4C: Correlation, Communication, Complexity, and Competition

4C

4C

- Correlation

4C

- Correlation
- Communication

4C

- Correlation
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4C

- Correlation
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- Correlation
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We can add

4C

- Correlation
- Communication
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- Competition

We can add

Cooperation, Coordination, Concealed Correlation,...,...

4C

- Correlation
- Communication
- Complexity
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We can add

Cooperation, Coordination, Concealed Correlation,...,...

and get a smoother topic:

4C

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and get a smoother topic: C^∞

General Introduction

The classical paradigm of game theory assumes full rationality of the interactive agents.

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In particular, it often assumes unlimited computational power.

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In particular, it often assumes unlimited computational power.

However, there are many decision problems and games for which it is impossible to assume that the agents (players) can either compute or implement an optimal (or best response or approximate optimal) strategy.

Design and Implementation

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Therefore, the complexity of finding an optimal (or approximate optimal) strategy is conceptually less disturbing.

However, the computational feasibility and the computational cost of implementing various strategies should be taken into account.

Design and Implementation

One can imagine scenarios where the design and choice of strategies is by rational agents with (essentially) unlimited computation power and the selected strategies need be implemented by players with restricted computational resources.

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One can imagine scenarios where the design and choice of strategies is by rational agents with (essentially) unlimited computation power and the selected strategies need be implemented by players with restricted computational resources.

- A corporation
- The USA Navy
- A soccer team
- A chess player
- A computer network

Pure Mixed and Behavioral

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- In practice, mixed and behavioral strategies are not equivalent.

Recall that

- A mixed strategy reflects uncertainty regarding the chosen pure strategy, and
- A behavioral strategies randomizes actions at the decision nodes.

Strategies in the Repeated Game

- The number of pure strategies of the repeated game grows at a double exponential rate in the number of repetitions.
- Many of the strategies are not implementable by reasonable sized computing agents.

General Objective

The impact on

strategic interactions
the value and equilibrium payoffs

of variations of the game where players are restricted to employ

Simple Strategies

Simple Strategies

- Computable Strategies
-
-
-

Simple Strategies

-
- Finite Automata
-
-

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Simple Strategies Recall

-
-
- Bounded Recall
-

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- Bavly and Neyman (forthcoming)
  Concealed Correlation by Boundedly Rational Players

Simple Strategies

-
-
-
- Bounded Strategic Entropy

References: Bounded Entropy

Neyman and Okada

- Strategic Entropy and Complexity in Repeated Games
  Games and Economic Behavior (1999)

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  Games and Economic Behavior (2000)

Simple Strategies

-
-
-
-
- Kolmogorov’s Complexity

References/Origin

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- Neyman (forthcoming) Finitely Repeated Games with Bounded Kolmogorov's Strategic Complexity

Simple Strategies

- Computable Strategies
- Finite Automata
- Bounded Recall
- Bounded Strategic Entropy
- Kolmogorov's Complexity

Notation-Finite Automata

Notation-Finite Automata

$$
\begin{aligned}
M &:= \max_{a \in A} \min_{b \in B} g(a,b) \\
V &:= \min_{y \in \Delta(B)} \max_{a \in A} g(a,y) \\
&= \max_{x \in \Delta(A)} \min_{b \in B} g(x,b) \\
\\
mm(k_1,k_2) &:= \min_{\tau \in \Sigma_2(k_2)} \max_{\sigma \in \Sigma_1(k_1)} G(\sigma,\tau) \\
&:= \min\max(k_1,k_2) \quad \geq \\
Mn(k_1,k_2) &:= \min_{\tau \in \Delta(\Sigma_2(k_2))} \max_{\sigma \in \Sigma_1(k_1)} G(\sigma,\tau) \\
&:= Min\max(k_1,k_2)
\end{aligned}
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2-P 0-sum FA: The Questions

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Assume $k_2 \geq k_1 \to \infty$

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Table-Finite Automata

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2-P 0-sum Finitely Repeated FA

Let $Mm(T;k_1,k_2)$ be the minmax the T-stage game when P2 minimizes over all mixtures of automata of size k_2 and P1 maximizes over all automata of size k_1. Similarly $mm(T;k_1,k_2)$

2-P 0-sum Finitely Repeated FA

The Questions

What are the asymptotic relations between the size of k_1 and k_2 of the automata of P1 and P2 and the number of repetitions T so that

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- $mm(T; k_1, k_2) = M$

2-P nonzerosum Finitely Repeated FA

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Let $G(T; k_1, k_2)$ be the T-stage game when P2 uses machines of size k_2 and P1 uses machines of size k_1.

The Questions

2-P nonzerosum Finitely Repeated FA

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The Questions

What are the asymptotic relations between the sizes k_1 and k_2 and the number of repetitions T so that

- The set of equilibrium payoffs of $G(T; k_1, k_2)$ converge to the equilibrium payoffs of the infinitely repeated game G^*.

n-person Finitely Repeated FA $n > 2$

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The objective is the study of the equilibrium of

$$G(k_1, \dots, k_n)$$

and of

$$G(T; k_1, \dots, k_n).$$

n-person Finitely Repeated FA $n > 2$

The objective is the study of the equilibrium of

$$G(k_1, \ldots, k_n)$$

and of

$$G(T; k_1, \ldots, k_n).$$

It requires the analysis of the individual rational payoff of say player 1, namely of

$$\text{Min Max } G(\sigma^{-1}, \sigma^1)$$

where the min is over all strategy profiles $\sigma^{-1} = (\sigma^j)_{j \neq 1}$ where σ^j is a mixture of automata of Pj of size k_j and the max is over all automata of P1 of size k_1.

Notation-Bounded Recall

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\begin{aligned}
M &= \max_{a \in A} \min_{b \in B} g(a, b) \\
V &= \min_{y \in \Delta(B)} \max_{a \in A} g(a, y) \\
&= \max_{x \in \Delta(A)} \min_{b \in B} g(x, b) \\
mm(k_1, k_2) &= \min \max(k_1, k_2) \\
&= \min_{\tau \in BR_2(k_2)} \max_{\sigma \in BR_1(k_1)} G(\sigma, \tau) \\
Mn(k_1, k_2) &= Min \max(k_1, k_2) \\
&= \min_{\tau \in \Delta(BR_2(k_2))} \max_{\sigma \in BR_1(k_1)} G(\sigma, \tau)
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	$mm(k_1, k_2)$	$Mn(k_1, k_2)$
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$\log k_2 = o(k_1)$	?	V Lehrer

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$k_2 >> \|A \times B\|^{k_1}$	M Neyman and Okada	M

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$k_2 >> \|A \times B\|^{k_1}$	M Neyman and Okada	M
$k_2 > Ck_1$	$\exists C$ such that $\leq V$	$\leq V$

Complexity and Competition

Complexity and Competition

- Ben-Porath 85

Complexity and Competition

- Ben-Porath 85
- Lehrer 88

Complexity and Competition

- Ben-Porath 85
- Lehrer 88
- Neyman 97

Complexity and Competition

- Ben-Porath 85
- Lehrer 88
- Neyman 97
- Stearns 97

Complexity and Competition

- Ben-Porath 85
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Complexity and Cooperation

2-person finitely repeated games

Complexity and Cooperation

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- Meggido and Wigderson 86

Complexity and Cooperation

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Complexity and Cooperation

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Complexity and Cooperation

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2-person infinitely repeated games

Complexity and Cooperation

n-person games ($n > 2$)

Complexity and Cooperation

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Complexity and Cooperation

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- Ben-Porath 92
- Lehrer 94
- Neyman 97...

Complexity and Cooperation

n-person games ($n > 2$)

- Ben-Porath 92
- Lehrer 94
- Neyman 97...
- Gossner Hernandez and Neyman

Complexity and Concealed Correlation

- Gossner (Polynomial time Turing Machines)

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  - 2 weak players conceal correlation from a stronger one)

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- Neyman and Bavly 03 (Bounded Recall and FA)
  - $n \geq 2$ weak and 1 strong conceal correlation from a median one

Concealed Correlation

Concealed Correlation

- Gossner and Tomala

Concealed Correlation

- Gossner and Tomala
- Gossner Tomala and Laraki

Online Concealed Correlation

by Boundedly Rational Players

Gilad Bavly and Abraham Neyman

Online Concealed Correlation

by Boundedly Rational Players

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Distributions on Cartesian Products

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Consider a stochastic process with values in A^∞

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Consider a stochastic process with values in A^∞ where A is a product set, e.g., $A = A^1 \times A^2 \times A^3$

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i.e., a probability distribution P over streams $a_1, a_2, \ldots, a_t, \ldots$ with

$$a_t = (a_t^1, a_t^2, a_t^3) \in A = A^1 \times A^2 \times A^3$$

Distributions on Cartesian Products

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The law P of the process is governed by a list of independent rules, σ^1, σ^2, and σ^3, each governing its own factor A^1, A^2, and A^3, respectively.

The independent rules = strategies

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The rule σ^i specifies, for each t, the coordinate a_t^i as a function of $a_1, \ldots, a_{t-1}$.

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The rule σ^i specifies, for each t, the coordinate a_t^i as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of A^i

The independent rules = strategies

The rule σ^i specifies, for each t, the coordinate a_t^i as a function of $a_1, \ldots, a_{t-1}$.

- A deterministic rule: $\sigma^i(a_1, \ldots, a_{t-1})$ an element of A^i
- A behavioral rule: $\sigma^i(a_1, \ldots, a_{t-1})$ a probability over A^i
- A mixed rule is a mixture of deterministic rules

The independent rules = strategies

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Product marginals

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when $m \to \infty$ ($k_i = k_i(m)$).

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If $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, then for every $(b_1, \ldots, b_m, b_{m+1})$ we compute

$$P_\sigma((a_{t-m}, \ldots, a_{t-1}, a_t) = (b_1, \ldots, b_m, b_{m+1}))$$

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We study the distribution of b_{m+1} conditional on $b_1, \ldots, b_m$

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- For a given asymptotic relation between m and k_1, k_2, k_3, what are the distributions Q on A that can be "realized" as the distribution of b_{m+1} given $b_1, \ldots, b_m$ w.r.t. some P_σ where σ has (k_1, k_2, k_3)-recall

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- (Bavley-N) If m is superexponential in k_1 and k_2 ($\exists C$ s.t. $m \geq e^{Ck_1+Ck_2}$) then the marginal on $A^1 \times A^2$ of the distribution of b_{m+1} given $b_1, \ldots, b_m$ is a product distribution.

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- More to come

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- Pure strategies of player 3:

$$z = (z_1, \ldots, z_n)$$

$$z_t : I^{t-1} \times J^{t-1} \to K$$

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n-stage payoff to the team:

$$G(x, y, z) = \frac{1}{n} \sum_{t=1}^{n} g(x_t, y_t, z_t)$$

Example

$I = J = K = \{0, 1\}$ and

$$g(i,j,k) = \begin{cases} 1 & if \;\; i = j = k \\ 0 & otherwise \end{cases}$$

1	0
0	0

0	0
0	1

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Can they do better?

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Resulting sequences of actions:

$$x = (x_1, x_2, x_3, x_4, \ldots, x_{80})$$

$$y = (x_2, x_2, x_4, x_4, \ldots, x_{80})$$

$$z = (z_1, x_2, z_3, x_4, \ldots, x_{80})$$

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- Against the worst possible case:
  - Payoff of 1 at even stages.
  - Payoff of zero at odd stages.
  - Average payoff of 0.5.

Question

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- In expected payoffs?

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- In expected payoffs?
- In the worst case?
- Can mixed strategies do better for the latter?

What is your answer?

Answer

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- There exist *pure* strategies for the team that guarantee $v^* - o(1)$ against *all* sequences.
- Against an i.d.d. sequence $(\frac{1}{2}, \frac{1}{2})$, no strategy of the team can obtain more than v^*.
- v^* is defined by

$$H(v^*) + (1 - v^*) \log 3 = 1$$

where H is the entropy function.

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- $\exists$ *pure* strategies for the team in the ∞-stage game with expected average payoff in the n-stages converging as $n \to \infty$ to $v^*(\mu)$ against a μ iid sequence.

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- $\exists$ a sequence $v_n^* = v^* - o(1)$ and *pure* strategies for the team in the ∞-stage game that achieve an average payoff in the n-stages $\geq v_n^* = v^* - o(1)$ against any sequence.
- $\exists \mu \in \Delta(I)$ s.t. when player 1's sequence of actions is i.i.d. according to μ, $\forall$ strategies of the forecaster and the follower, their payoff in the n-stage version of the game does not exceed v^*.

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- the existence of ε-optimal pure strategies for the team.

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and

$$v^* = \min_{\mu} v^*(\mu) = \min_{\mu} \max_{Q \in \mathcal{Q}(\mu)} \mathbf{E}_Q(g(i,j,k))$$

More forecasters and/or followers?

Existence of ε-optimal *pure* strategies for the team enables the extension of the result to $1 + s + f = n$ - person games where there are s forecasters and f followers.
Replace the set of s forecasters by a single forecaster with an action set equal to the cartesian product of the action sets of the forecasters, and the f followers by a single follower with an action set equal to the product of the action sets of the followers.

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Proof in the special case of the example.

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$$g_t = \mathbf{E}_\mu \left(\mathbb{I}(X_t = Z_t = Y_t) \mid \mathcal{F}_{t-1} \right)$$

is $\mathcal{F}_{t-1}$-measurable.

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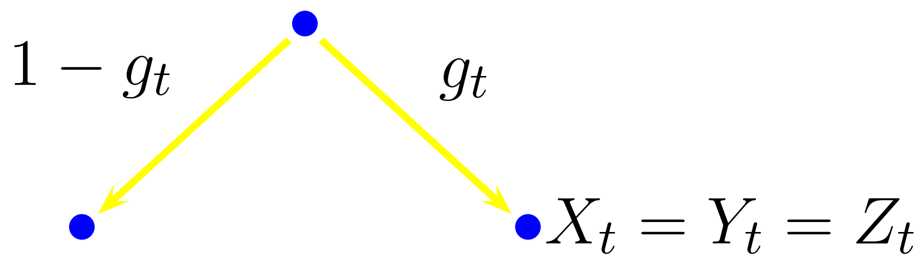

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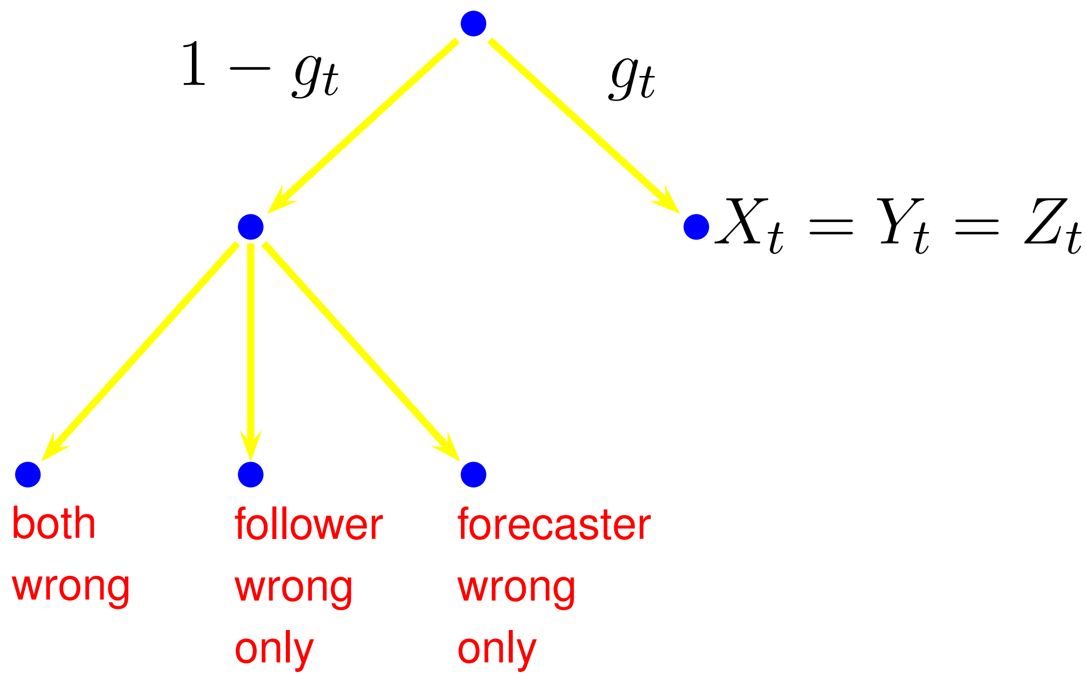

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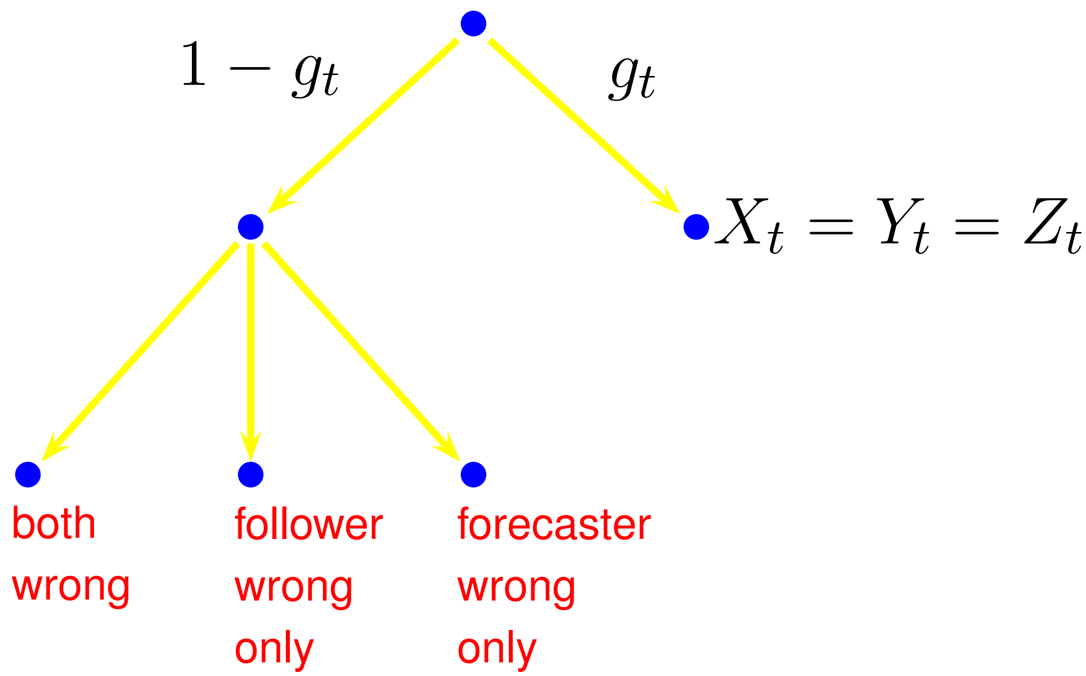

$$h(X_t, Y_t \mid X_1 \dots Y_{t-1}) \leq H(g_t) + (1 - g_t) \log 3$$

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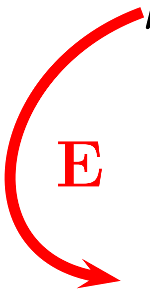

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Sum over t

$$nh \leq \sum_{1}^{n} \mathbf{E}_\mu \left(H(g_t) + (1 - g_t) \log 3\right)$$

Conclusion of the first part

With $g = \mathbf{E}_\mu \left(\frac{1}{n}\sum_{t=1}^n g_t\right)$, (g, h) is in the convex hull of $V = \{(x, y) \leq (x, H(x) + (1 - x)\log 3)\}$

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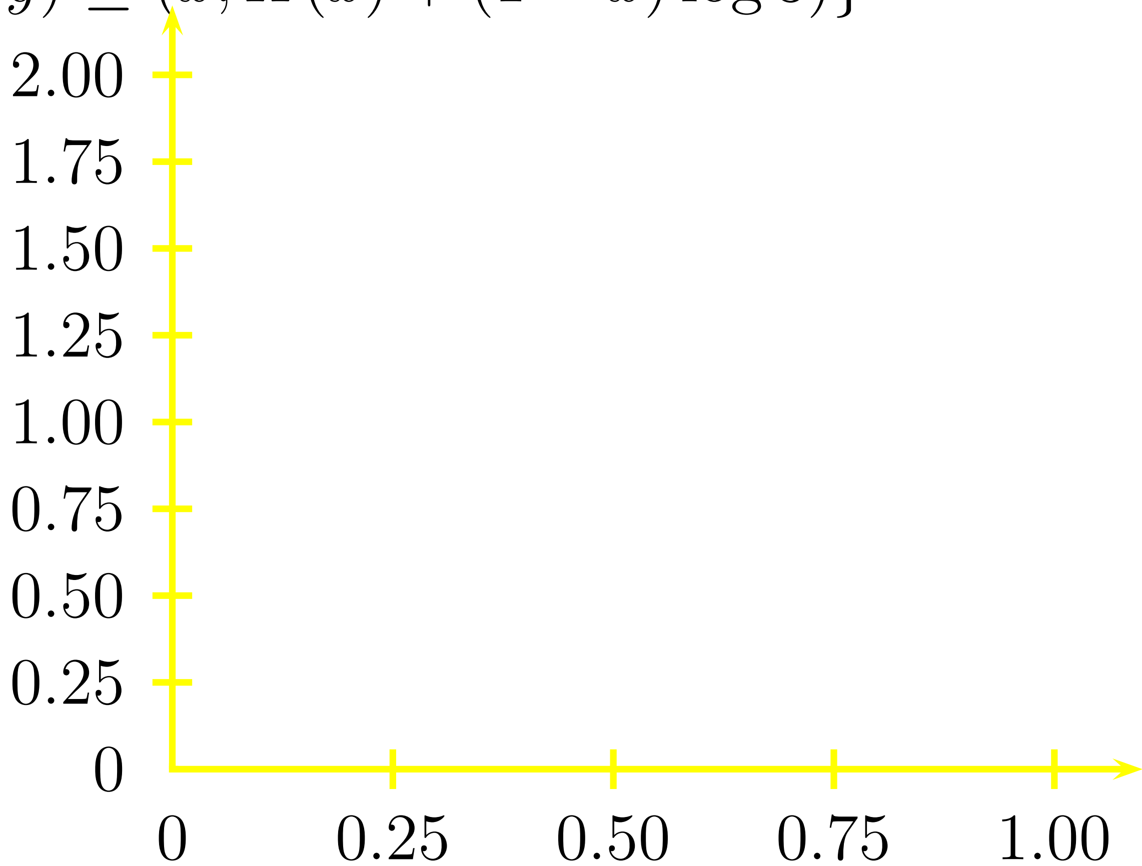

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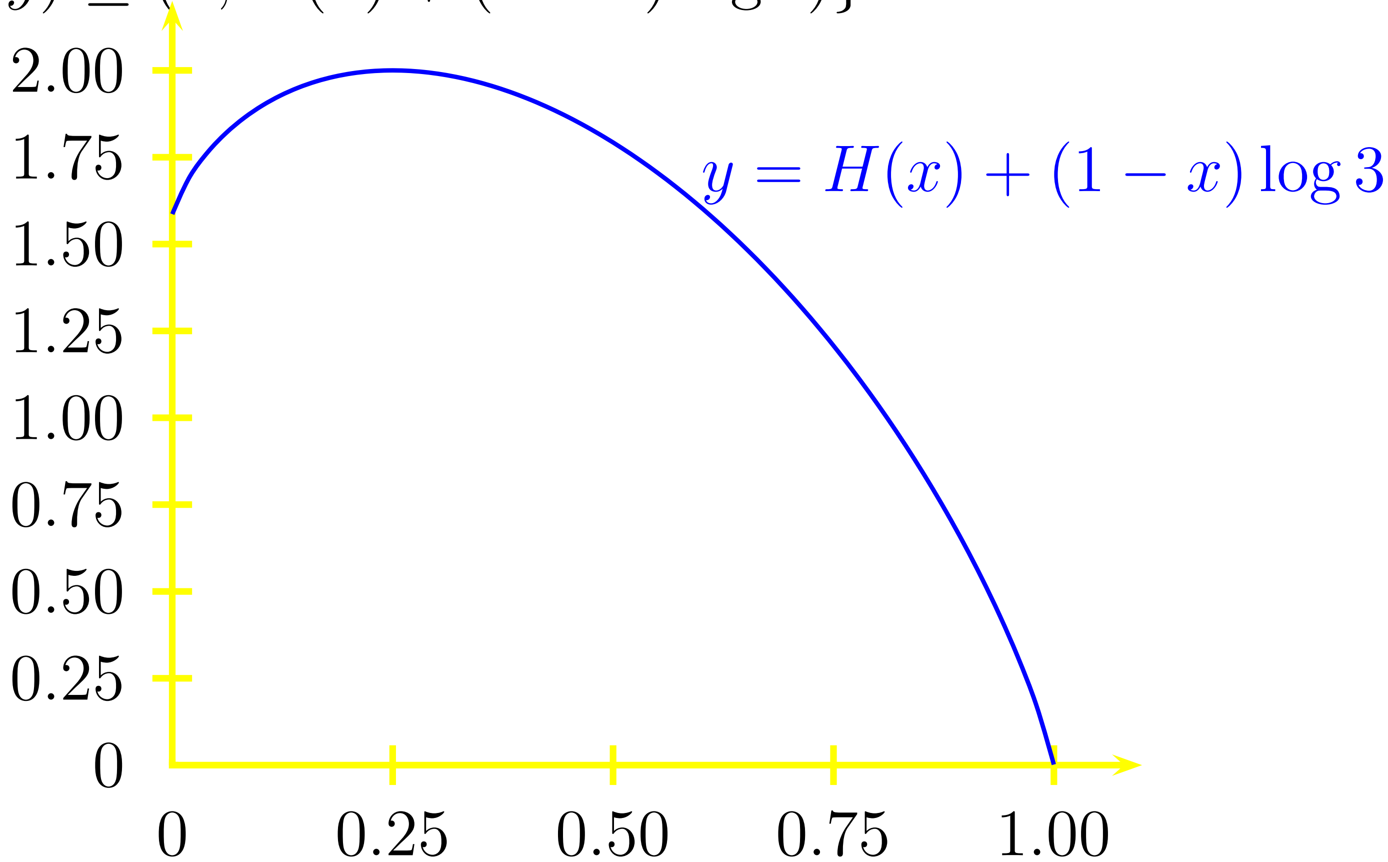

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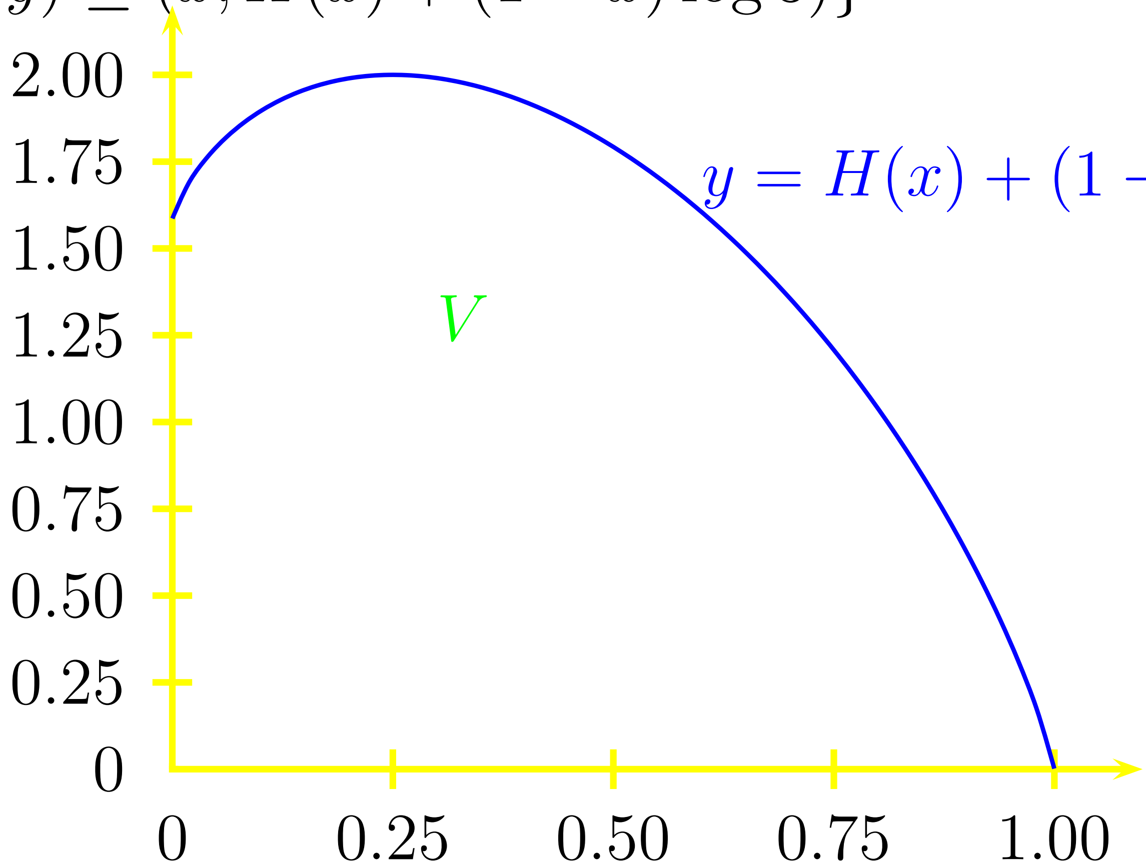

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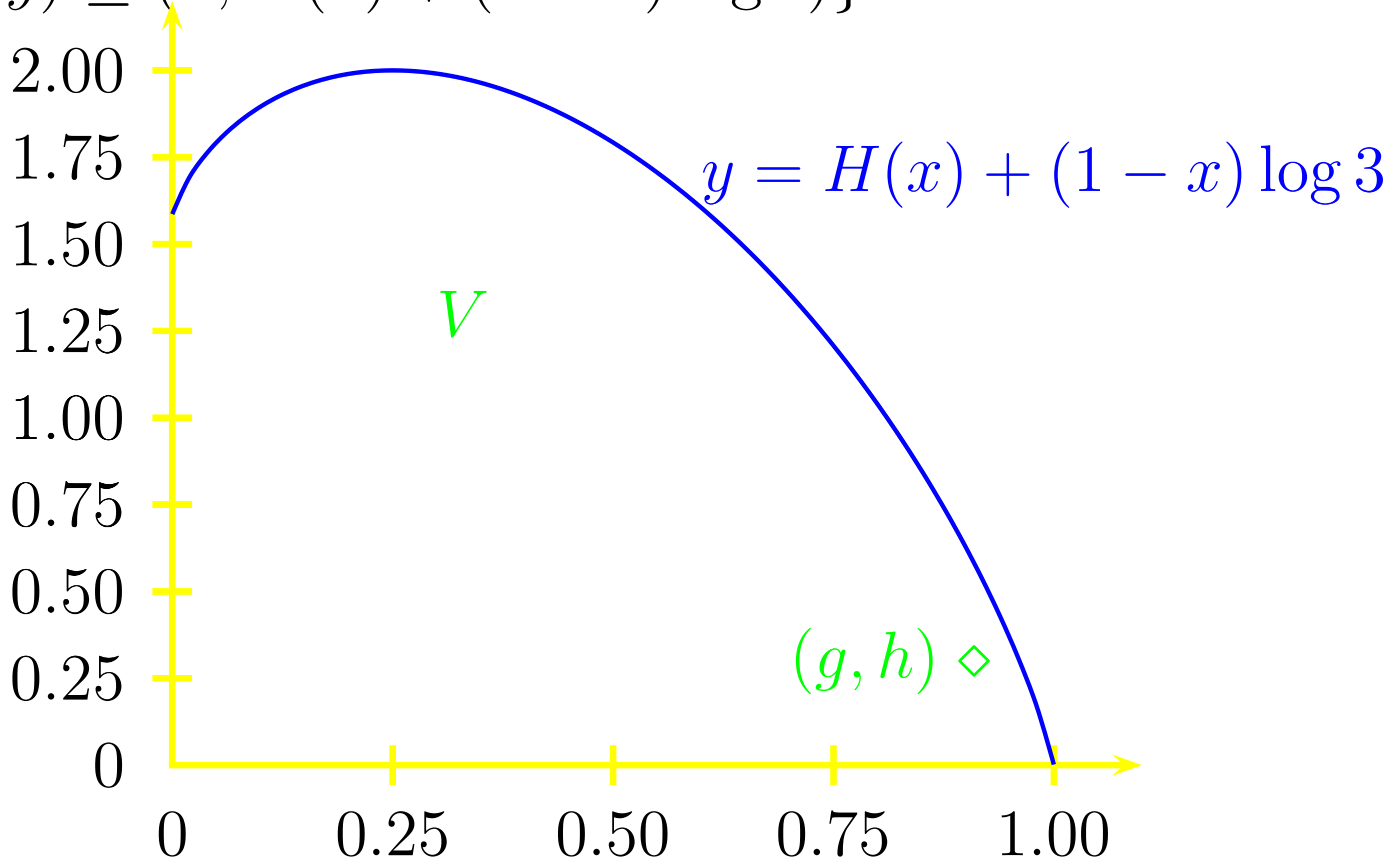

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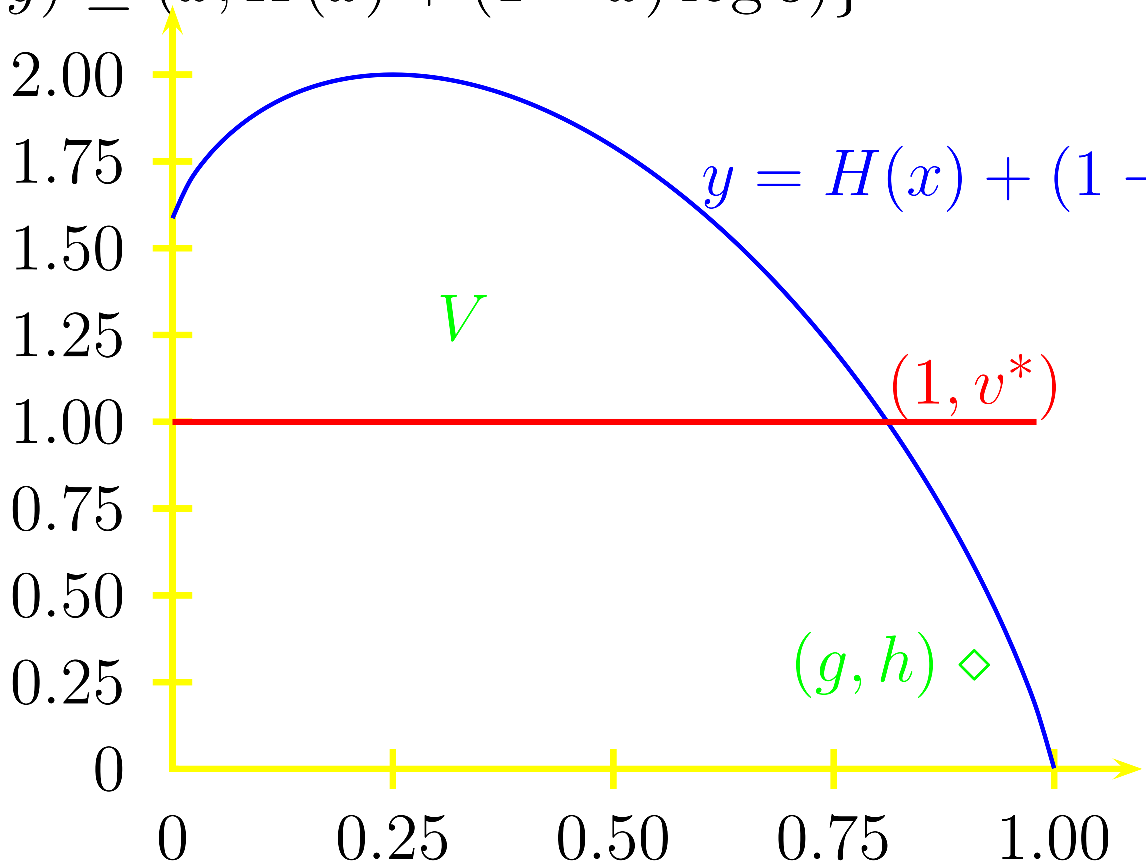

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- q: % of stages at which the follower is wrong.
- p is the % of stages at which the forecaster is wrong, conditional on the follower right.

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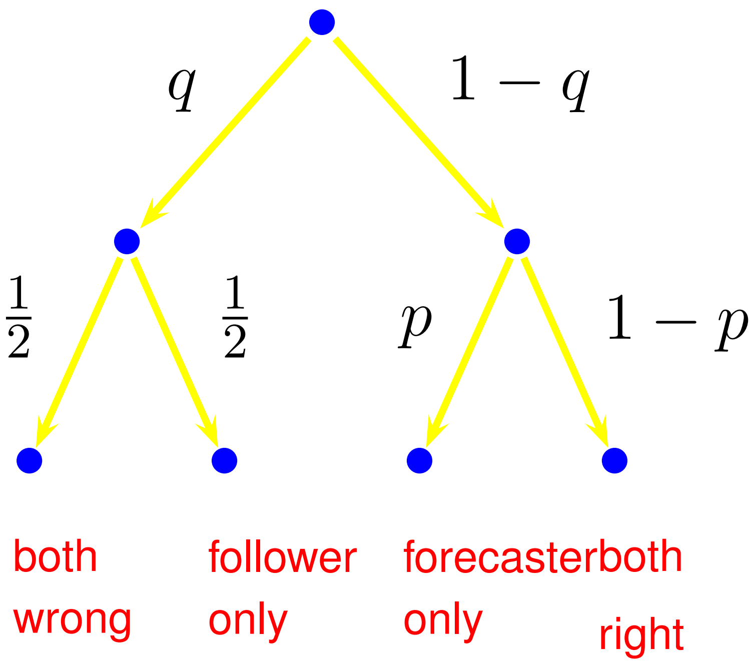

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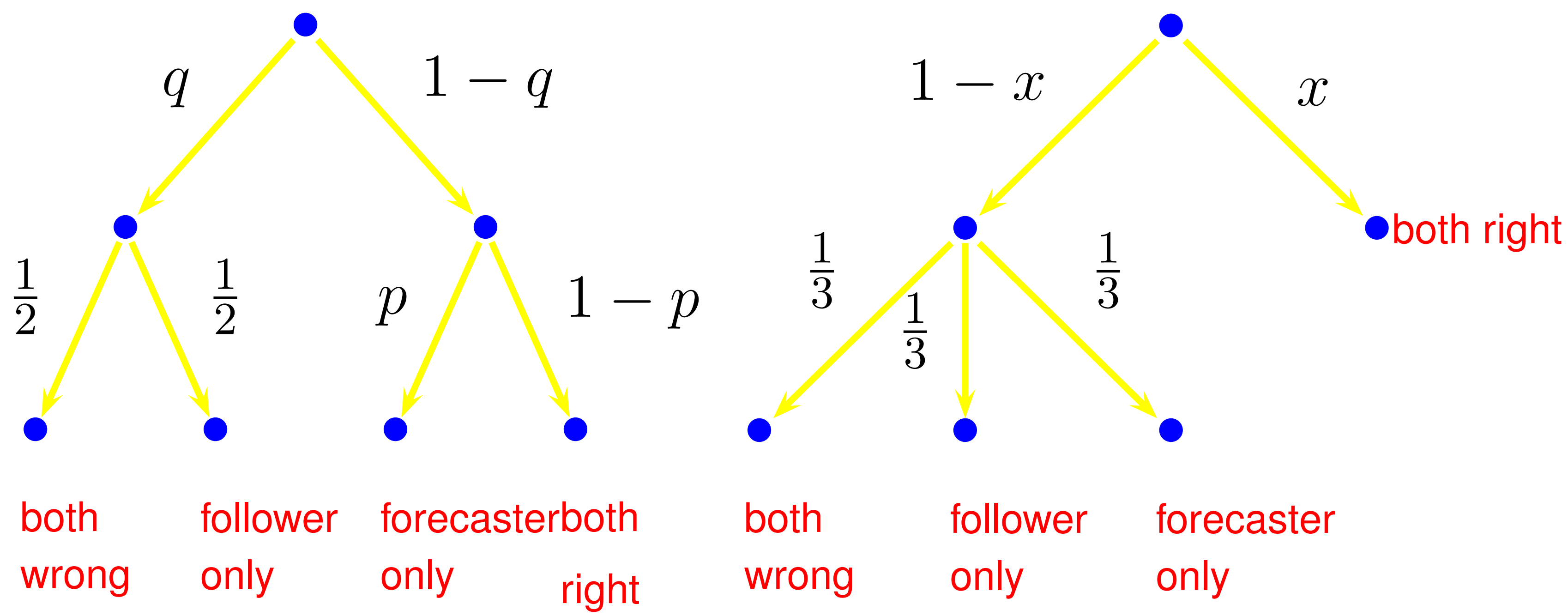

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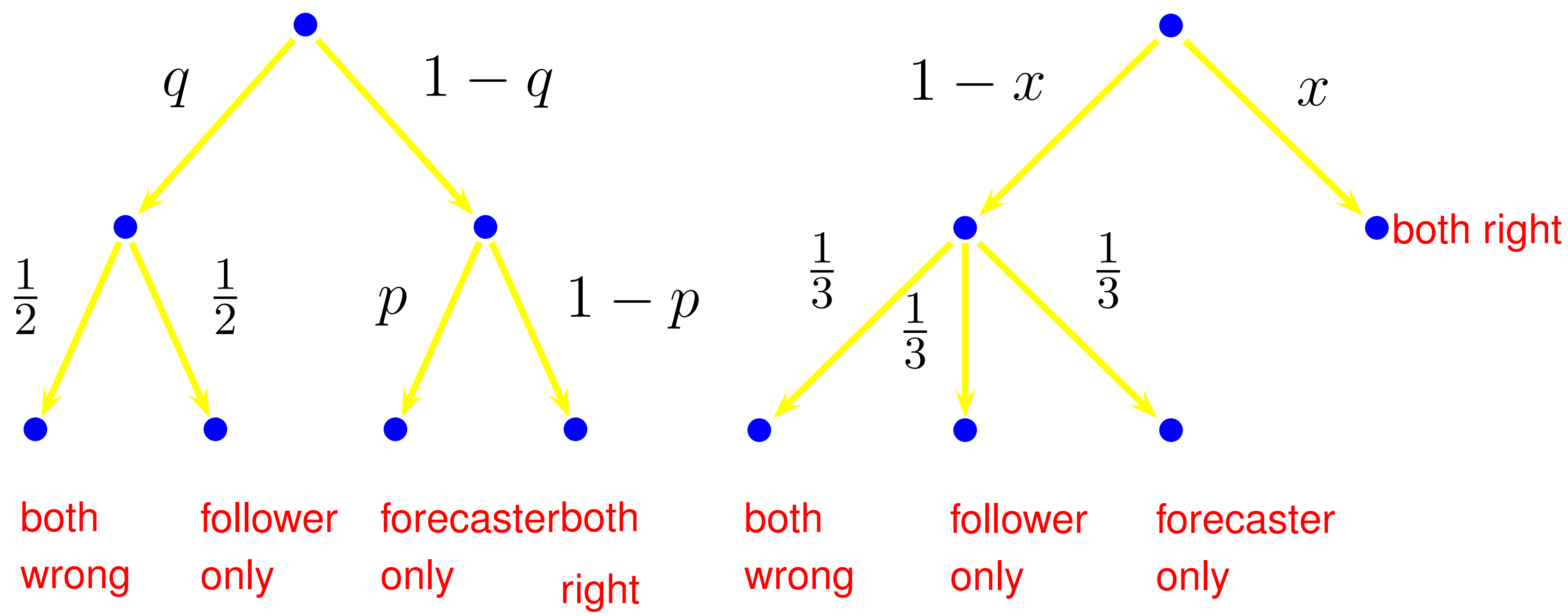

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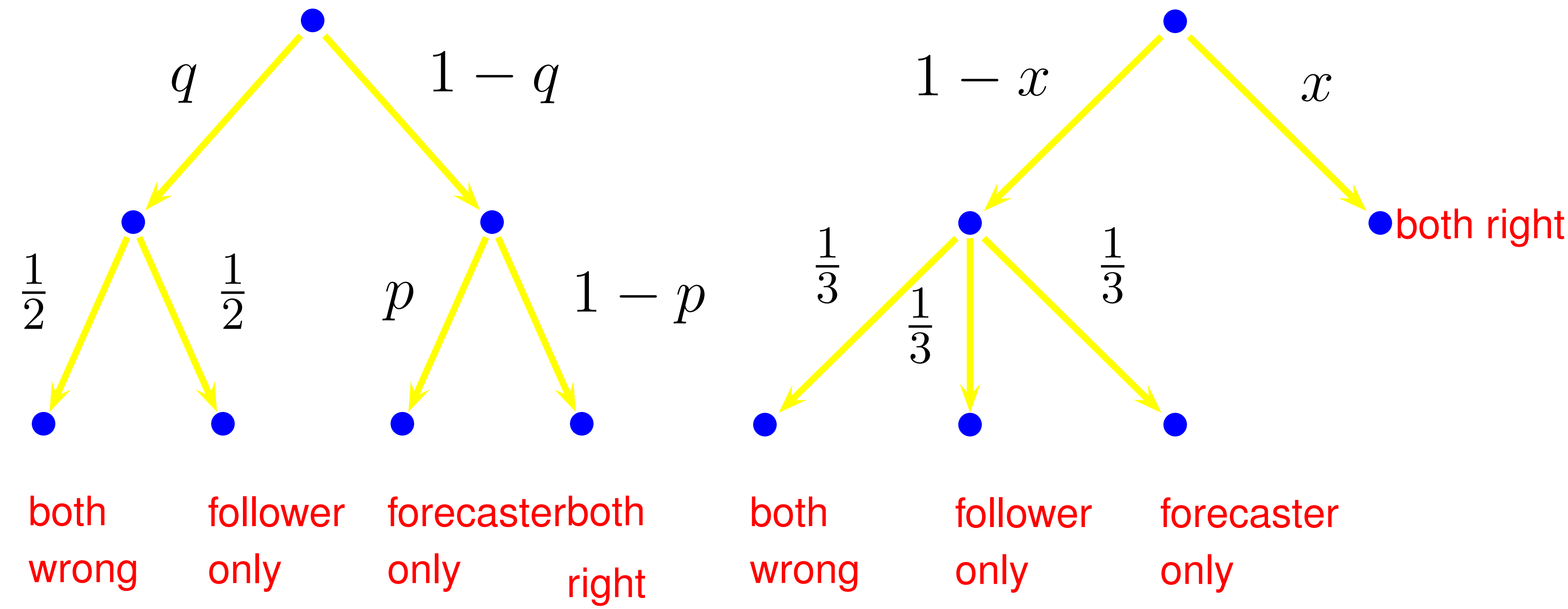

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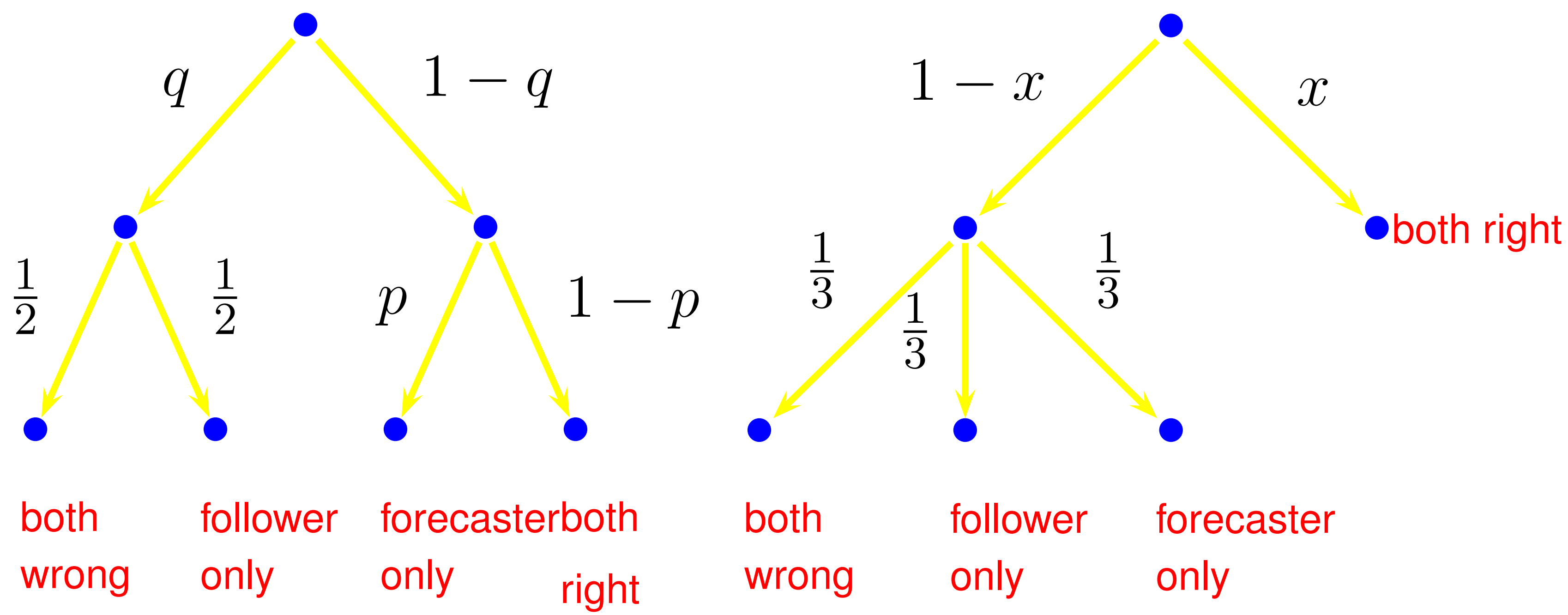

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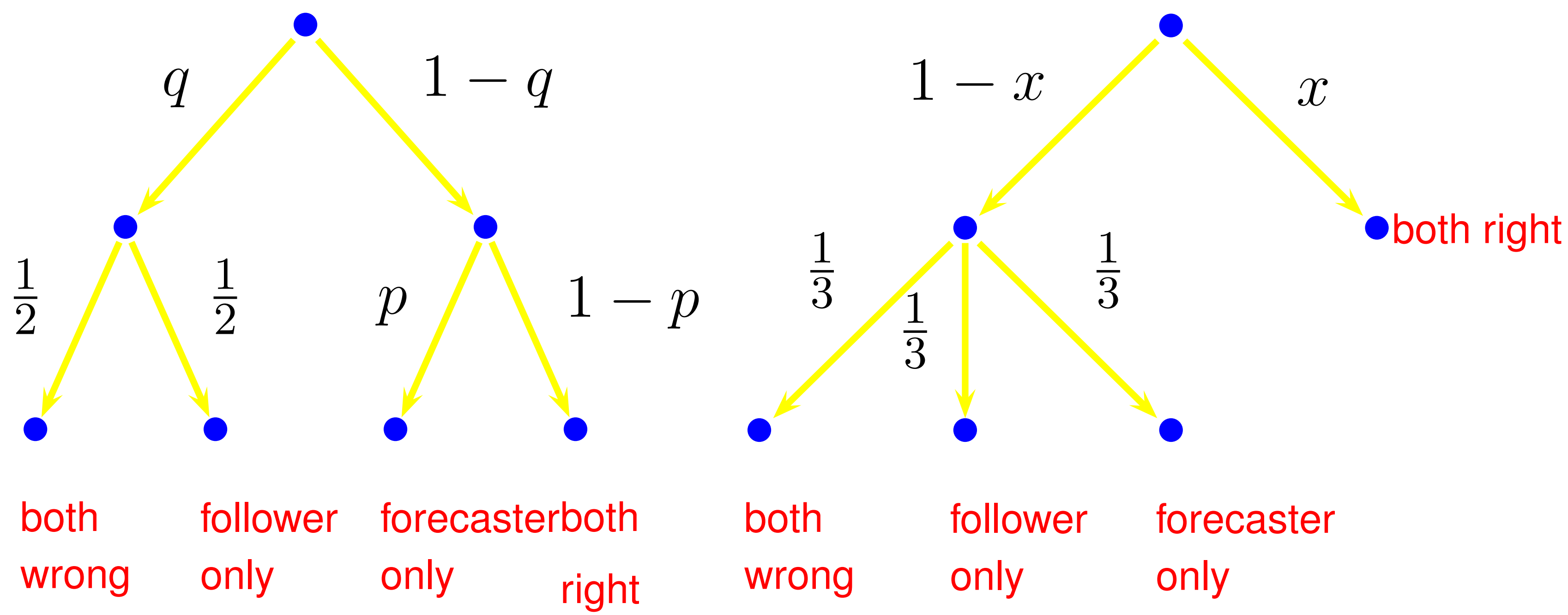

$$H(q) + q + (1-q)H(p) \quad = \quad H(x) + (1-x)\log 3 \quad = \quad 1.$$

Therefore $q + (1-q)H(p) = 1 - H(q)$ and thus $2^{n(q+(1-q)H(p))} = 2^{n(1-H(q))}$ messages can be sent.

Question

Does there exist a set $A \subset 2^n$ such that

$$|A| = 2^{(1-H(q)+o(1))n}$$

and s.t.: $\forall x \in 2^n \;\; \exists y \in A$ s.t.

$$d_H(x, y) = (1-q)n.$$

where d_H is the Hamming distance?

Existence of A

Probabilistic proof:

Take a set $A = \{a_i\}$ of $2^{(1-H(q))n}$ points taken randomly i.i.d. uniformly in 2^n.

For every fixed $x \in 2^n$ the probability that there is no $z \in 2^n$ so that $d_H(x, y) = [qn]$ is

$$\leq (1 - \binom{n}{[qn]}/2^n)^{2^{(1-H(q))n}} \leq \exp -2^{n(H(q)+1-H(q)}$$

We prove that the probablity that A feeds our needs is positive.

Hence, such A exists.

Example 1

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T		
B		

T		
B		

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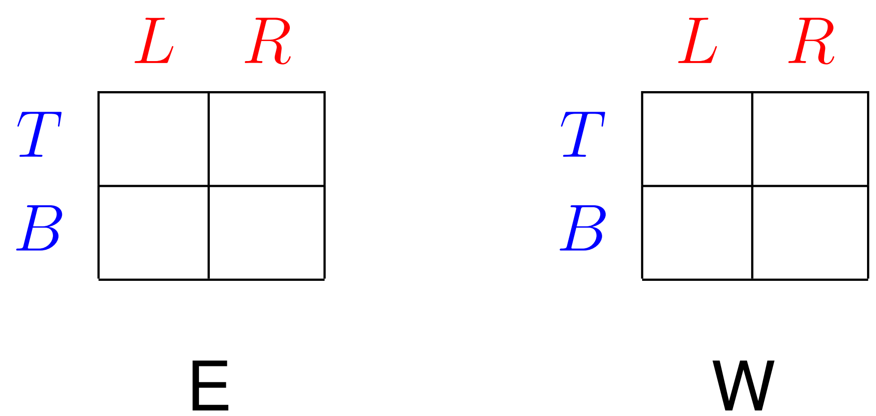

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	L	R
T	.2	.1
B	.1	.1

E

	L	R
T	.1	.1
B	.1	.2

W

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T	$.2$	$.1$
B	$.1$	$.1$

W:

	L	R
T	$.1$	$.1$
B	$.1$	$.2$

$H(\mathbf{i}) = 1 = H(\mathbf{k})$ and $H(\mathbf{i}, \mathbf{j}, \mathbf{k}) = 1 + H(.4, .6) + .6 \log 3 > 2$

Example 2

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B		

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T		
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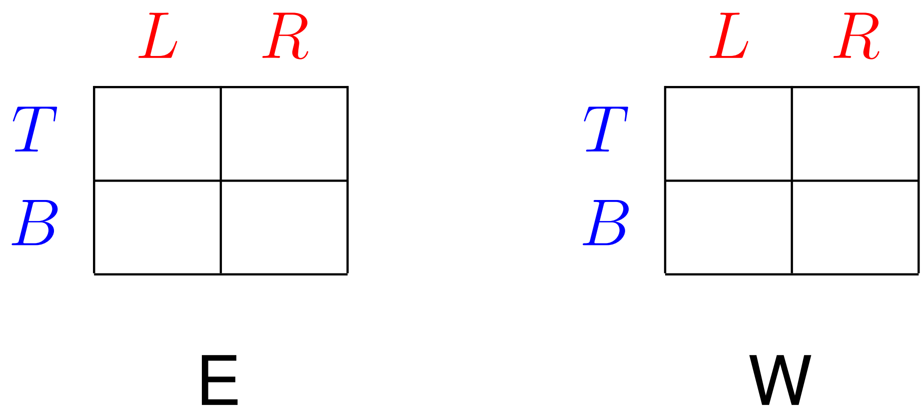

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E

	L	R
T	$.35$	$.05$
B	$.05$	$.05$

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$H(\mathbf{i}) = 1 = H(\mathbf{k})$ and $H(\mathbf{i}, \mathbf{j}, \mathbf{k}) = 1 + H(.7, .3) + .3 \log 3 > 2$

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T		
B		

T		
B		

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	L	R
T		
B		

	L	R
T		
B		

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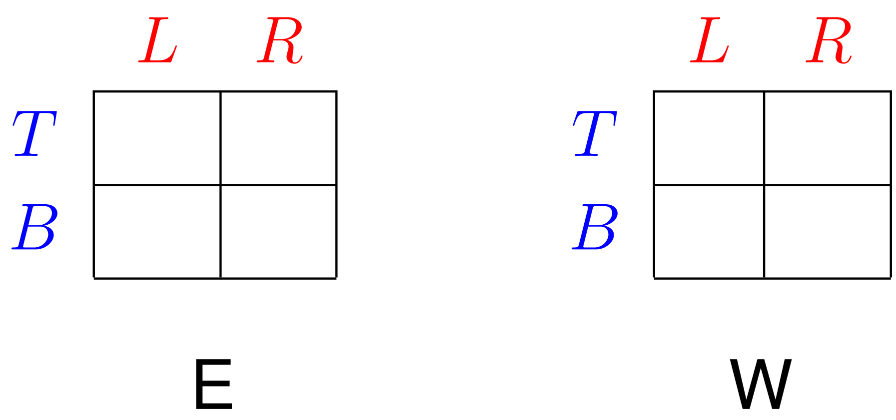
Q described in the Figure below, P2 chooses the rows (T or B), P3 chooses the columns (L or R), Temporal state of nature is E or W iid 1/2,1/2.



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	L	R
T	$.41$	x_1
B	x_2	x_3

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$$x_1 + x_2 + x_3 = .09$$

$H(\mathbf{i}) = 1 = H(\mathbf{k})$ and $H(\mathbf{i}, \mathbf{j}, \mathbf{k}) \leq 1 + H(.41, .59) + .18 \log 3 < 2$

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- $i_1, i_2, \ldots$ follow a Markov chain
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Let $\mu \in \Delta(I)$ be the invariant distribution and $\hat{\mu} \in \Delta(I \times I)$ where the first coordinate has distribution μ and the transition from the first to the second is given by the transition of the markov chain. As the distribution of $\mathbf{i}_t$ conditional on i_{t-1} is given by the Markov chain transitions we consider the implementation of distributions over $I \times I \times J \times K$ that represents the expected long-run average of (i_{t-1}, i_t, j_t, k_t).

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An implicit conclusion that appears “between the lines” of this inequality is that the optimization of the forecaster and the agent needs ‘banking’ with entropy
Information/entropy banking appears also in Neyman and Okada 98 and Gossner and Tomala

Resuls for Finite State Machines

We study repeated games where players strategies are implementable by finite state machines like finite automata or bounded recall strategies. We are interested in the analysis of such interaction where the power of the machines are differentiated.

In particular, we wish to study to what extent can a powerful machine that breaks a complicated code of a simple machine share its codes with a simple machine.

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remark

If μ, σ, and τ are strategies of players 1, 2, and 3 respectively that are implementable by finite automata then the play of a repeated game enters a cycle and thus the expectation of the limiting average payoff is well defined and denoted by $g(\mu, \sigma, \tau)$.

Main result: Finite state machines

$$\bar{V}(m_1, m_2, m_3) = \min_{\mu \in X_1^*(m_1))} \max_{\substack{\sigma \in X_2(m_2) \\ \tau \in X_3(m_3)}} G(\mu, \sigma, \tau) \quad (1)$$

$$V(m_1, m_2, m_3) = \max_{\substack{\sigma \in X_2(m_2) \\ \tau \in X_3(m_3)}} \min_{\mu \in X_1^*(m_1)} G(\mu, \sigma, \tau) \quad (2)$$

where $G(\mu, \sigma, \tau) = g_2(\mu, \sigma, \tau)$. Note that $\bar{V}(m_1, m_2, m_3) \geq V(m_1, m_2, m_3)$. The main result specifies asymptotic conditions on m_1, m_2, m_3 for which the limits of $\bar{V}(m_1, m_2, m_3)$ and $V(m_1, m_2, m_3)$ exist and are equal. Moreover, we characterize the limit.

Formula

Given $x \in \Delta(I)$ we denote by $\mathcal{Q}(x)$ the set of all probability measures Q on $I \times J \times K$ such that

$$H_Q(i,j,k) \geq H_Q(i) + H_Q(k).$$

$$v^* = \min_{x \in \Delta(I)} \max_{Q \in \mathcal{Q}(x)} g_2(Q).$$

Theorem

Theorem 1

$$\limsup_{\log m_3 = o(m_1) \to \infty} \bar{V}(m_1, m_2, m_3) \le v^* \tag{3}$$

and

$$\liminf_{\substack{m_2 > |I|^{2m_1} 2m_1 \to \infty \\ m_3 \to \infty}} V(m_1, m_2, m_3) \ge v^* \tag{4}$$

Special cases of the result are of interest and generalize earlier know results. Consider for example the case where $|J| = 1$. It follows that $\mathcal{Q}(x)$ consists of product distributions and thus $v^* = \min_{x \in \Delta(I)} \max_{z \in \Delta(K)} g(x, z)$ and thus the result implies the result of Ben-Porath.