On Networks of Two-Way Channels

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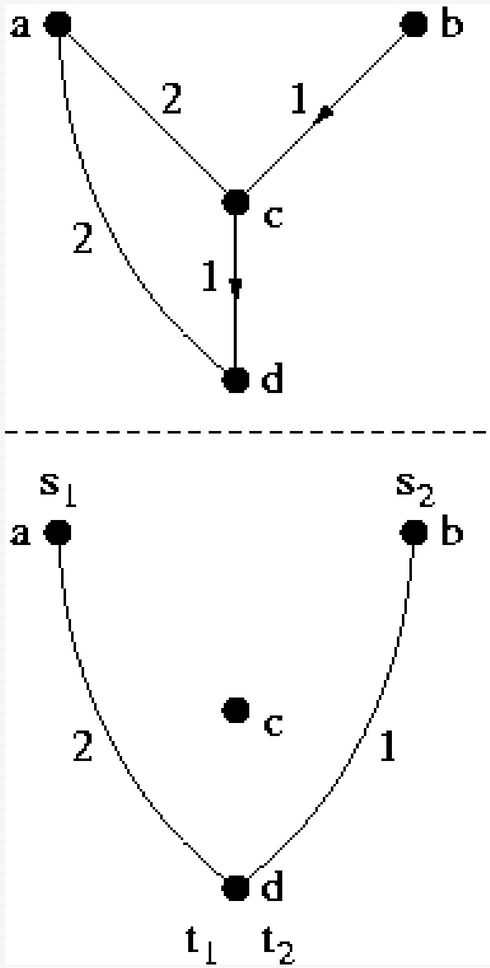
- Outline
  1. Network model
  2. Cut set bounds
  3. Implications for network coding
  4. Disconnecting set bounds

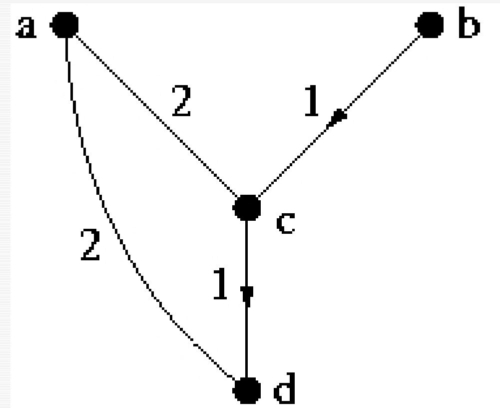
1. Network Model

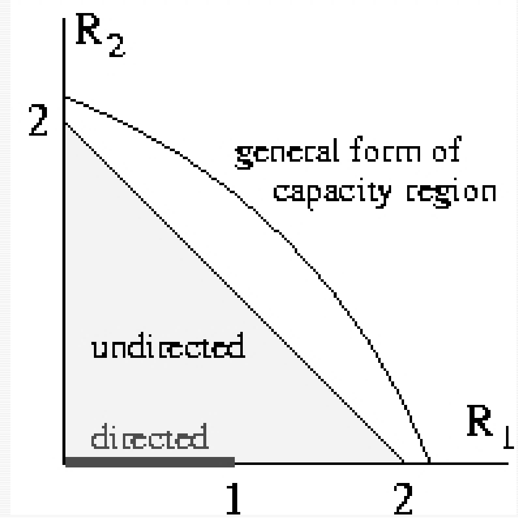
- Network
  - graph G=(V,E)
  - vertices V: “terminals”
  - edges E: “channels”
- Channels:
  - directed/undirected
  - capacity restrictions
- Demand (sources and destinations)
  - multi-commodity flow
  - multi-casting



Communication Networks

- Edges: cables, wireless channels, etc.
  - two-way channels (TWCs)
    edge bc: $P(y_b,y_c|x_b,x_c)$
- Capacity Region
  - the set of rate pairs (R_1,R_2) achievable with coding
  - convex if time-sharing permitted
  - consider ε–error capacity region
- Network capacity:
  what can the vertices can do?





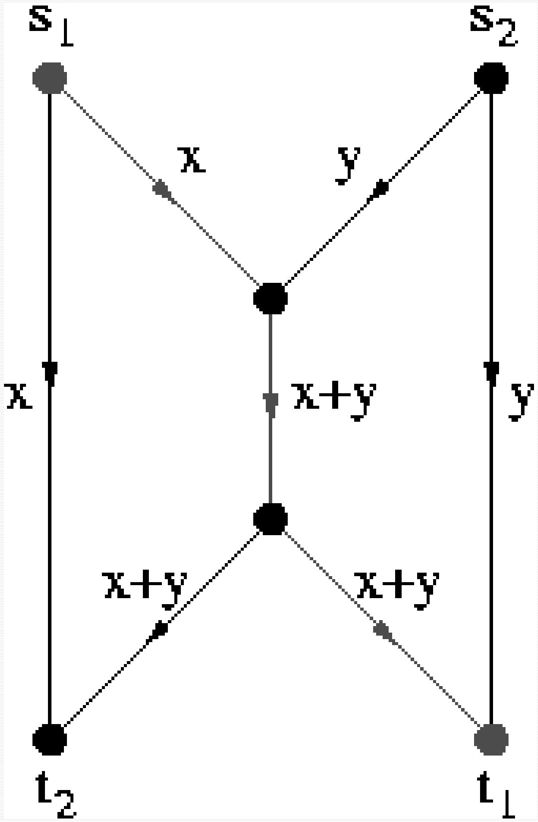
Networks of TWCs

- Model:
  - messages $W_1,...,W_M$ available at $s_1,...,s_L$, where $L \leq M$
  - network is clocked, i.e., a universal clock ticks N times
  - vertex v can transmit one symbol into its TWCs after clock tick n and before clock tick n+1 for all n =1,2,...,N
  - symbols are received at clock tick n+1 for all n
  - flow or routing: vertices can collect, store and forward symbols (including local message symbols)
  - here: network coding is allowed, i.e., for all clock ticks n, vertex v transmits (let $W_{M(v)}$ be the set of messages at v)

$$X_v[n] = f_n(W_{M(v)}, Y_v[1,2,...,n-1])$$

Network Coding Gains

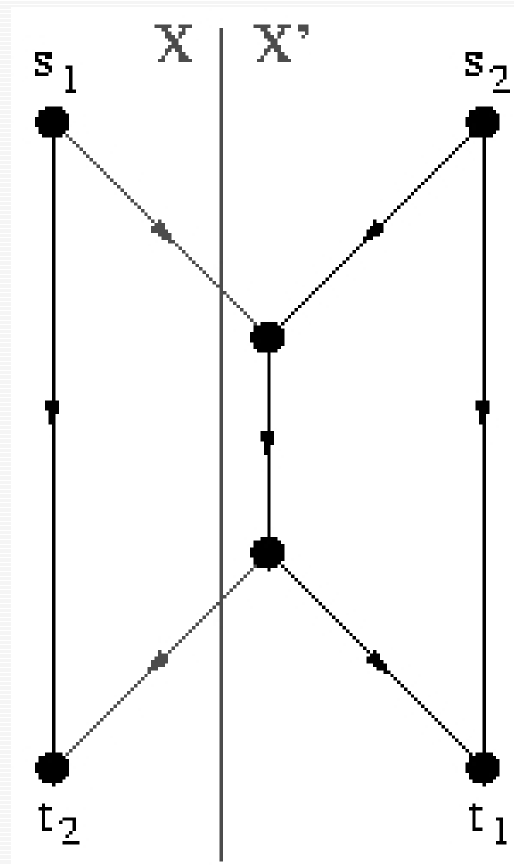
- A standard example (Ahlswede, Cai, Li, Yeung, 2000):
  - a two-flow problem with directed, unit capacity edges
  - max flow: 1
  - max coded sum rate: 2 can even decode both messages at both nodes
  - avg. resources used: flow: 3 edges/clock tick coding: 7 edges/clock tick



2. Cut Set Bounds

- Cut set E’: edges that <u>disconnect</u> each of a set of sources from (one of) its sinks, and that <u>divide</u> V into (X,X’)
- R_X : sum of rates of flows starting in X with a sink in X’
- $C_{X\rightarrow X'}$: sum of capacities of edges in E’ going from X to X’
- C_X : sum of capacities of edges in E’
- Bounds: $R_X \leq C_{X\rightarrow X'}$

  $R_X + R_{X'} \leq C_X$



Information Theory (IT) Cut Set Bound

- Cut set: same as above
- Need bound to apply to network coding
- Optimization of a standard IT cut set bound:

  1) convert every edge (TWC) into a pair of directed edges (one-way channels) whose rate pair is a boundary point of the capacity region of this edge

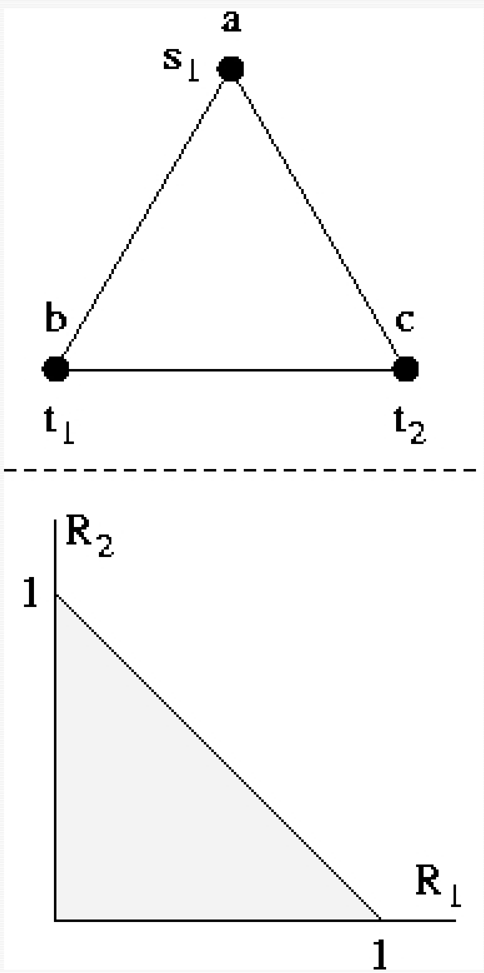
  2) apply the flow cut set bound

  3) repeat 1) and 2) for all boundary points on all edges

- IT cut set bound implies the above flow cut set bound

Example 1: undirected edges

– unit capacity, undirected edges, multi-casting with two sinks
– flow cut set bound: $R \leq 2$
– IT cut set bound: $0 \leq R_{ij}$, $R_{ij}+R_{ji} \leq 1$
  $R \leq R_{ab}+R_{ac}$, $R_{ab}+R_{cb}$, $R_{ac}+R_{bc}$
  The last two bounds give
  $2R \leq R_{ab}+R_{ac}+1 \leq 3$
– IT bound is stronger and tight
– rings with 1 source and K separate sinks: $R=(K+1)/K$ is best



Example 2: symmetric TWCs

- suppose capacity regions are the set of (R_1,R_2) satisfying

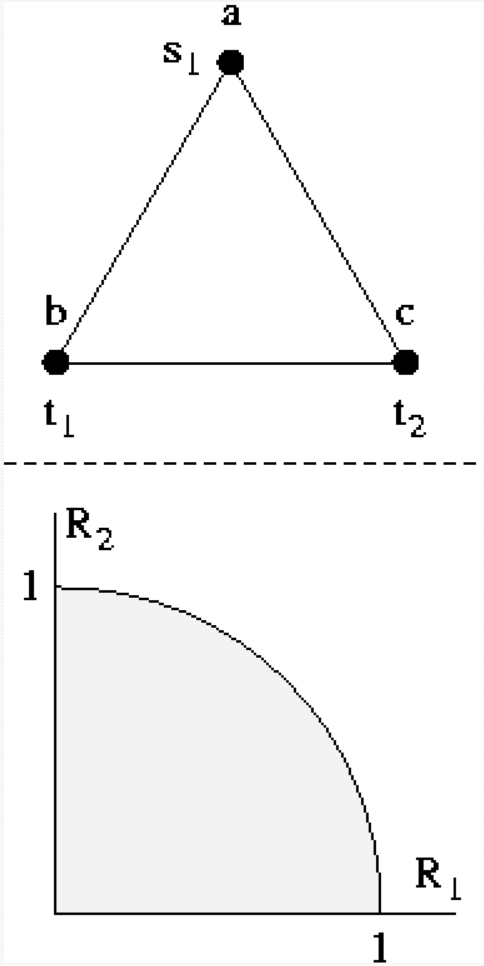
$$0 \leq R_1^2+R_2^2 \leq 1$$

- flow cut set bound: $R \leq 2$
- IT cut set bound: $R_{ij}^2+R_{ji}^2=1$

  $R \leq R_{ab}+R_{ac}, R_{ab}+R_{cb}, R_{ac}+R_{bc}$

  The last two bounds give

  $2R \leq R_{ab}+R_{ac}+(R_{cb}+R_{bc}) \leq 2+2^{1/2}$
- IT bound is again stronger and tight



Example 3: bidirected edges

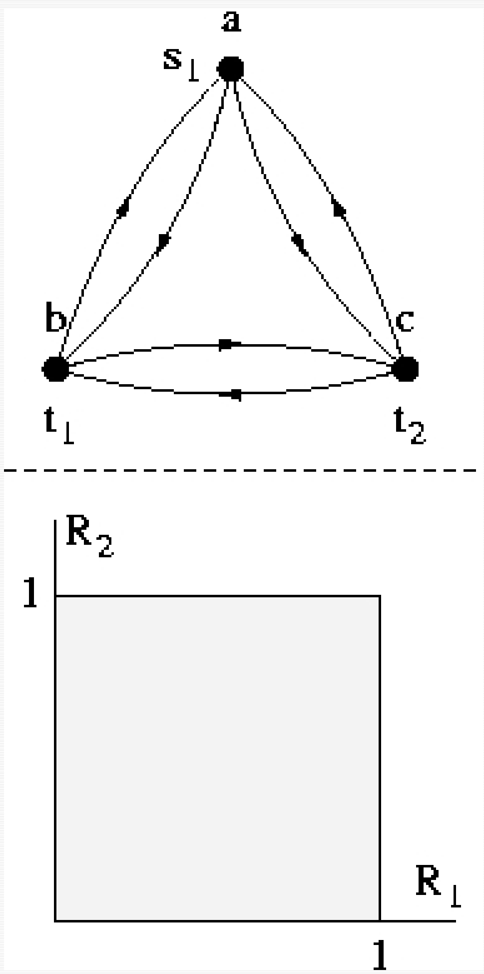
- suppose capacity region is the set of (R_1,R_2) satisfying

$$0 \le R_1 \le 1,\ 0 \le R_2 \le 1$$

- flow cut set bound: $R \le 2$
- IT cut set bound: $R_{ij}=1$, $R_{ji}=1$

$$R \le 2$$

- Flow and IT cut set bounds are the same for networks with directed edges
- multi-casting capacity is known for directed graphs (Koetter, Médard 2003)

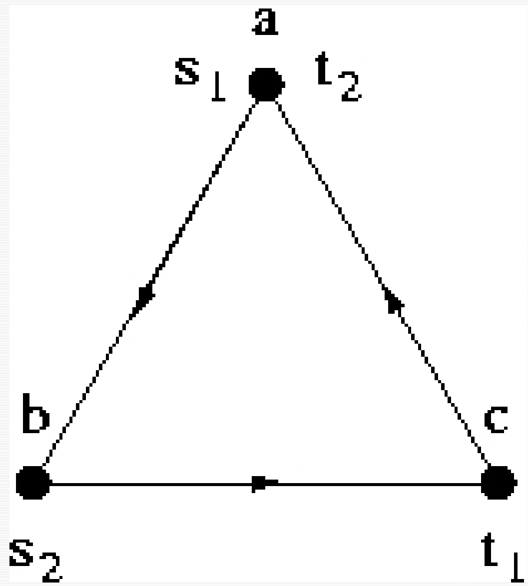


3. Implications for Network Coding

- If max-flow=flow-min-cut, routing is optimal
  - single commodity flow (Ford-Fulkerson, 1956)
  - two commodities in an undirected graph (Hu, 1963)
    - <u>not</u> true more generally (see standard example)
  - undirected planar graphs, multi-commody flow, sources and sinks on boundary of infinite region (Okamura, Seymour, 1981)
- Flow/routing questions:
  - when is max-flow=IT-min-cut for undirected networks?
  - when is max-flow=IT-min-cut for mixed networks?
  - do there exist, e.g., <u>disconnecting set</u> bounds for coding?

4. A Disconnecting Set Bound

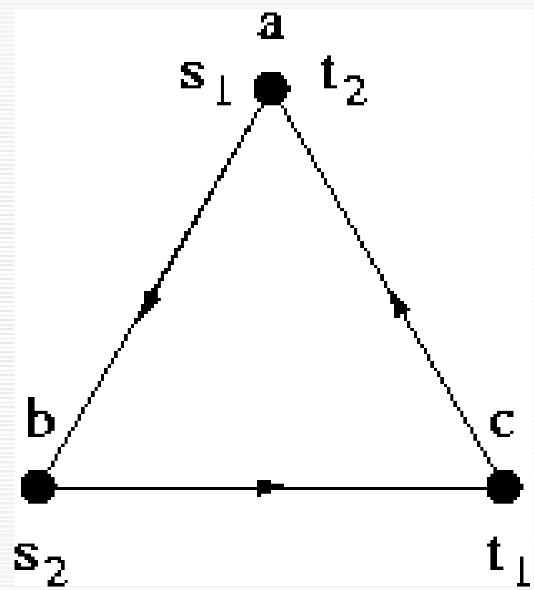
- Example: directed triangle
  - unit capacity edges
  - two commodities
  - max-flow is 1
- Disconnecting set: edge bc
  - IT cut set bound permits sum rate of 2!
  - Is this rate achievable with coding?



An improved IT bound

- We have the IT inequalities:

$N(R_1+R_2)$
$\leq I(W_1;\underline{X}_{bc}) + I(W_2;\underline{X}_{ca}W_1)$
$= I(W_1;\underline{X}_{bc}) + I(W_2;\underline{X}_{ca}|W_1)$
$\leq I(W_1;\underline{X}_{bc}) + I(W_2;\underline{X}_{bc}|W_1)$
$= I(W_1W_2;\underline{X}_{bc})$
$\leq H(\underline{X}_{bc}) \leq N$



- A simple disconnecting set bound.
  Can one generalize it?
  Yes, but in a limited way.

Summary and Some Open Problems

- Summary
  - model: network of TWCs
  - IT cut set bound needed for network coding
- Open Problems
  - what can flow/routing achieve for TWC edges?
  - when is max flow=flow-min-cut for TWC edges?
  - when is max flow=IT-min-cut (even for basic TWCs)?
  - what kinds of network codes are needed for general TWC capacity regions? Linear/nonlinear?
  - does a symmetric TWC capacity region simplify things?