BIPLANAR CROSSING NUMBERS

ÈVA CZABARKA

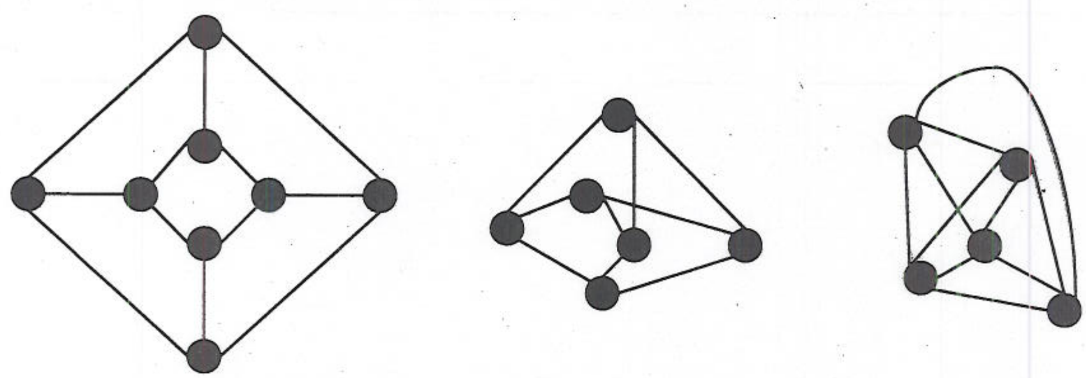
ONDREJ SÝKORA

LÁSZLÓ SZÉKELY

IMRICH VŔTO

Planar graphs:

- Vertices ~ points in the plane
- Edges ~ curves connecting the two endpoints of the edge
- Edges do not cross each other
- Edges do not go through other points of the graph than their endpoints.

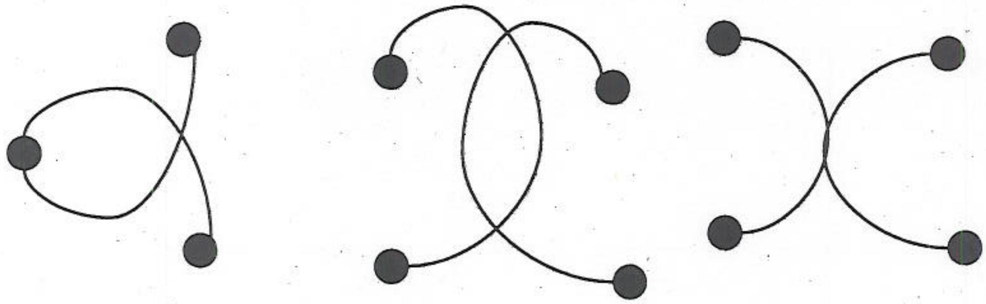


Crossing number: cr(G) (Turán, 1942)
The smallest possible number of edge-crossings in a drawing of the graph G in the plane.

$$G \text{ is planar} \leftrightarrow \mathrm{cr}(G) = 0$$

Facts about crossing number:

- cr(G) does not change if the following are forbidden:



- From Euler's Formula:

$$e > 3n - 6 \quad \rightarrow \quad \text{cr}(G) \geq 1$$

iteration yields:

$$\text{cr}(G) \geq e - 3n + 6$$

- Lemma (Ajtai, Chvátal, Newborn, Szemerédi 1982; Leighton 1983):

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{e^3}{n^2} \quad \text{or} \quad e \leq 4n \,.$$

$cr_2(G)$ = BIPLANAR CROSSING NUMBER

$$= \min_{G_1 \cup G_2 = G} cr(G_1) + cr(G_2)$$ (OWENS)

G BIPLANAR: $cr_2(G) = 0$ (or $\theta(G)=2$)
(BEINEKE)

IS G PLANAR? $\in$ P

IS $cr(G) \leq k$? $\in$ NP-COMPLETE

IS G BIPLANAR? $\in$ NP-COMPLETE (MANSFIELD)

HENCE

IS $\theta(G) \leq k$ $\in$ NP-COMPLETE

IS $cr_2(G) \leq k$ $\in$ NP-COMPLETE

BIPLANARITY, $cr_2(G)$

NOT SUBDIVISION INVARIANT (TUTTE)

$cr_2(K_{5,n})$ exactly

$cr_2(K_{p,q})$ CONJECTURED FOR SOME p,q (CSSV)

FOR $p > \frac{c_0}{n}$

$$cr_2(G(n,p)) > c_1(n^2p)^2 \quad \text{(SPENCER*)}$$

---

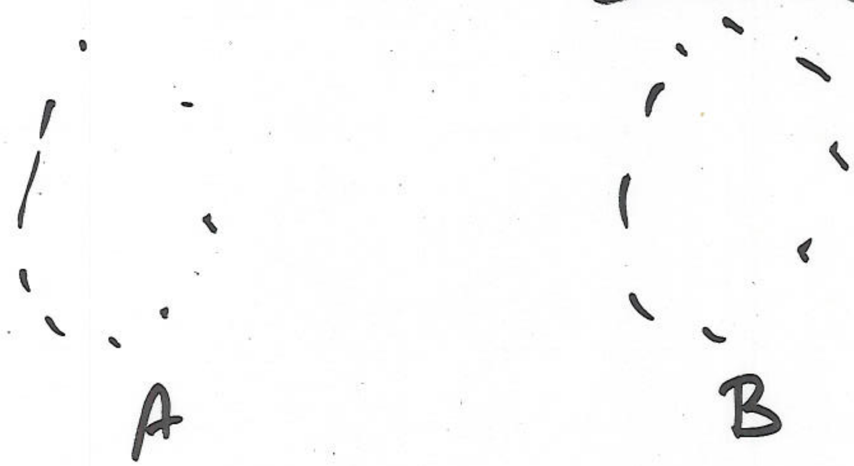
* HAPPY BIRTHDAY TO JOEL SPENCER!

3/8 THEOREM (SPLITTING THEOREM)

$$cr_2(G) \le \frac{3}{8}\, cr(G)$$

PROOF RANDOM BIPARTITION OF $V(G)$

"SPLITTING"



PLANE 2:

AB EDGES

PLANE 1:

AA AND BB EDGES – PULL APART VERTEX SETS

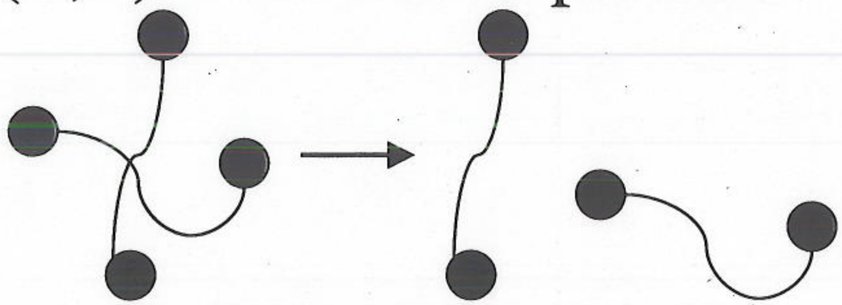
PROBLEM WHAT IS

$$\inf\{c>0 : \forall G \ cr_2(G) \le c \cdot cr(G)\} \geq \frac{1}{36}$$

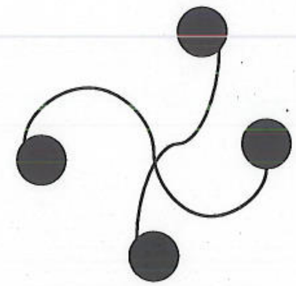
PERHAPS $\frac{7}{24}$? (OWENS, 1970)

D – drawing of G realizing $\mathrm{cr}(G)$

(A,B) = random 2-partition of $V(G)$



plane 1



plane 2

$$\mathbb{E}(\mathrm{cr}(D_1)) = \frac{1}{8}\mathrm{cr}(D) \qquad \mathbb{E}(\mathrm{cr}(D_2)) = \frac{1}{4}\mathrm{cr}(D)$$

(pull apart classes!)

e,f non-adjacent edges:

$$X_{ef} = \begin{cases} 1 & e,f \text{ cross in } D_1 \\ 0 & \text{otherwise} \end{cases} \qquad Y_{ef} = \begin{cases} 1 & e,f \text{ cross in } D_2 \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum X_{ef} \quad \mathbb{E}(X) = \frac{\mathrm{cr}(G)}{8} \qquad Y = \sum Y_{ef} \quad \mathbb{E}(Y) = \frac{\mathrm{cr}(G)}{4}$$

THEOREM $\exists\, c_1, c_2 > 0,\ k_1, n_1$ such that

$\forall\, n \geq n_1\ \forall\, m \geq k_1 n\ \exists\, G$ order n size m

$$cr(G) \geq c_1 m^2$$

$$cr_2(G) \leq c_2 \frac{m^3}{n^2}$$

THEOREM x_0 BE REAL ROOT OF $x^3 = x+1$

$\forall \gamma > \frac{\ln x_0}{\ln 2} \approx .4057$

$$\Theta(G) - 2 = O\left(\operatorname{cr}_2(G)^{\gamma} \log n \right)$$

AND SUCH A DRAWING CAN BE OBTAINED BY A RANDOMIZED ALGORITHM WITH PROB $> 1 - \delta$ IN TIME

$$\operatorname{POLY}\left(\ln \frac{1}{\delta},\ n(G)\right)$$

PROBLEM DOES .4057 GO DOWN TO .25 ?

RELATED:

$\Theta(G) \le \lfloor \sqrt{\frac{m}{3}} + \frac{7}{6} \rfloor$ DEAN-HUTCHINSON-SCHEINERMAN

$\Theta(G) \le \lceil \frac{\Delta}{2} \rceil$ WESSEL, HALTON

$\Theta(G) = O(\sqrt{\operatorname{genus}(G)})$ MALITZ

$$X_{e,f} = \begin{cases} 1 \Longleftrightarrow \text{non-adjacent edges } e,f \\ \quad \text{cross in the } 1^{st} \text{ plane} \\ 0 \Longleftrightarrow \text{otherwise} \end{cases}$$

$$Y_{e,f} = \qquad \text{on } 2^{nd} \text{ plane}$$

$$X = \sum_{\{e,f\}} X_{e,f} \qquad Y = \sum_{\{e,f\}} Y_{e,f}$$

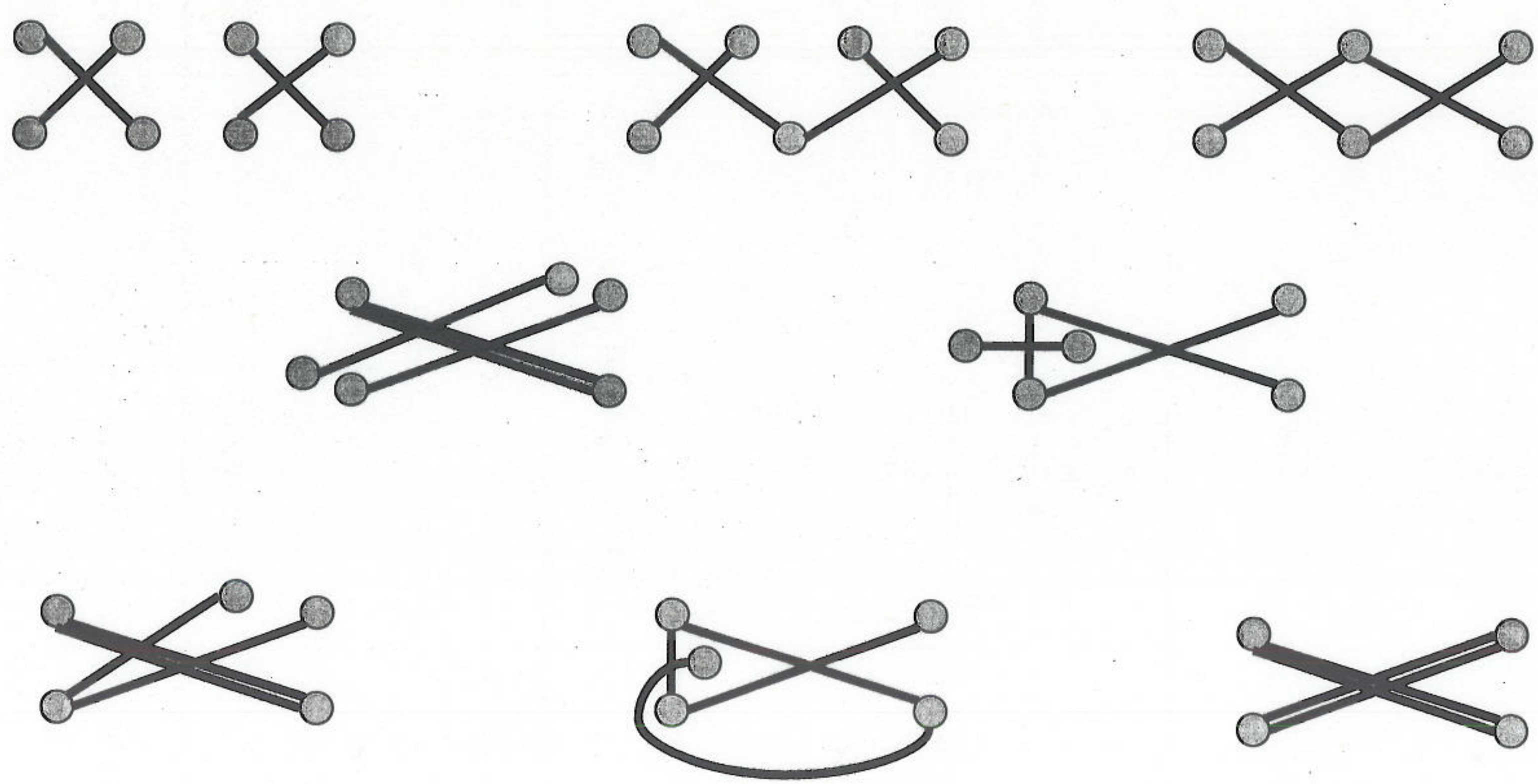
$$E(X) = \frac{cr(D)}{8} \qquad E(Y) = \frac{cr(D)}{4}$$

$$\sigma^2(Y) = \sum_{\{e,f\}} \sum_{\{a,b\}} E(Y_{e,f} Y_{a,b}) - E(Y_{e,f}) E(Y_{a,b})$$

$$\sigma^2(Y) \leq cr(D)(2m + 12n + 3)$$

Lemma: $\sigma^2(Y) \le (2m + 12n + 3) \cdot \mathrm{cr}(D)$

Proof: $\sigma^2(Y) = \sum_{\{e,f\}} \sum_{\{a,b\}} \left\{ \mathbb{E}(Y_{ef} Y_{ab}) - \mathbb{E}(Y_{ef}) \mathbb{E}(Y_{ab}) \right\}$



CASE 1. IF $cr(D) > \frac{100}{\varepsilon^3}(2m + 12n + 3)$

THEN RUN ($\leq N$) TIMES SPLITTING

IF SUCCEED : OUTPUT 2 DRAWINGS

OTERWISE : OUTPUT "FAIL"

CASE 2. IF $cr(D) \leq \frac{100}{\varepsilon^3}(2m + 12n + 3)$

THEN THROW WITH GREEDY ALG.

INTO $116 \cdot \frac{100}{\varepsilon^3}$ PLANES NONCROSSING EDGES

ELIMINATE ISOLATED VERTICES

FROM THE REST, AND RUN THE

PROCEDURE ON IT

MAIN ALGORITHM:

ITERATE THE PROCEDURE

LEMMA $\varepsilon > 0$ SMALL ($< 1/3$)

$$cr(D) > \frac{100}{\varepsilon^3}(2m + 12n + 3)$$

$$P\left[X \leq (1+\varepsilon)E(X) \wedge Y \leq (1+\varepsilon)E(Y)\right] \geq \frac{\varepsilon}{2}$$

PROCEDURE (N, ε GIVEN)

INPUT: NICE DRAWING IN ONE PLANE (D)
SIZE m, ORDER n, $cr(D)$ CROSSINGS

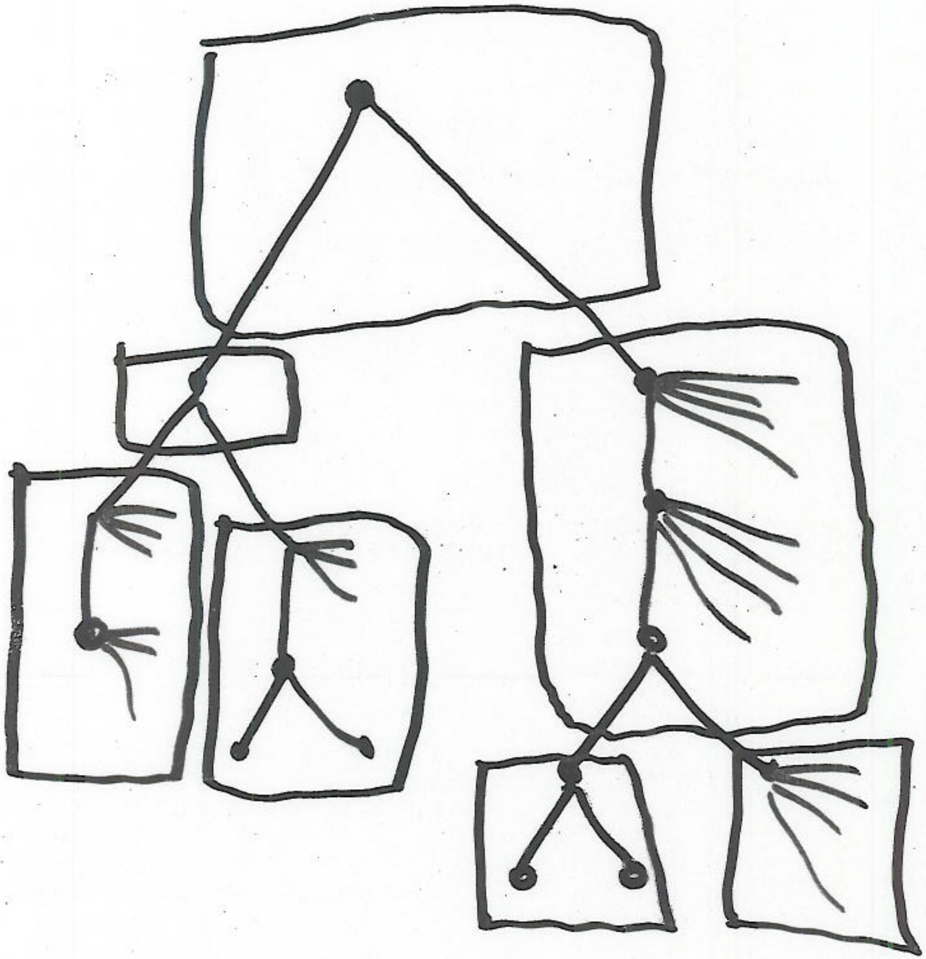
OUTPUT: DRAWINGS D_1, D_2 ON 2 PLANES

$$cr(D_1) < \left(\frac{1}{8} + \varepsilon\right) cr(D)$$

$$cr(D_2) < \left(\frac{1}{4} + \varepsilon\right) cr(D)$$

AND SOME PLANAR
DRAWINGS (NO CROSSING)

OR
FAIL



A RUN OF PROCEDURE

CASE 1

CASE 2

THEOREM $\exists\, c_1, c_2 > 0,\ k_1, n_1$ such that

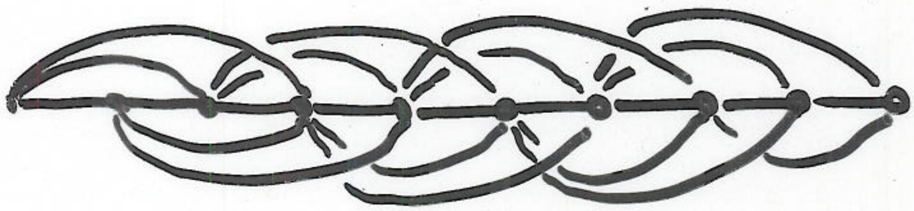
$\forall n \geq n_1\ \forall m \geq k_1 n\ \exists G$ order n size m

$$cr(G) \geq c_1 m^2$$

$$cr_2(G) \leq c_2 \frac{m^3}{n^2}$$

NOTE: TRIVIAL FOR $m \geq \frac{n^2}{20}$

DEF: $H(n,k)$ — n-path
— EXTRA EDGES FOR DISTANCE $\leq k-1$



$H(10,4)$

DEF $\Pi(H_1, H_2)$

$V = V(H_1) = V(H_2)$, H_1, H_2 GRAPHS

$$\Pi(H_1, H_2) = E(H_1) \cup E(H_2)^{\pi}$$

π RANDOM PERMUTATION OF V

MAIN LEMMA.

$\exists k_0 \; \exists n_0 \quad \forall n \geq n_0 \quad \forall k_0 \leq k \leq \frac{n}{3}$

$G^* = \Pi\left(H(n,k), H(n,k)\right)$ with prob. $1-o(1)$

$cr(G^*) \geq c_3 \left(m(G^*)\right)^2$

$cr_2(G^*) \leq c_4 \dfrac{m(G^*)^3}{n^2}$

PROOF OF THEOREM

m IS TARGET SIZE

largest k : $2nk < m$

G^* with fixed π from lemma

G^{***} $\Pi\left(H(n,3k), H(n,3k)\right)$ with same π

$\exists\, G^{**}$ of size m $\quad G^* < G^{**} < G^{***}$

PROBLEM What about other graphs than $H(n,k)$?

LEMMA $k \geq 4$ → PROVES CLAIM ON cr_2

$$cr(H(n,k)) \leq 48 \frac{m(H)^3}{n^2}$$

PROOF: CIRCULAR DRAWING

DEF $H_1, H_2, \ldots, H_k \subseteq H(n,k)$

H_i = EDGES GOING UP FROM $i + zk$ $(z = 1, 2, \ldots)$

$H(10,4)$

H_2

Lemma
- H_i PARTITIONS $E(H(n,k))$
- H_i IS UNION OF w_i STARS

$$\frac{n}{k} - 1 \leq w_i \leq \frac{n}{k} + 1$$

- H_i HAS MAXDEG $\leq k-1$
- $(n-k)(k-1) < m(H(n,k)) < n(k-1)$

DEF $c(a, s, G) = \left|\left\{A \subseteq V(G) : |A| = a, |E(A, V-A)| \le s\right\}\right|$

#CUTS: 

LEMMA $\exists \varepsilon > 0 \ \forall n \geq n_2$

$\forall a \quad \frac{n}{3} \le a \le \frac{n}{2}$

$\forall i \quad 1 \le i \le k \qquad \left(k_0 \le k \le \frac{n}{3}\right)$

$$c(a, \varepsilon n, H_i) < 1.1^n$$

LEMMA $\forall n \geq n_3 \ \forall \frac{1}{3} \le c \le \frac{2}{3}$, cn INTEGER

$$\frac{1}{n}\binom{n}{cn} > 1.5^n$$

Bisection width lower bound

$$E(A,B) = \#\text{edges betwen } A, B$$

$$b(G) = \min_{\substack{A \cup B = V \\ |A|,|B| \geq \frac{n}{3}}} E(A,B)$$

Theorem (Leighton 1982; Sykora-Vrto 1993; Pach-Shahroki-Szegedi 1994):

$$b(G) \leq 10\sqrt{\mathrm{cr}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}$$

LEMMA G_1, G_2 GRAPHS ON SAME n VERTICES

$\exists s \ \forall a \ \frac{n}{3} \leq a \leq \frac{n}{2}$

$$c(a, s, G_1) \cdot c(a, s, G_2) \leq \frac{g(n)}{n} \binom{n}{a}$$

$\Rightarrow b(\Pi(G_1, G_2)) \geq s$ with prob. $\geq 1 - g(n)$

PROOF $\quad c_i(a) = c(a, s, G_i)$

$$\mathbb{P}[b \leq s] \leq \sum_{a=n/3}^{n/2} c_1(a) \frac{c_2(a)\, a!\,(n-a)!}{n!} =$$

$$= \sum_{a=n/3}^{n/2} \frac{c_1(a)\, c_2(a)}{\binom{n}{a}} \leq g(n)$$

LEMMA FIX $\varepsilon > 0$ ACCORDING TO 1.1^n LEMMA.

$$G^* = \pi[H(n,k), H(n,k)]$$

$b(G^*) \geq \varepsilon n k$ HOLDS WITH PROB $\geq 1 - \left(\frac{1.21}{1.5}\right)^{\frac{n}{k^2}}$

PROOF:

LET $s = \varepsilon n k$

a arbitrary $\frac{n}{3} \leq a \leq \frac{n}{2}$

ASSUME $A, V-A$ $< s$-CUT in G^*

$< s$-CUT IN $H(n,k)$ AND $H(n,k)^\pi$

$< \frac{s}{k} = \varepsilon n$ CUT IN H_i AND H_j^π

$$C^2(a,s,H) < (k \cdot 1.1^n)^2 < \left(\frac{1.21\, k^{2/n}}{1.5}\right)^n \frac{1}{n}\binom{n}{a}$$

EARLIER LEMMAS FINISH THIS ONE.

BISECTION WIDTH LOWER BOUND FINISHES THE MAIN LEMMA