Approximation Algorithms for Aligning Points *

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Abstract

We study the problem of aligning horizontally, vertically, or diagonally, as many points as possible when each point is allowed to be placed anywhere in its own, given region. Different shapes of placement regions and different sets of alignment orientations are considered. More generally, we assume that a graph is given on the points, and only the alignments of points that are connected in the graph count. For the case of trees and planar graphs, we give approximation algorithms whose performance depends upon the shape of the given regions and the set of orientations. When the orientations to consider are the ones given by the axes and the regions are axis-parallel rectangles, we obtain a polynomial time approximation scheme.

1 Introduction

This paper studies a placement problem for multiple objects, motivated by cartography. In the design of schematic networks, like subway maps, a strongly simplified depiction of a transportation system should be computed. The connection between two major locations or junctions is shown in stylized manner where the exact geometry of the connection is unimportant. In particular, it is common to displace the important locations or junctions of a schematic network such that the segments connecting those points are horizontal, vertical, or diagonal.

Given a fixed set of orientations O, we define a function χ_O that assigns to pairs of points the value 1 if the line through them has its orientation in O, and 0 otherwise. Abstracting from our application into cartography, the problem can be stated as follows: given a set of n convex regions, $\mathcal{S} = \{S_0, \dots, S_{n-1}\}$, a graph $G = (\mathcal{S}, E)$ and a set of orientations O, place n points $p_1, \dots, p_n$, with $p_i \in S_i$, to maximize the function

$$\sum_{\{S_i,S_j\}\in E} \chi_O(p_i, p_j).$$

(Note that we abuse notation slightly by denoting both the regions and the nodes of G by $\mathcal{S}$, because they are in one-to-one correspondence.) We denote the maximum value by $M_O(G)$, or simply $M(G)$, as we consider the given orientations O to be fixed. When the regions overlap, the placement of two points can coincide, and in this case we also assume that they are aligned.

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For the application to cartography, the orientations will typically be axis-parallel ($|O| = 2$) or also including diagonal lines (with slope 1 or -1, so $|O| = 4$). We also consider G to be planar, but it is possible that the computed placement does not give a planar embedding. In fact we are not assuming that an embedding is given initially. If this would be the case, the new embedding may be non-equivalent to the original one.

2 Results

Using results from [3] we can prove that the problem is hard.

Theorem 1 *Let O be the vertical orientation, let $\mathcal{S} = \{S_0, \ldots S_{n-1}\}$ be a set of horizontal segments and let $G = (\mathcal{S}, E)$ be a planar graph. It is NP-hard to compute $M(G)$.*
When the graph G is non-planar, there is a constant $C < 1$ such that it is NP-hard to get C-approximation of $M(G)$.

When the graph is a tree, there is a PTAS using dynamic programming from the leaves to the root. Furthermore, it is known that a planar graph G can be decomposed in $O(n \log n)$ time into three trees (or forests), such that every edge of G appears in exactly one tree (or forest) [2]. Therefore, an approximation for trees gives an approximation for any planar graph.

Theorem 2 *Given a set of n convex regions, $\mathcal{S} = \{S_0, \ldots, S_{n-1}\}$ with total complexity $O(n)$, a set O of two orientations, and a tree $\mathcal{T} = (\mathcal{S}, E)$, we can place, for any natural number k, a point $p_i \in S_i$ for every $i = 0, 1, \ldots, n-1$ that yields a $\frac{k}{k+1}$-approximation of the maximum number of alignments $M(\mathcal{T})$ in $O(k9^k n^2)$ time.*
If $G = (\mathcal{S}, E)$ is a planar graph, we obtain a $\frac{k}{3(k+1)}$-approximation of $M(G)$ in the same time.

When $|O| > 2$ we get the same approximation ratio, but in $n^{O(2^k)}$ time.
For "axis-aligned problems", we can use [1] to get the following result.

Theorem 3 *Let O be the orientations of the coordinate axes, let $\mathcal{S} = \{S_0, \ldots, S_{n-1}\}$ be a set of n disjoint, axis-parallel rectangles, and let $G = (\mathcal{S}, E)$ be a planar graph. For any natural number k, we can place a point $p_i \in S_i$ for every $i = 0, \ldots, n-1$ that yields a $\frac{k}{k+1}$-approximation of $M(G)$ in $O(k(2n)^{3k+1})$ time.*

For this case, there is also a $\frac{1}{3}$-approximation algorithm that runs in quadratic time.

References

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