
Real Number Labellings
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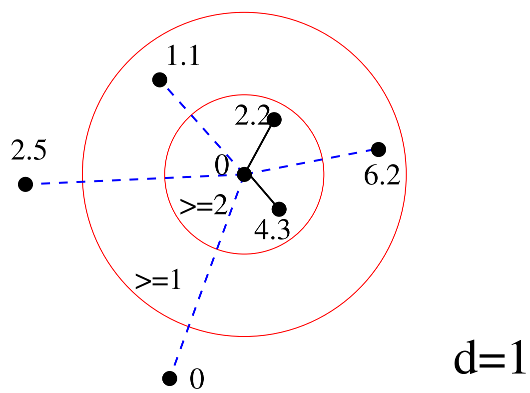
A Channel Assignment Problem [F. Roberts, 1988]

Find an efficient assignment of channels $f(x) \in \mathbb{R}$ to sites $x \in \mathbb{R}^2$ so that two levels of interference are avoided:

$$|f(x) - f(y)| \geq \begin{cases} 2d & \text{if } \|x-y\| \leq A \\ d & \text{if } \|x-y\| \leq 2A \end{cases}$$



We must minimize $\text{span}(f) := \max_x f(x) - \min_x f(x)$.

We consider the analogous problem for graphs $G = (V, E)$ [G., 1989]. The problem can be reduced to the case $d = 1$ and labelings $f : V \to \{0, 1, 2, \ldots\}$ such that

$$|f(x) - f(y)| \geq \begin{cases} 2 & \text{if } \mathrm{dist}(x, y) = 1 \\ 1 & \text{if } \mathrm{dist}(x, y) = 2 \end{cases}$$

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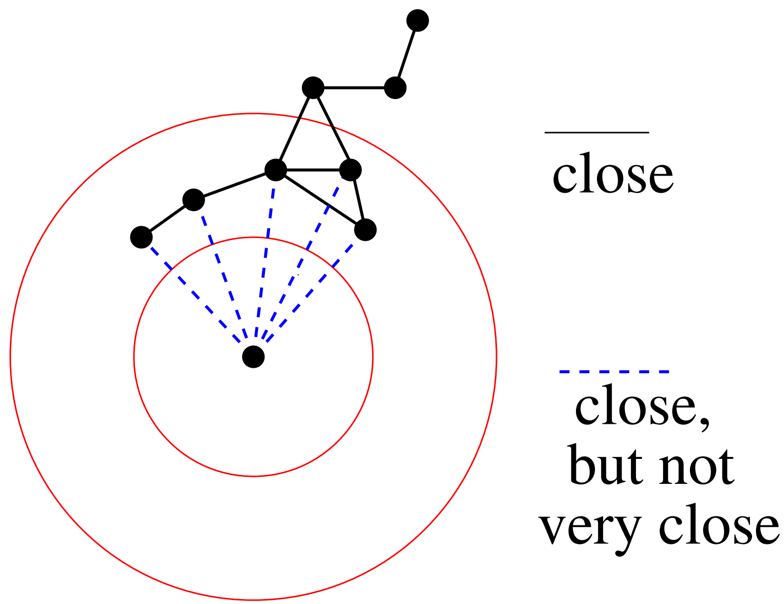
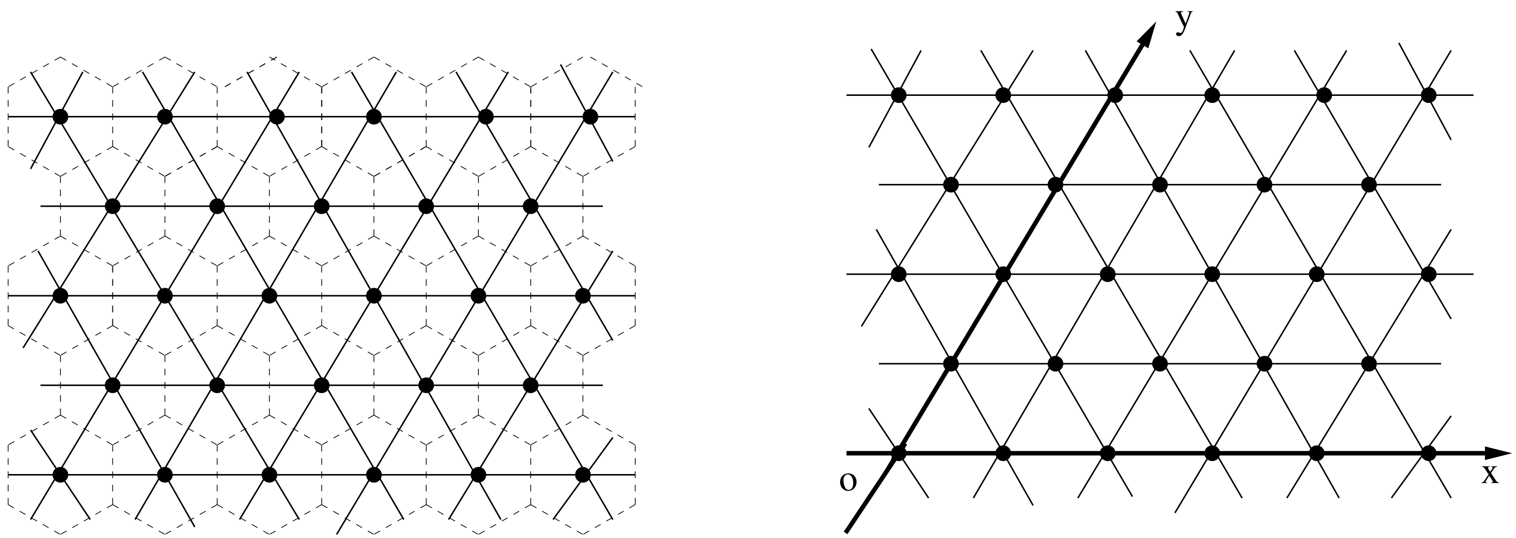
Such an f is called a **λ-labeling** and $\lambda(G)$:=$\min_f \mathrm{span}(f)$.

The graph problem differs from the “real” one when putting vertices $u \sim v$ corresponding to “very close” locations u, v.





A Network of Transmitters with a Hexagonal Cell Covering
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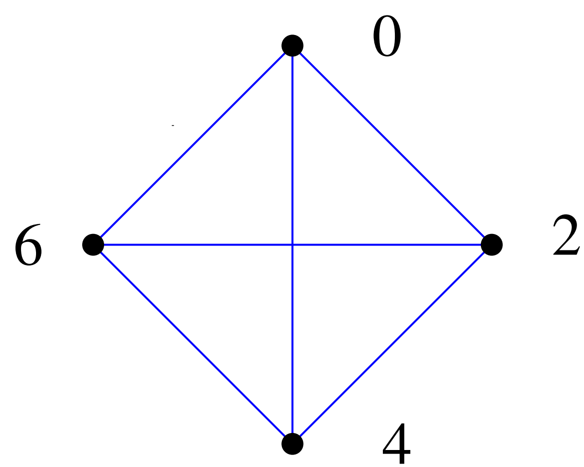
Complete Graphs K_n.



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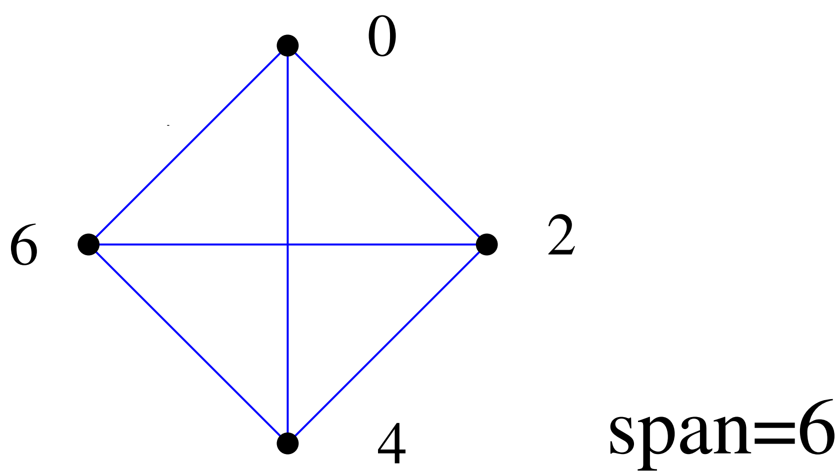
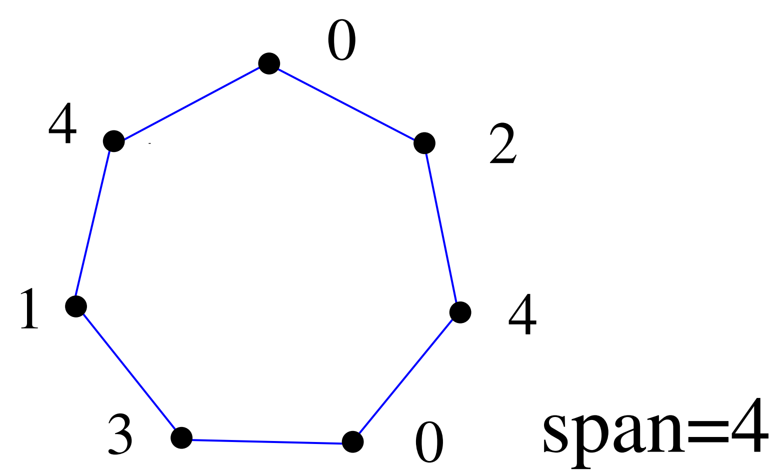


$$\lambda(K_n) = 2n - 2$$

Cycles C_n.



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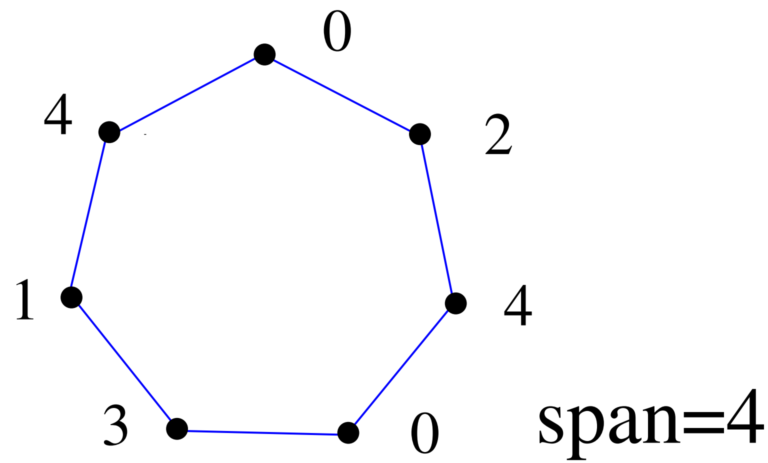

$$\lambda(C_n) = 4 \text{ for } n \geq 3.$$

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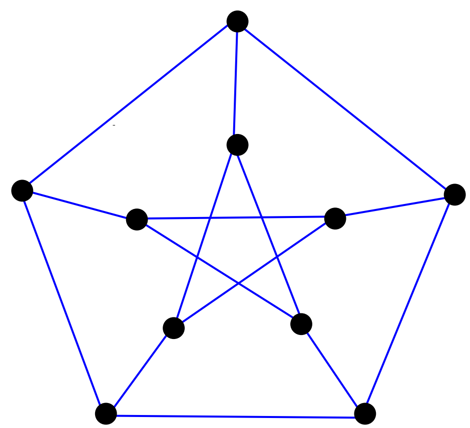
Example Petersen Graph.



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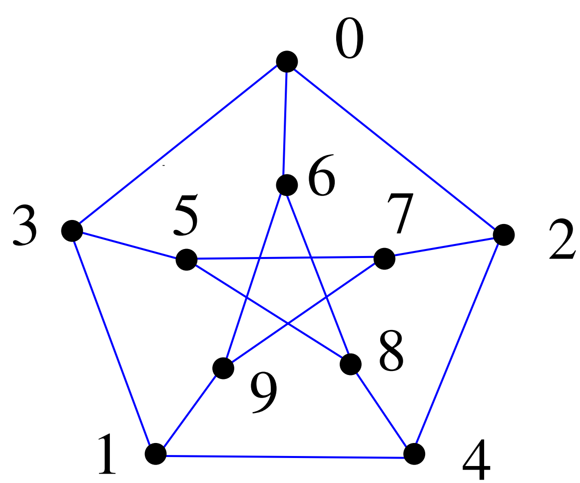

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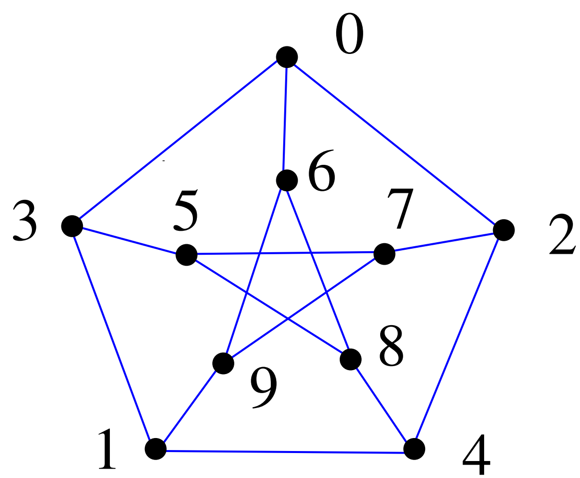
<u>Example</u> Petersen Graph. $\lambda = 9$.



Conjecture. $\Delta = 3 \implies \lambda \leq 9$.

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More generally, we have the

Δ^2 Conjecture. [G.-Yeh, 1989]

For all graphs of maximum degree $\Delta \ge 2$,

$$\lambda(G) \le \Delta^2.$$

Results. Δ-Bounds on λ:

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- In particular, $\lambda \leq 10$ for $\Delta = 3$

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Kang verified $\lambda \leq 9$ when G is cubic and Hamiltonian.

Among many results verifying the conjecture for special classes of graphs, we have

Theorem [G-Yeh, 1992].

For graphs G of diameter 2, $\lambda \leq \Delta^2$,

and this is sharp iff $\Delta = 2, 3, 7, 57(?)$.

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[Fiala, Kloks, and Kratochvíl, 2001] Fix k. Is $\lambda = k$?

Polynomial: $k \leq 3$.

NP-Complete: $k \geq 4$. via homomorphisms to multigraphs.

Trees. Let $\Delta :=$ maximum degree ($= A$ in Figures).

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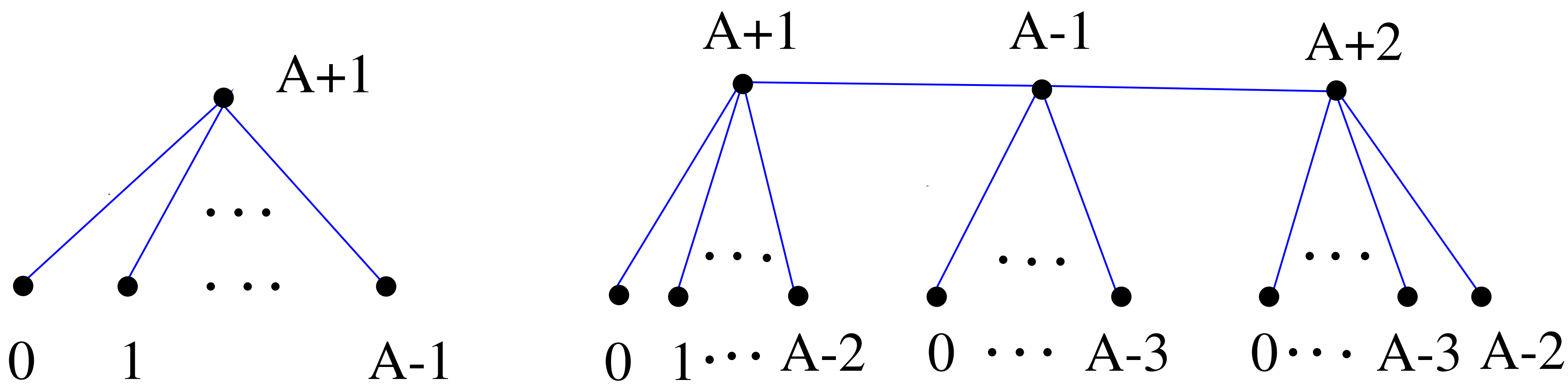
Example. $\lambda = \Delta + 1$ (left) and $\lambda = \Delta + 2$ (right).



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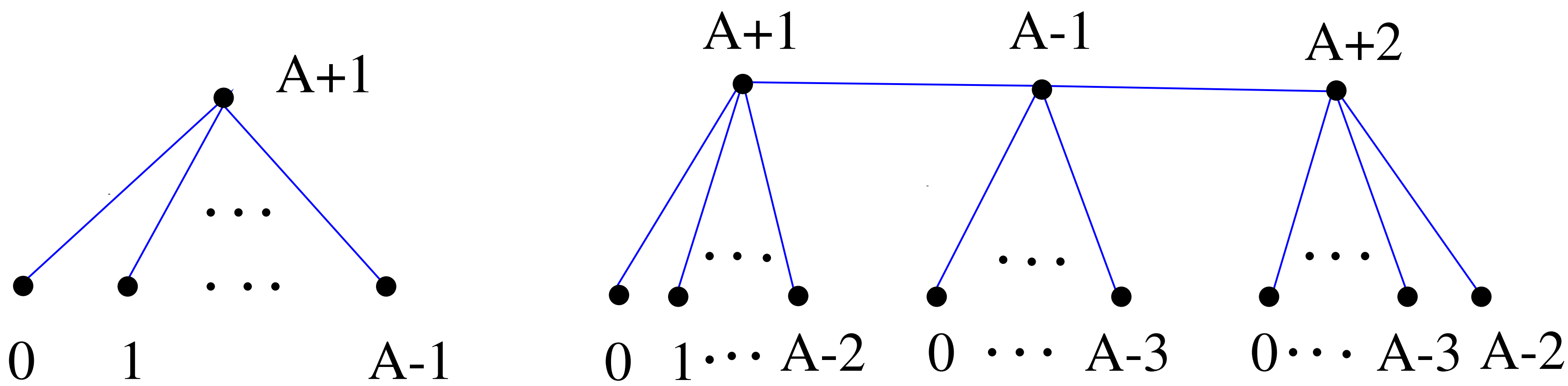

Theorem [Yeh, 1992]. For a tree T, $\lambda(T) = \Delta + 1$ or $\Delta + 2$.

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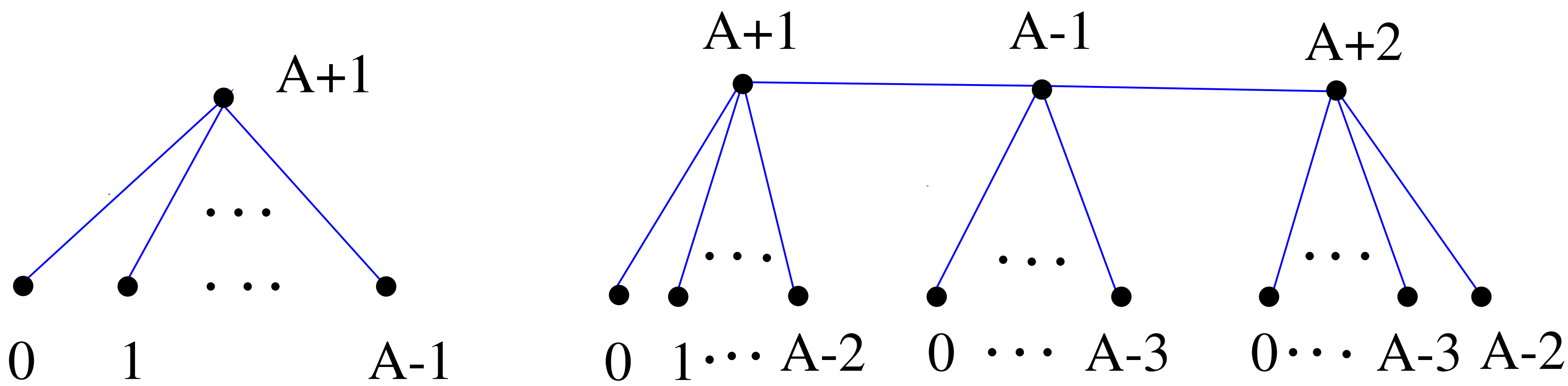

Theorem [Yeh, 1992]. For a tree T, $\lambda(T) = \Delta + 1$ or $\Delta + 2$.

It is difficult to determine which, though there is a polynomial algorithm [Chang-Kuo 1995].

General Version [G. 1992].

Integer $L(k_1, k_2, \cdots, k_p)$-labelings of a graph G:

- $k_1, k_2, \ldots, k_p \geq 0$ are integers.
- A labeling f: vertex set $V(G) \rightarrow \{0, 1, 2, \ldots\}$ such that
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The minimum span $\lambda(G; k_1, k_2, \cdots, k_p) := \min_f \mathrm{span}(f)$.

More History of the Distance Labeling Problem

- Hale (1980) :
  Models radio channel assignment problems
  by graph theory.
- Georges, Mauro, Calamoneri, Sakai, Chang, Kuo, Liu, Jha, Klavzar, Vesel et al.
  investigate $L(2,1)$-labelings, and more general integer $L(k_1, k_2)$-labelings with $k_1 \geq k_2$.

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Let $\overrightarrow{k} = (k_1, \ldots, k_p)$ with each $k_i \geq 0$ real.
Given graph $G = (V, E)$, possibly infinite, define
$L(G; \overrightarrow{k})$ to be the set of labelings $f : V(G) \to [0, \infty)$ such that $|f(u) - f(v)| \geq k_d$ whenever $d = \text{dist}_G(u, v)$.

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$\lambda(G; k_1, k_2, \cdots, k_p) = \inf_{f \in L(G; \overrightarrow{k})} \mathrm{span}(f)$.

An advantage of the concept of real number labelings.

SCALING PROPERTY. For real numbers $d, k_i \geq 0$, $\lambda(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) = d \cdot \lambda(G; k_1, k_2, \ldots, k_p)$.

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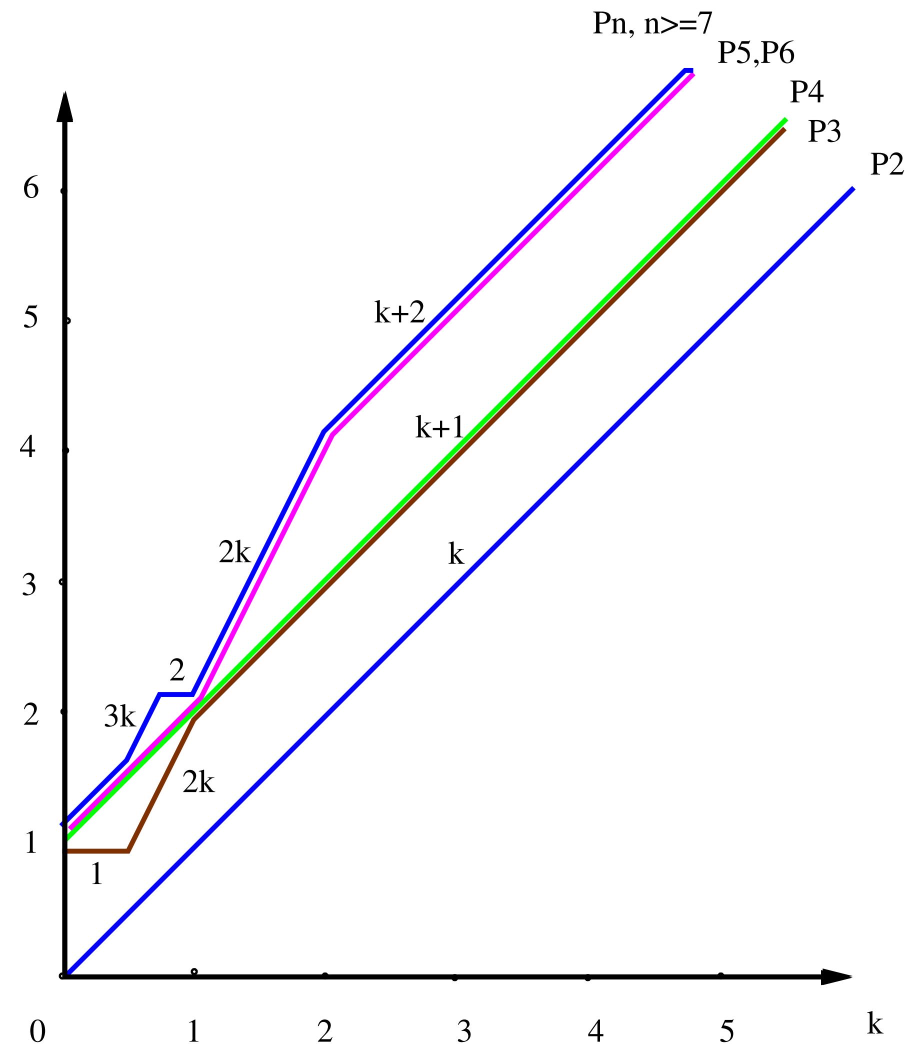
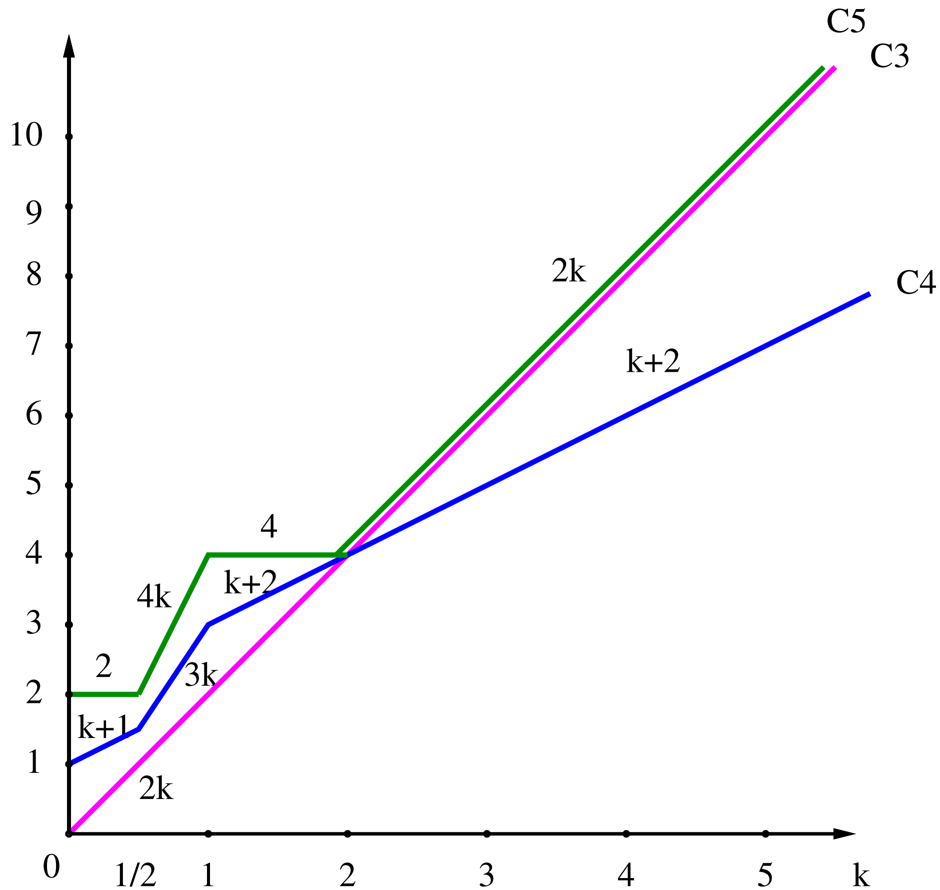
Example. $\lambda(G; k_1, k_2) = k_2 \lambda(G; k, 1)$
where $k = k_1/k_2$, $k_2 > 0$,
reduces it from two parameters k_1, k_2 to just one, k.

Theorem. [G-J; cf. Georges-Mauro 1995] For the path P_n on n vertices, we have the minimum span $\lambda(P_n; k, 1)$.

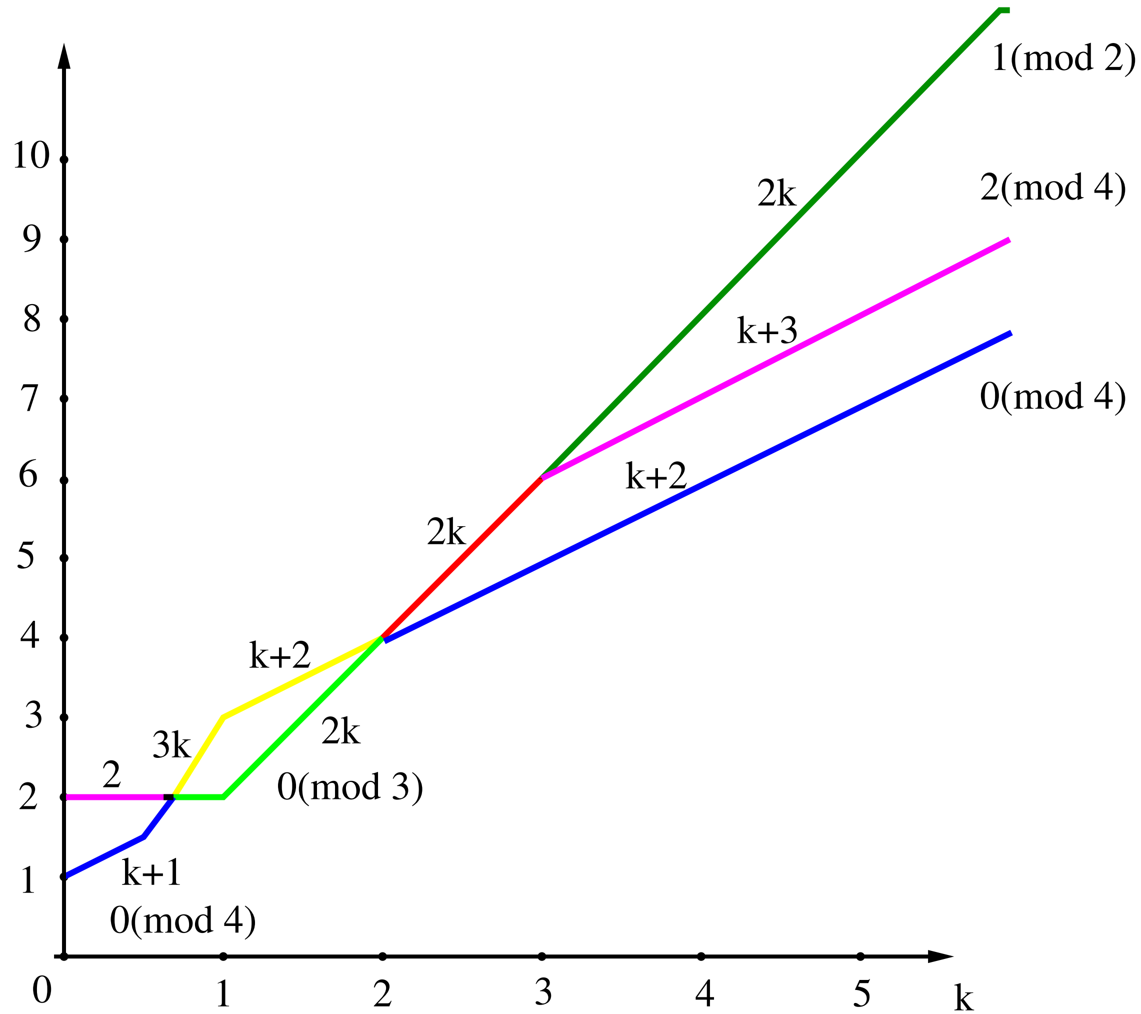


Theorem. [G-J; cf. Georges-Mauro 1995] For the cycle C_n on n vertices, we have the minimum span $\lambda(C_n; k, 1)$



$\lambda(C_n; k, 1), n = 3, 4, 5.$

(ctd.) The minimum span $\lambda(C_n; k, 1), n \geq 6$, depending on $n \pmod 3$ and $\pmod 4$.



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$$D_{k_1,k_2,\ldots,k_p} := \{\sum_{i=1}^{p} a_i k_i : a_i \in \{0, 1, 2, \ldots\}\}.$$

Hence, $\lambda(G; k_1, k_2, \ldots, k_p) \in D_{k_1,k_2,\ldots,k_p}$.

(ctd.) Moreover, if G is finite, each label of f^* is of the form $\sum_i a_i k_i$, where the coefficients $a_i \in \{0, 1, 2, \cdots\}$ *and* $\sum_i a_i < n$, the number of vertices.

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Corollary. If all k_i are integers, then $\lambda(G; k_1, k_2, \ldots, k_p)$ agrees with the former integer λ's.

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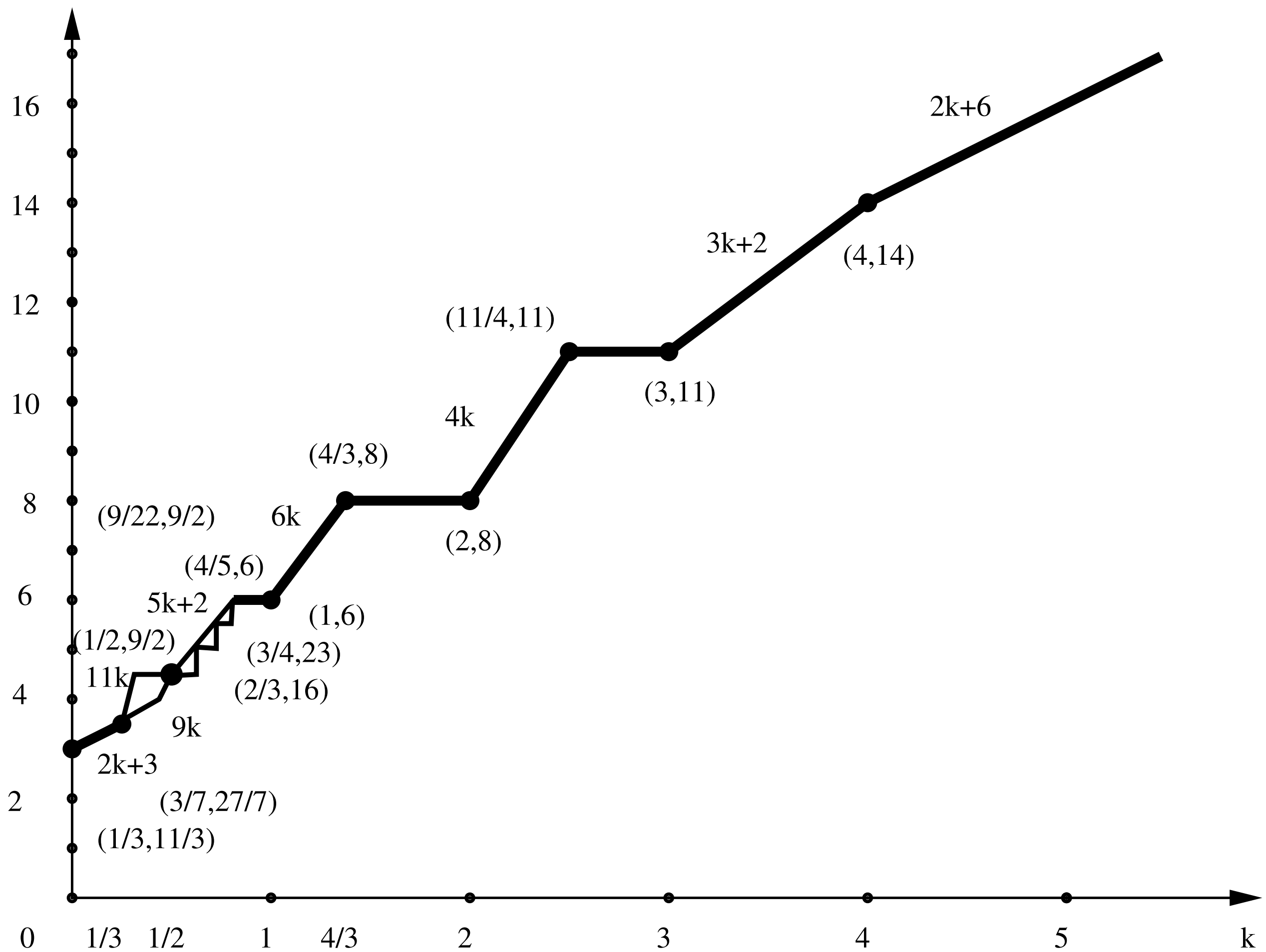
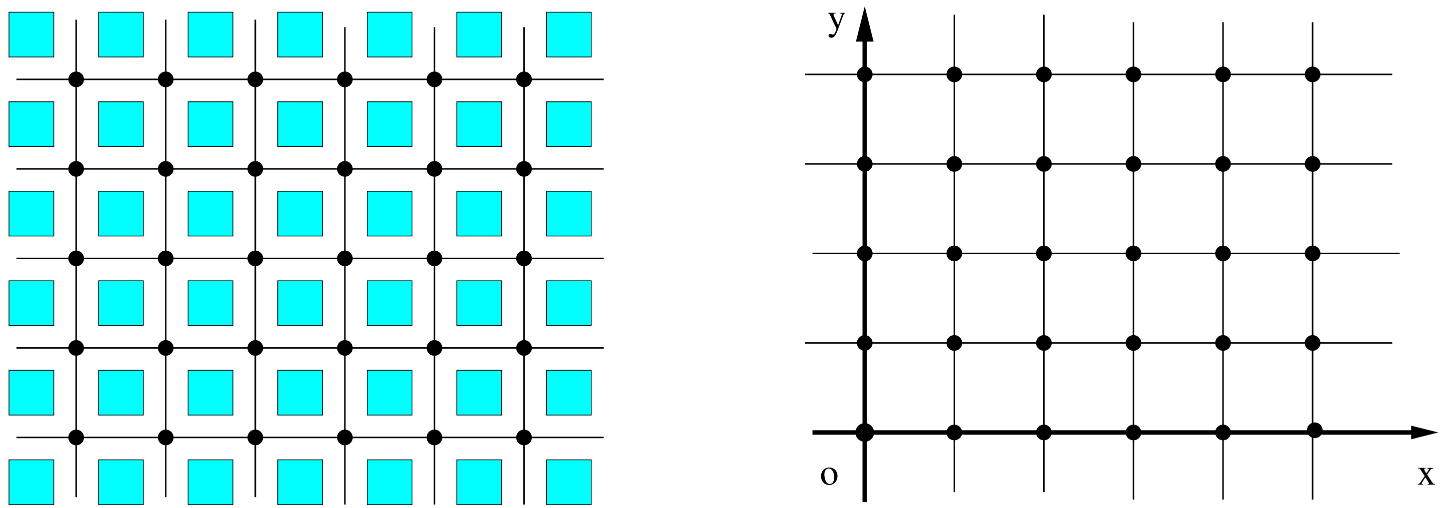
Note. The D-set Thm. allows us to ignore some labels. Example. For $(k_1, k_2) = (5, 3)$, it suffices to consider labels $f(v)$ in $D_{5,3} = \{0, 3, 5, 6, 8, 9, 10, \ldots\}$.

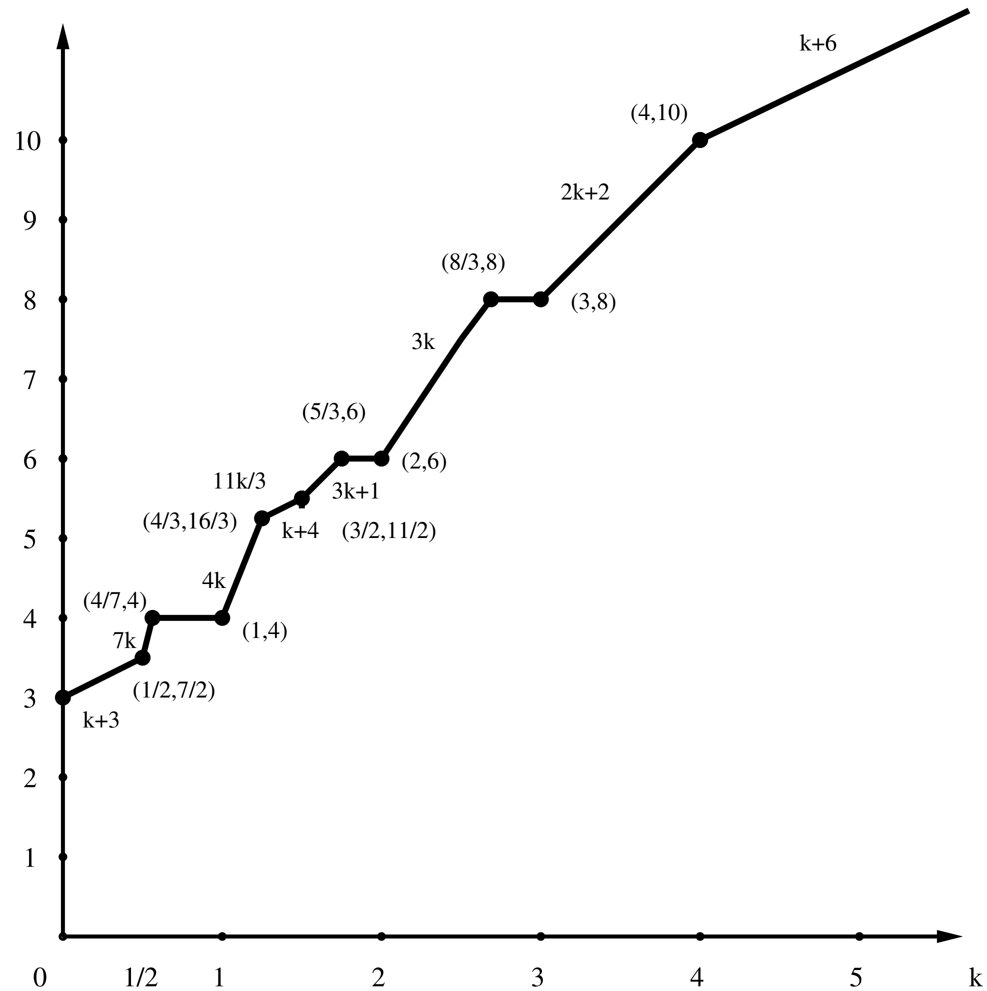
Theorem. For the triangular lattice we have $\lambda(\Gamma_{\triangle}; k, 1)$:

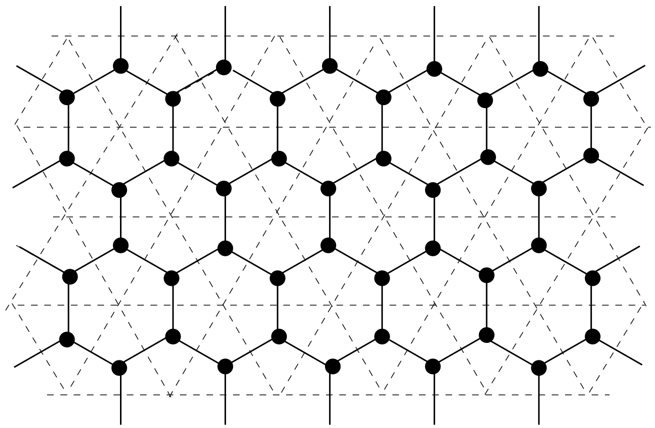


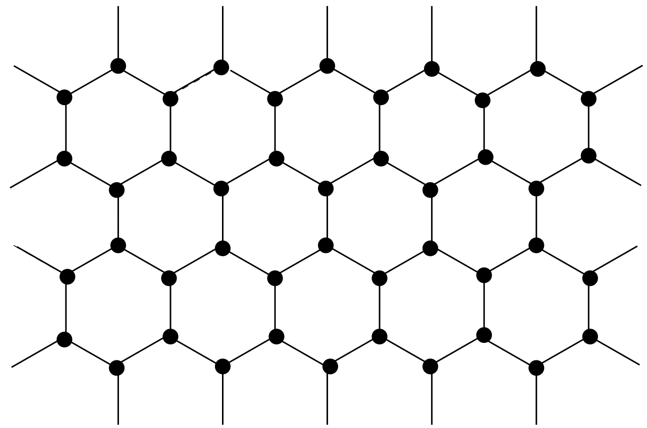


A Manhattan Network and the Square Lattice $\Gamma_{\Box}$

Theorem. For the square lattice we have $\lambda(\Gamma_\square; k, 1)$:

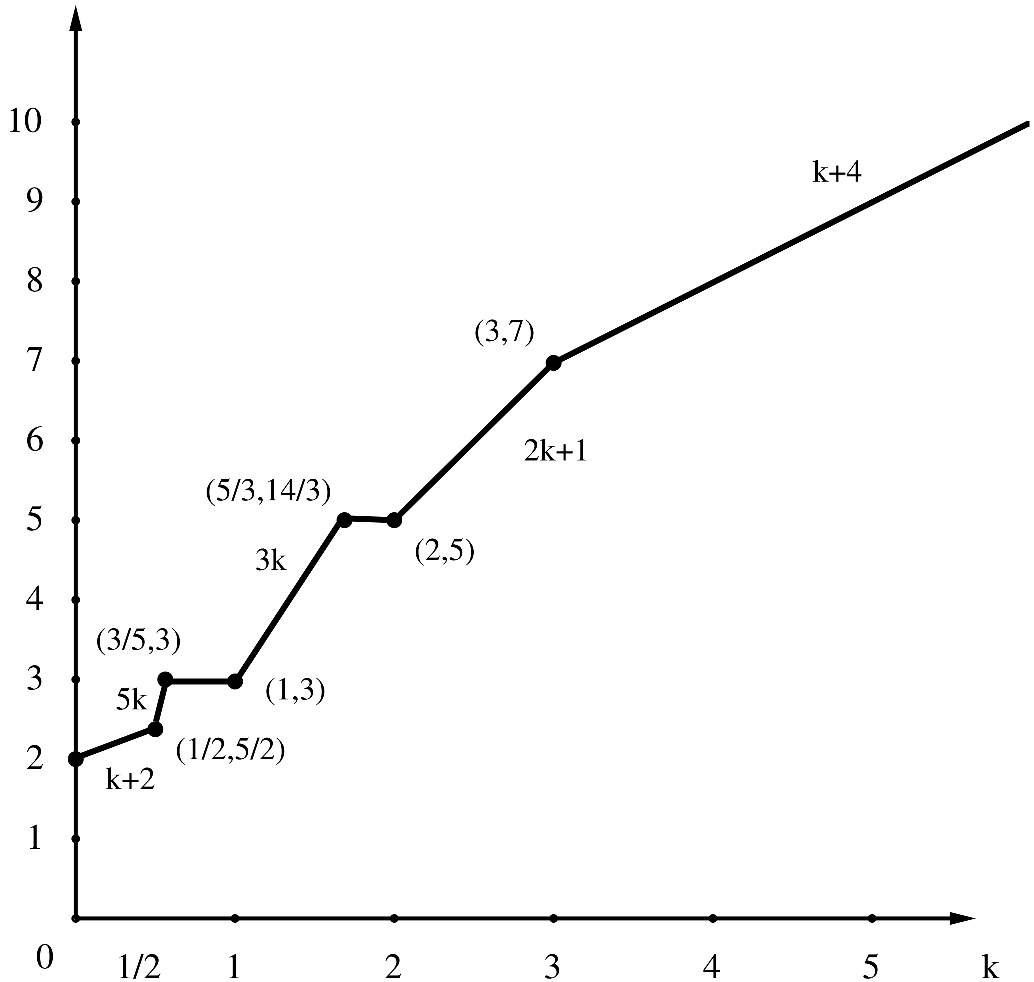






Equilateral Triangle Cell Covering and the Hexagonal Lattice Γ_H

Theorem. For the hexagonal lattice we have $\lambda(\Gamma_H; k, 1)$:



Piecewise Linearity

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PL Conjecture. For any integer $p \geq 1$ and any graph G of bounded maximum degree, $\lambda(G; \overrightarrow{k})$ is **PL**, i.e., continuous and piecewise-linear, with finitely many pieces as a function of $\overrightarrow{k} = (k_1, k_2, \ldots, k_p) \in [0, \infty)^p$.

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Finite Graph PL Theorem. For any integer $p \geq 1$ and any finite graph G, $\lambda(G; \overrightarrow{k})$ is PL.

Theorem ($p = 2$). For any graph G, possibly infinite, with finite maximum degree, $\lambda(G; k, 1)$ is a piecewise linear function of k with finitely many linear pieces.

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Moreover,

$$\lambda(G; k, 1) = \begin{cases} ak + \chi(G^2 - G) - 1 & \text{if } 0 \le k \le 1/\Delta^3 \\ (\chi(G) - 1)k + b & \text{if } k \ge \Delta^3 \end{cases}$$

for some constants $a, b \in \{0, 1, \ldots, \Delta^3 - 1\}$, where $G^2 - G$ is the graph on $V(G)$ in which edges join vertices that are at distance two in G.

We make the stronger

Delta Bound Conjecture For all p and Δ, there is a constant $c := c(\Delta, p)$ such that for all graphs G of maximum degree Δ and all $k_1, \ldots, k_p$, there is an optimal labeling $f \in L(k_1, \ldots, k_p)$ in which the smallest label is 0, all labels are in $D(k_1, \ldots, k_p)$ and of the form $\sum_i a_i k_i$ where all coefficients $a_i \leq c$.

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Theorem This holds for $p = 2$.

Lambda Graphs.

A more general model for graph labelling has been introduced recently by Babilon, Jelínek, Král', and Valtr. A λ-graph $G = (V, E)$ is a multigraph in which each edge is of one of p types. Given reals $k_1, \ldots, k_p \geq 0$, a labelling $f : V \to [0, \infty)$ is proper if for every edge $e \in E$, say it is type i, the labels at the ends of e differ by at least k_i.

The infimum of the spans of the proper labellings of G is denoted by $\lambda_G(k_1, \ldots, k_p)$.

We assume implicitly that for every choice of the parameters k_i, the optimal span $\lambda_G(k_1, \ldots, k_p)$ is finite. For example, this holds when $\chi(G) < \infty$.

Given a graph G, form λ-graph $H = G^p$ in which an edge joining vertices u, v has type $i = \mathrm{dist}_G(u, v)$, $1 \leq i \leq p$. Thus, the real number distance labelling is a special case of λ-graphs.

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Results on distance-labelling, concerning continuity, piecewise-linearity, and the D-Set Theorem, can be extended to λ-graphs [Babilon et al.].

Král’ has managed to prove the PL and Delta Bound Conjectures, in the more general setting of λ-graphs, in a stronger form:

Theorem For every $p, \chi \geq 1$, there exist constants $C_{p,\chi}, D_{p,\chi}$ such that the space $[0, \infty)^p$ can be partitioned into at most $C_{p,\chi}$ polyhedral cones K, on each of which the optimal span $\lambda_G(k_1, \ldots, k_p)$ of every lambda graph G, with p types of edges and chromatic number at most χ, is a linear function of $k_1, \ldots, k_p$.

Moreover, for each K and G, there is a proper labelling f of λ-graph G in the form $f(v) = \sum_i a_i(v) k_i$ at every vertex v, which is optimal for all $(k_1, \ldots, k_p) \in K$, where the integer coefficients $0 \leq a_i(v) \leq D_{p,\chi}$.

A surprising consequence is

Corollary [Král'] There exist only finitely many piecewise-linear functions that can be the λ-function of a λ-graph with given number of edges k and chromatic number at most χ.

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- Symmetry properties of optimal labellings of lattices.

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