THE DISCREPANCY OF QUASI-ARITHMETIC PROGRESSIONS: POWER OF N OR POWER OF LOG N?

SUJITH VIJAY

RUTGERS UNIVERSITY

(Dissertation work in progress; supervised by Prof. József Beck.)

Van der Waerden's Theorem: For all k, there exists $N=N(k)$ such that every 2-colouring of $\{1,2,\ldots,N\}$ yields a monochromatic k-term arithmetic progression.

Bounds: For all colourings, $k \geq \log\log\log\log\log N$ (Gowers)
There exists a colouring with $k \leq \log_2 N$ (Local Lemma)

Roth's Theorem: For all k, there exists $N=N(k)$ such that every 2-colouring of $\{1,2,\ldots,N\}$ yields an arithmetic progression of discrepancy at least k.

Discrepancy = |Number of reds − Number of blues|

Bounds: For all colourings, $k \geq c_1 N^{1/4}$ (Roth)
There exists a colouring with $k \leq c_2 N^{1/4}$ (Matoušek & Spencer)
[Sárközy: $N^{1/3+\epsilon}$, Beck: $N^{1/4+\epsilon}$]

Homogeneous Arithmetic Progressions: $\{0, d, 2d, 3d, \ldots\}$

Is there a colouring of $\mathbb{N}$ such that all homogeneous arithmetic progressions have bounded discrepancy? (Erdős)

Upper Bound: $O(\log n) \longrightarrow \chi(3^a b) = \begin{cases} 1 & \text{if } b \equiv 1 \pmod 3 \\ -1 & \text{if } b \equiv -1 \pmod 3 \end{cases}$

Partial Colouring: $\chi(3k+1) = 1 \quad \chi(3k) = 0 \quad \chi(3k-1) = -1$

Upper bound in terms of d alone: $O(d^{4+\epsilon})$ (Reimer)

Quasi-arithmetic Progressions: $\{0, \lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}$

e.g.: $\alpha = \sqrt{5} \rightarrow \{0, 2, 4, 6, 8, 11, 13, \ldots\}$

There are $O(N^2)$ distinct quasi-arithmetic progressions contained in $\{0, 1, 2, \ldots, N\}$ (Farey Sequence)

Polynomial-size family, random colouring upper bound on discrepancy: $O(\sqrt{N \log N})$

Lower Bounds on the discrepancy of quasi-progressions

For any 2-colouring of $\{0, 1, \ldots, N\}$, there exists a quasi-progression of discrepancy at least $c(\log N)^{1/4}$
(Hochberg '94)

Improvement: For any 2-colouring of $\{0, 1, \ldots, N\}$, there exists a quasi-progression of discrepancy at least $\frac{1}{50} N^{1/6}$

Proof Sketch: → Pick a subinterval $\{N-M, N-M+1, \ldots, N\}$
→ Take the A.P. of discrepancy $\frac{1}{20} M^{1/4}$ and common difference d (fact: $d \le \sqrt{6M}$) inside $\{N-M, N-M+1, \ldots, N\}$
→ Show that this A.P. can be realised as a quasi-progression.

$$\sqrt{8} \longrightarrow \{0, \underbrace{2, 5, 8, 11, 14}_{\equiv 2 \pmod 3}, \underbrace{16, 19, 22, 25, 28}_{\equiv 1 \pmod 3}, \ldots\}$$

$\alpha = d - \epsilon$

$\lfloor k\alpha \rfloor \equiv -1 \pmod d$ for the first $(1/\epsilon)$ terms

$\equiv -2 \pmod d$ for the next $(1/\epsilon)$ terms and so on.

For $M = N^{2/3}$, all arithmetic progressions inside $\{N-M, N-M+1, \ldots, N\}$ can be realised as quasi-progressions.

Typical Behaviour of Quasi-progressions

Given any 2-colouring of the non-negative integers, for almost every $\alpha \in [1,\infty)$, there are infinitely many n such that $\mathbf{D_\alpha(n) \geqslant \log^* n}$ (Beck '86)

$$D_\alpha(n) \doteq \max_{1 \leqslant k \leqslant n} \left| \sum_{j=0}^{k} \chi(\lfloor \alpha j \rfloor) \right|$$

Proof Sketch: Let $N_t(k)$ be the smallest integer such that any 2-colouring of an interval J of length $N_t(k)$ yields, for "most" $\alpha \in [t, t+1)$, $D_\alpha(j) \geqslant k$ for some $j \in J$

Now consider $2^{N_t(k)}$ blocks of length $N_t(k)$. There must be two blocks coloured identically

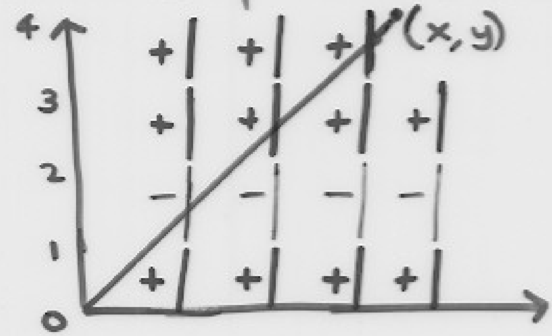
L : Number of terms in $[a_1, a_2] \approx \dfrac{a_2 - a_1}{\alpha}$

If L is odd, either $[b_1, a_2]$ or $[a_1, b_2]$ yields discrepancy at least $k+1$.

Thus $N_t(k+1) \leqslant 2^{N_t(k)}$ i.e., $D_\alpha(n) \gtrsim \log^* n$

Improvement: Given any 2-colouring of the non-negative integers, for almost every $\alpha \in [1,\infty)$, there are infinitely many n such that $D_\alpha(n) \geqslant (\log n)^{1/5}$

(Also works for partial colourings of positive density)

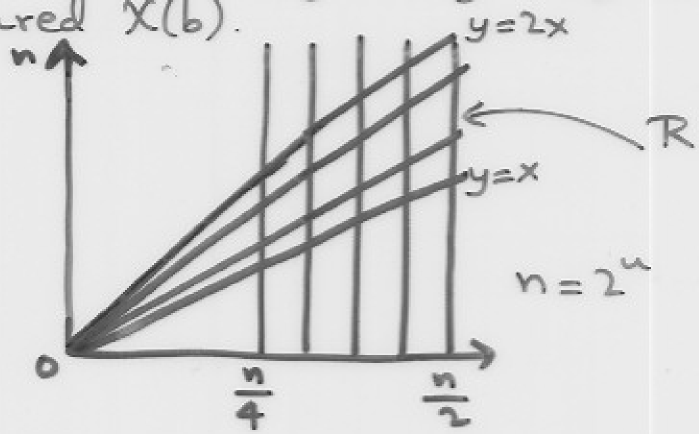


$\alpha \in [1,2)$

$$D(x,y) = \sum_{k \leq x} \chi(\lfloor k\alpha \rfloor)$$

$\alpha = y/x$

The vertical line segment joining (a,b) with $(a,b+1)$ is coloured $\chi(b)$.



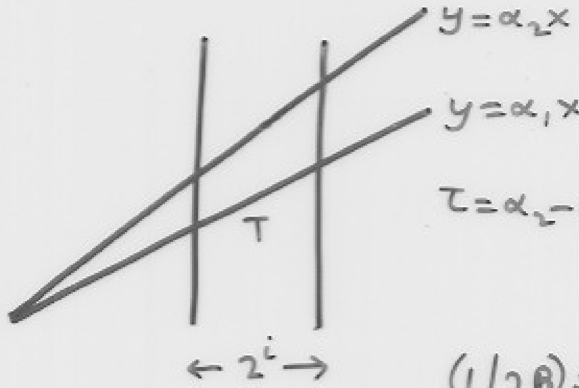
Defn: α is <u>balanced</u> iff $D_\alpha(n) < (\log n)^{1/5}$

$$H(x,y) = \begin{cases} D(x,y) & \text{if } y/x \text{ is balanced} \\ 0 & \text{otherwise} \end{cases}$$

Plan: Suppose the set of balanced α has measure $\delta > 0$.
Construct orthonormal functions $g_1, g_2, \dots, g_r$ where $r = \frac{\log n}{8}$ and $\sum_{i=1}^{r} \langle H, g_i \rangle^2 \geqslant c_0 n^2 (\log n)^{3/5}$

Since R has area $O(n^2)$, Bessel's inequality yields a contradiction.

The i^{th} trapezoidal grid



$\tau = \alpha_2 - \alpha_1 = \frac{1}{2^i n\beta}$ (β to be fixed later; around $(\log n)^{1/5}$)

$(1/2\beta) \leq$ area of $T \leq (1/\beta)$

→ The position of the leftmost vertical line is chosen randomly and uniformly in $\left(\frac{n}{4}, \frac{n}{4}+2^i\right)$

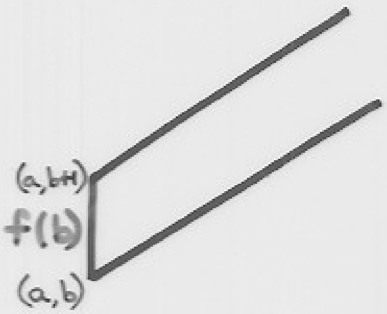
→ The slope of the lowermost slanting line is chosen randomly and uniformly in $(1, 1+\tau)$

Switch values: $\{b \mid X(b) \neq X(b-1)\}$

Switch points: $\{(a,b) \mid b$ is a switch value$\}$

Observation: If X is constant on some interval of length $2(\log n)^{1/5}$, then for all $\alpha \in [1,2)$, we have $D_\alpha(n) \geq (\log n)^{1/5}$.

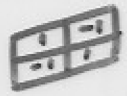
If not, there are at least $\frac{n}{2(\log n)^{1/5}}$ switch values and $\frac{n^2}{4(\log n)^{1/5}}$ switch points.



$$H_{a,b}(x,y) = \begin{cases} f(b), & \text{if } \frac{b}{a} \leq \frac{y}{x} < \frac{b+1}{a} \text{ and } \frac{y}{x} \text{ is balanced} \\ 0 & \text{otherwise.} \end{cases}$$

$$H(x,y) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} H_{a,b}(x,y) \quad \text{[Finite sum for fixed } (x,y)\text{]}$$

Good switch point: Does not share its trapezoid with another lattice point, for any positioning of the grid.

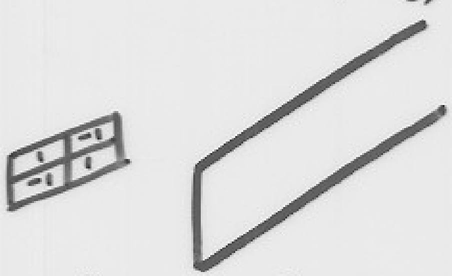
$G_i =$  or  if T contains a good switch point; 0 otherwise.

Vertical Dividing Line: Passes through the geometric centre of T

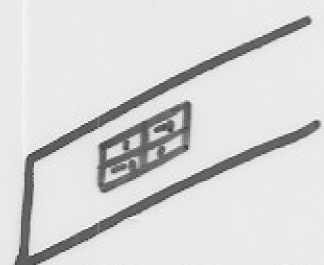
Slanting Dividing Line: Chosen such that the measure of balanced α above and below the line are equal

Fact: $\{G_i\}$ are orthogonal

$$\langle G_i, H \rangle = \sum \langle G_i, H_{a,b} \rangle$$



$\langle G_i, H_{a,b} \rangle = 0$



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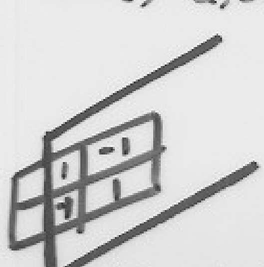

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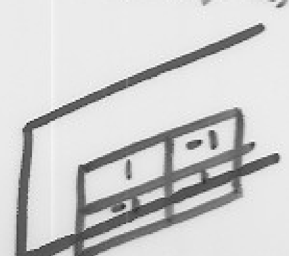
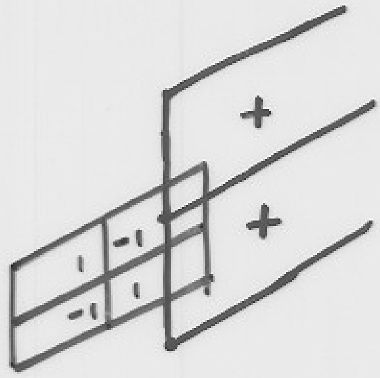
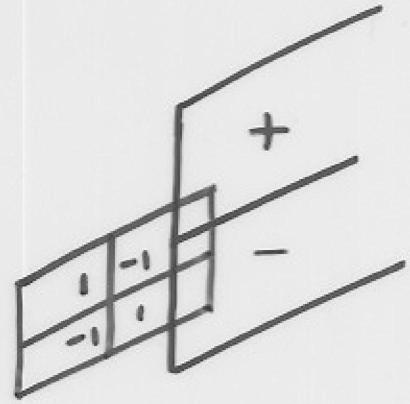


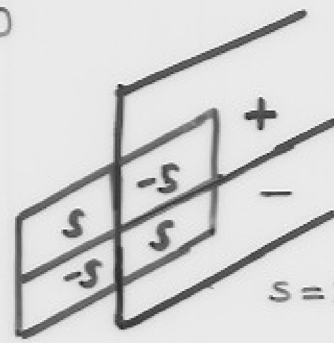
$\langle G_i, H_{a,b} \rangle = 0$



$\langle G_\iota, H_{a,b}\rangle = 0$



$\langle G_\iota, H_{a,b}\rangle \neq 0$



$s = \pm 1$

Switch point at the centre: $\langle G_\iota, H_{a,b}\rangle \geqslant \dfrac{\mu_j^*}{8\beta}$

General position: $E(\langle G_\iota, H_{a,b}\rangle) \geqslant \dfrac{(\mu_j^*)^2}{64\beta}$

$\mu_j^* = \dfrac{\mu_j}{\tau}$ where μ_j is the measure of balanced α in the j^{th} sector

$$E(\langle G_\iota, H\rangle) \geqslant \frac{\sum (\mu_j^*)^2 s_j^*}{64\beta}$$

where s_j^* is the number of good switch points in the j^{th} sector.

Question: μ_j^* and s_j^* are large on average, but how well do they overlap?

Lemma: Let $J \subseteq [0,1)$ be an arbitrary interval of length λ, and let $b_1, b_2, \ldots, b_q$ be integers. Let $N(\alpha, J) = |\{j : \{b_j \alpha\} \in J, 1 \leq j \leq q\}|$. If $q \geq \lambda^{-6}$, then

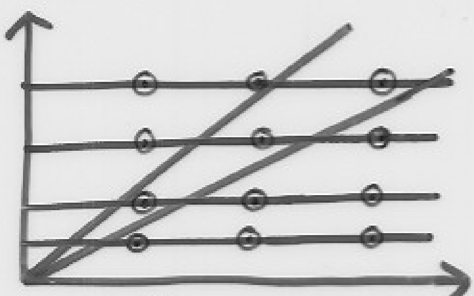
$$\mu(\alpha \in [0,1] : N(\alpha, J) \geq 1) \geq 1 - \frac{1}{\sqrt{q}} \quad \text{(Beck '86)}$$

Fact: Under the same hypotheses,

$$\mu\left(\alpha \in [0,1] : N(\alpha, J) \geq \frac{q\lambda}{2}\right) \geq 1 - \frac{8}{\sqrt{q}}$$

Upshot: s_j^* is concentrated around its mean.

We confine our attention to "rich" sectors ($\mu_j^* > \frac{\delta}{2}$)



The measure of α with fewer than half the expected number of switch points is small.

So "most" rich sectors contain "enough" switch points.

Can be shown: There are at most $\frac{n^2}{4\beta}$ bad switch points.

(No lattice point: Typical ; Two or more: Quite rare)

Putting it all together

$$E(\langle H, G_i\rangle) \geqslant \frac{c_1 nq}{\beta}$$

$$E(\langle H, G_i\rangle^2) \geqslant [E(\langle H, G_i\rangle)]^2 \geqslant \frac{c_1^2 n^2 q^2}{\beta^2}$$

$$\|G_i\| \leqslant \frac{nq}{2\beta}$$

$$E(\langle H, g_i\rangle^2) \geqslant \frac{c_1^2 nq}{2\beta} = \Omega\left(\frac{n^2}{(\log n)^{3/5}}\right)$$

$$\|H\|^2 \geqslant \frac{E\left(\sum_{i=1}^{r}\langle H, g_i\rangle^2\right)}{(3n^2/32)} = \Omega\left((\log n)^{2/5}\right)$$

$\|H\| > (\log n)^{1/5}$ for sufficiently large n; a contradiction.

Open Questions

→ Better lower bound?

→ Unbounded discrepancy for all but countably many α?

→ Upper bound?