On the chromatic number of a random 5-regular graph

Graeme Kemkes
Nick Wormald
U. of Waterloo

Definitions

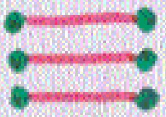
$\chi(G)$: chromatic number

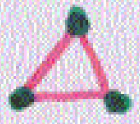
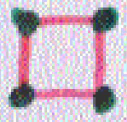
$\mathcal{G}_{n,d}$: n-vertex d-regular graphs

a.a.s.: $A \subseteq \mathcal{G}_{n,d}$ occurs a.a.s.
if $\frac{|A|}{|\mathcal{G}_{n,d}|} \to 1$ as $n \to \infty$.

$\mathbb{P}, \mathbb{E}$: probability, expectation

Examples

$d=1$  $\chi = 2$

$d=2$   $\chi = 3$ a.a.s.

odd cycle a.a.s.

$d=3$ odd cycle a.a.s. $\chi = 3$ a.a.s.

no K_4 a.a.s.

Previous Work

Achlioptas & Moore: $k = k(d)$

χ is a.a.s. k, $k+1$, or $k+2$.

e.g. $d = 4$: $\chi = 3$ or 4 a.a.s.

$d = 5$: $\chi = 3, 4,$ or 5 a.a.s.

e.g. Shi & Wormald:

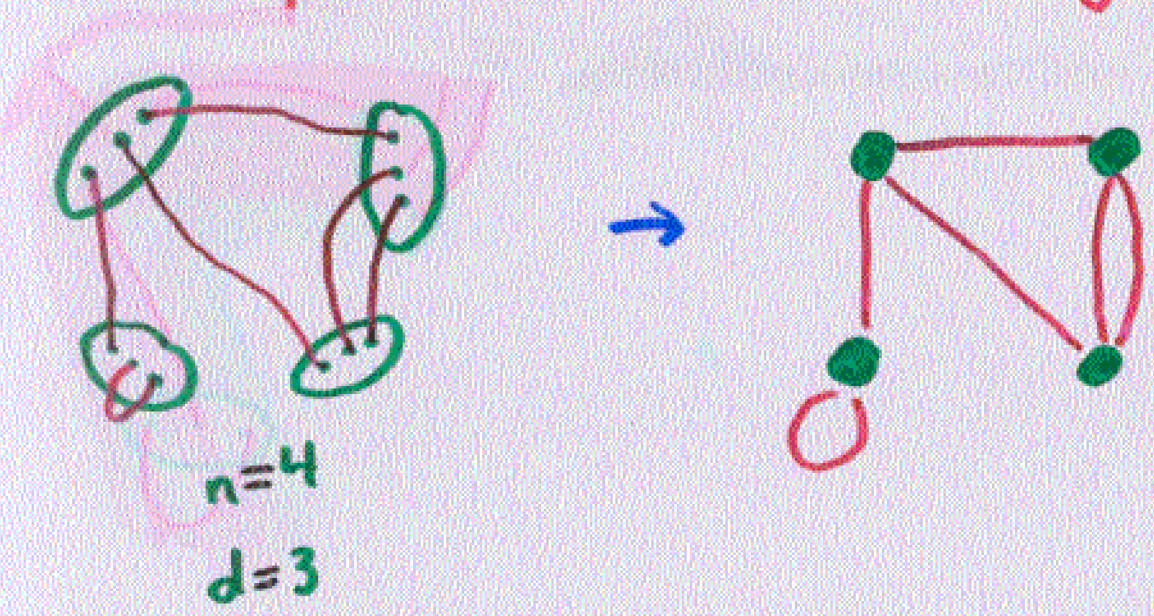
$d = 4$: $\chi = 3$ a.a.s.

$d = 5$: $\chi = 3$ or 4 a.a.s.

Pairing Model

n groups of d points

$P_{n,d}$: perfect matchings



→ random d-regular multigraphs

Second Moment

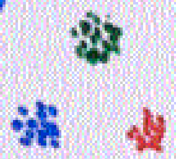
$X \geqslant 0$ random variable

$$\mathbb{P}[X>0] \geqslant \frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]}$$

5

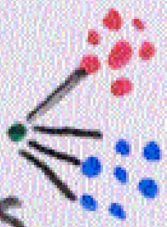
Balanced Colorings

equal-sized classes



Stable Colorings

every vertex is adjacent to vertices of all other colors.



(Díaz et al.)

$$\frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]} \overset{?}{\sim} \frac{7^6\, 11^4\, 79^2 \sqrt{13 \cdot 17}}{3^{13}\, 5^{13}} \approx 0.082$$

Small Subgraph Conditioning

(Robinson & Wormald)

$Y \geqslant 0$ random variable on $\mathbb{P}_{n,d}$

X_k = number of k-cycles

If $\dfrac{\mathbb{E}\left(Y[X_1]_{m_1} \cdots [X_j]_{m_j}\right)}{\mathbb{E}Y} \to \prod_{k=1}^{j} \left(\dfrac{(d-1)^k}{2k}(1+\delta_k)\right)^{m_k}$,

$\dfrac{\mathbb{E}(Y^2)}{(\mathbb{E}Y)^2} \leqslant \exp\left(\sum_k \dfrac{(d-1)^k}{2k} \delta_k^2\right) + o(1) < \infty$

then $\mathbb{P}\left[Y > 0 \,\middle|\, \bigwedge_{\delta_k = -1} \{X_k = 0\}\right] \to 1$.

Our Application

Y = number of stable balanced 3-colorings in $\mathcal{P}_{n,5}$

If $$\frac{\mathbb{E}(Y^2)}{(\mathbb{E}Y)^2} \le \frac{3^{13}\,5^{13}}{7^6\,11^4\,79^2\sqrt{221}}$$

then $\chi(\mathcal{G}_{n,5}) = 3$ a.a.s.

Díaz et al. report numerical evidence of this inequality!