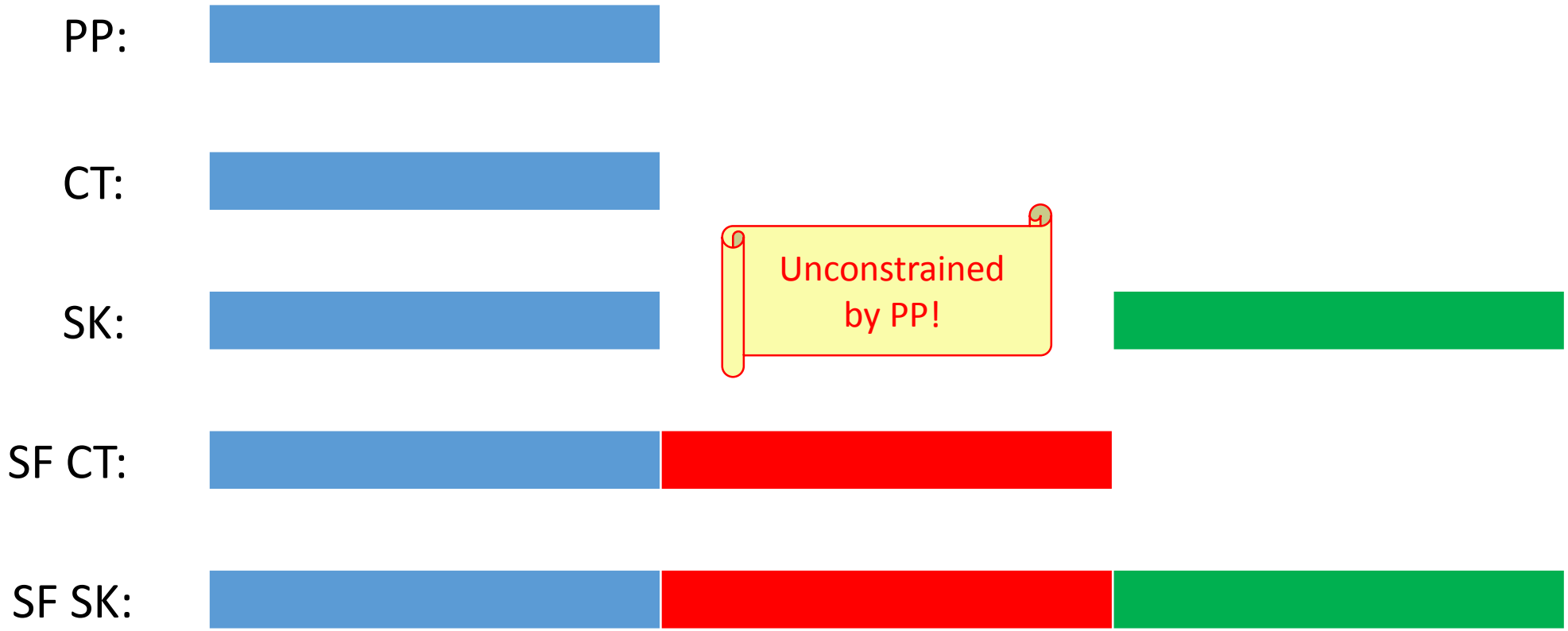
$$b_1^* = a^{-1}x_1^* \quad b_2^* = x_2^* - ax_1^*$$

$$\text{can make } g^{b_1}, g^{b_2}$$

$$\text{can make } h^{rb_1^*}, \text{ not } h^{rb_2^*}$$

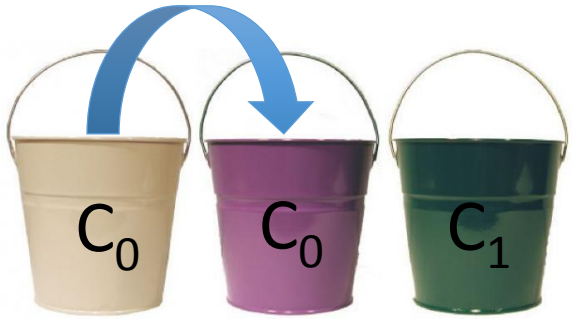
Dual System – Using Subgroup Assumptions for Functional Encryption [W09 + too many to cite*]

Most Basic Template:



Using Subgroup Assumptions for Obfuscation [GBSW 15]

- Reduction will ***isolate*** each input.
- Main idea:
  - Have poly many "parallel" obfuscations, each responsible for a bucket of inputs
  - Hybrid Type 1: Allocate/Transfer inputs among different buckets, ***but*** programs do not change at all. **Assumption used here.**



Ok. So what are these buckets really like?

Simple example:

Want to implement:

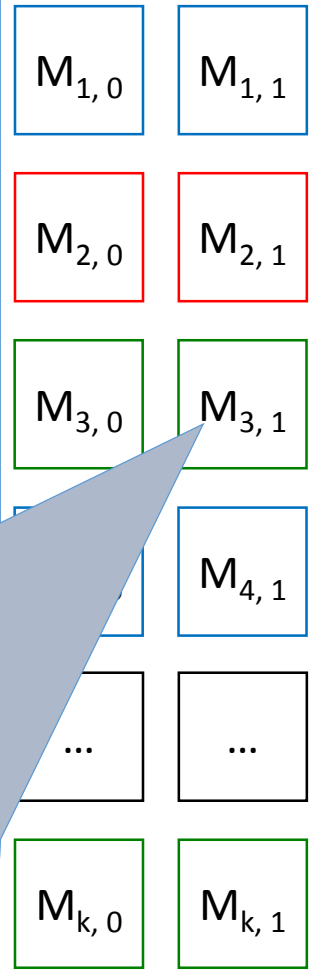
$F(x_1\ x_2) = XOR(\ x_1, x_2\ )$

$$M_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M_{2,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M_{2,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[Barrington]: All log-depth (NC^1) circuits have poly-size Matrix Branching Programs

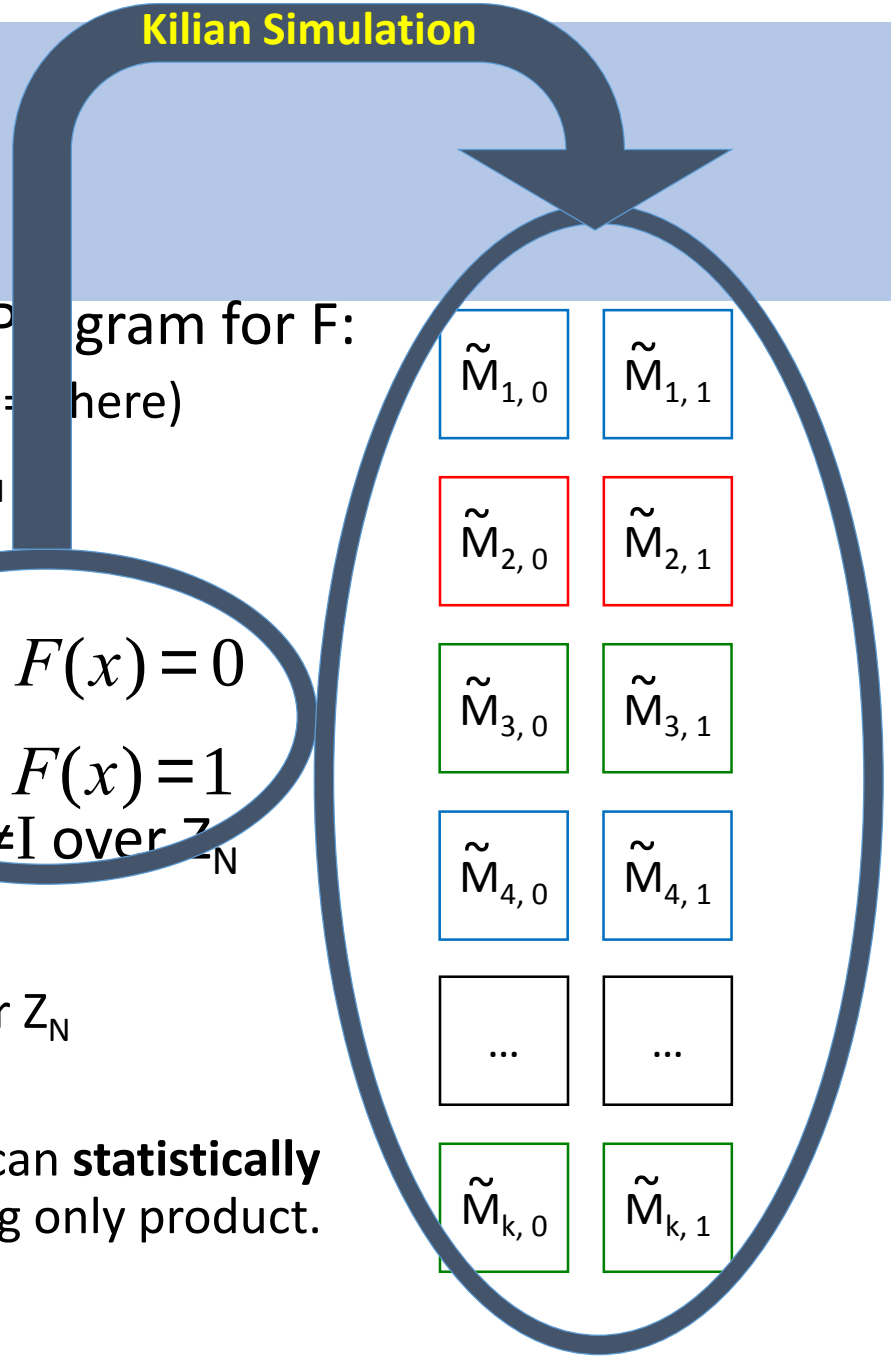


Towards Obfuscation

- Oblivious Matrix Branching P gram for F:
  - n-bit input $x=x_1x_2\ldots x_n$ (e.g. n= here)
  - 2k invertible matrices over Z_N
  - Evaluation on x:

$$\tilde{O}\ M_{i,x_{(i \bmod n)}} = \begin{cases} I & if\ F(x)=0 \\ B & if\ F(x)=1 \end{cases}$$

  - Where B is fixed matrix $\neq I$ over Z_N
- Kilian Randomization:
  - Chose $R_1, \ldots, R_{k-1}$ random over Z_N
  - $\tilde{M}_{i,b} = R_{i-1} \cdot M_{i,b} \cdot R_i^{-1}$
  - Kilian shows that for each x, can **statistically** simulate M_x matrices knowing only product.



Hybrids Intuition