The Limits of Ex Post Implementation Revisited

by

Sushil Bikhchandani[†]

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Abstract

Existence of ex post incentive compatible mechanisms in models with multi-dimensional signals and interdependent values is possible when there are no consumption externalities. An ex post incentive compatible mechanism is constructed in a model where a single indivisible object is allocated among several buyers with multi-dimensional information and interdependent values. It is the combination of informational externalities (i.e., interdependent values) *and* consumption externalities, rather than informational externalities alone, that leads to non-existence of ex post equilibrium when agents have multi-dimensional signals. As ex post equilibrium has been employed mostly in models with private goods, this is not a significant limitation.

Keywords: ex post incentive compatibility, multi-dimensional information, interdependent values.

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[†]Anderson School of Management, UCLA, CA 90095.

1 Introduction

In models of mechanism design with interdependent values, each player's information is usually modeled as a real number. While this is convenient, it might not capture a significant element of the setting. For instance, suppose that agent A's reservation value for an object is the sum of a private value, which is idiosyncratic to this agent, and a common value, which is the same for all agents in the model. Agent A's private information consists of an estimate of the common value and a separate estimate of his private value. As other agents care only about A's estimate of the common value, a single dimensional statistic will not capture all of A's private information that is relevant to every agent (including A).[1]

Therefore, it is essential to test whether insights from the literature are robust to relaxing the assumption that an agent's private information is a real number. This research agenda has been pursued in two important papers by Jehiel and Moldovanu (2001) and Jehiel, Meyer-ter-Vehn, and Moldovanu (2004). Jehiel and Moldovanu (2001) show that if agents have multi-dimensional information, interdependent values, and independent signals then, unlike in models with single dimensional information, every Bayesian Nash equilibrium is inefficient.[2]

Jehiel, Meyer-ter-Vehn, and Moldovanu (2004) call into question the existence of ex post equilibrium when agents have multi-dimensional information. They show that in a model with two players and two outcomes, ex post incentive compatible mechanisms do not generically exist (except, of course, trivial mechanisms which disregard the reports of players). Jehiel et al. claim that this non-existence result generalizes to any mechanism design setting that has at least two players and two outcomes.

The purpose of this paper is to show that ex post equilibrium exists in a wider class of models with multi-dimensional signals than might seem possible. In particular, Jehiel et al.'s theorem does not rule out existence in settings with private goods. The Jehiel et al. result depends on the assumption that neither agent is indifferent between the two outcomes; its extension to many players and outcomes depends on the assumption that for any pair of outcomes there exist at least two players who are not indifferent between that pair of outcomes. These assumptions are not tenable in private goods economies, i.e., when there are no consumption externalities. Consider

[1] A d-dimensional, $d \geq 2$, private signal s_A can be mapped without any loss of information into a single dimension using a one-to-one function $f : \Re^d \to \Re$. However, agents' values will not be non-decreasing in the signal $f(s_A)$. Hence, the assumption of single dimensional signals is a limitation only in conjunction with the assumption (commonly made in the literature) that signals are ordered so that a higher realization is more favorable.

[2] Postlewaithe & McLean (2004) show that efficient Bayesian implementation is possible when signals are correlated.

the allocation of one indivisible object to two buyers, 1 and 2. There are three possible outcomes: a_i, the good is assigned to buyer i, $i = 1, 2$, and a_0, neither gets the good. Buyer 1 is indifferent between a_2 and a_0 and buyer 2 is indifferent between a_1 and a_0. Therefore, even if buyers have multi-dimensional signals, the possibility that there exists a non-trivial selling mechanism in which truth-telling is ex post incentive compatible for the buyers is not precluded. What is ruled out by Jehiel et al.'s theorem is the existence of a non-trivial ex post incentive compatible mechanism with outcomes a_1 and a_2 only, as neither buyer is indifferent between these two outcomes.

We prove by construction an existence result for ex post incentive compatible mechanisms for the sale of a single indivisible object to n buyers with multi-dimensional signals and interdependent values. In the construction, the rule for deciding whether some buyer A should be assigned the object is as follows. Fix the other buyers' signals at some realization. Partition buyer A's set of possible signal realizations into equivalence classes such that A's reservation value is constant on an equivalence class. These equivalence classes are completely ordered by the buyer A's value. If (a generalization of) the single crossing property is satisfied then there exists a pivotal equivalence class with the property that it is ex post incentive compatible to award the object to buyer A if and only if A's signal is in an equivalence class which is greater than the pivotal one. If he wins, the price paid by A is equal to his value in the pivotal equivalence class (which is also equal to the maximum of other buyers' values on A's pivotal equivalence class).[3] This mechanism is non-trivial if the efficient allocation rule is non-trivial.

There are no consumption externalities in our model. Thus, it is the combination of informational externalities (i.e., interdependent values) with consumption externalities, rather than informational externalities alone, that lead to non-existence of ex post equilibrium when agents have multi-dimensional signals. As ex post equilibrium has been employed mostly in models with private goods, this is not a significant limitation.[4]

The paper is organized as follows. A model with two buyers is presented in Section 2. In Section 3, we construct an ex post incentive compatible mechanism in an example from Jehiel et al. and illustrate why their theorem does not imply non-existence in economies with private goods. An existence result for ex post incentive compatible mechanisms is proved in Section 4. This result is generalized to n buyers in Section 4.1.

[3]This generalizes the usual mechanism in single dimensional signal models, where each equivalence class is a singleton; the price paid is equal to the winning buyer's value at the pivotal signal which is equal to the maximum among the buyers' values, computed at the winner's pivotal signal.

[4]See, for example, Cremer and McLean (1985), Ausubel (1999), Dasgupta and Maskin (2000), Perry and Reny (2002), and Bergemann and Valimaki (2002).

2 The model

The main idea can be seen in a model with two buyers and one indivisible object. Each buyer i, $i = 1, 2$, receives a $d^i \geq 2$ dimensional private signal, denoted $s_i = (s_{i1}, s_{i2}, ..., s_{id_i})$. The domain of s_i is $S_i = [0,1]^{d_i}$, the unit cube in $\Re_+^{d_i}$. The buyers' signals are denoted $s = (s_1, s_2)$ with domain $S = S_1 \times S_2$. We also refer to s as an information state. Bidder i's valuation is $V_i(s_i, s_j)$ (also denoted $V_i(s)$). Buyers have quasilinear utility. If buyer i gets the object in state s and pays t, then his utility is $V_i(s) - t$; if he does not get the object and pays t, his utility is $-t$.

Denote by a_i, $i = 1, 2$, the outcome in which buyer i is allocated the object. The outcome in which no buyer gets the object is denoted a_0. A (deterministic) mechanism consists of an allocation rule h and two payment functions $\hat{t}_i$, $i = 1, 2$. The allocation rule $h : S \to \{a_0, a_1, a_2\}$ is a function from the buyers' reported signals to an allocation of the indivisible object to either no buyer or buyer 1 or buyer 2; the payment function $\hat{t}_i : S \to \Re$ is a function from the buyers' reported signals to a money payment by buyer i. A mechanism is *ex post incentive compatible* if for $i = 1, 2$, $i \neq j$,

$$V_i(s_i, s_j)1_{\{h(s_i,s_j)=a_i\}} - \hat{t}_i(s_i, s_j) \;\geq\; V_i(s_i, s_j)1_{\{h(s_i',s_j)=a_i\}} - \hat{t}_i(s_i', s_j), \qquad \forall s_i,\ \forall s_i',\ \forall s_j \tag{1}$$

where 1_A denotes the indicator function of the event A. In other words, at each information state if buyer j truthfully reports his signal then buyer i can do no better than truthfully report his signal.[5]

It is well-known that ex post incentive compatibility implies that the money payment made by buyer i depends on (i) whether or not buyer i is assigned the object and (ii) buyer j's reported signal, $j \neq i$. Further, we restrict attention to mechanisms in which a buyer pays nothing if he does not get the object; that is, $\hat{t}_i(s_i, s_j) = 0$ if $h(s) \neq a_i$.[6] Consequently, we write the money payment function as

$$\hat{t}_i(s_i, s_j) \;\equiv\; \begin{cases} t_i(s_j), & \text{if } h(s_i, s_j) = a_i, \\ 0, & \text{otherwise.} \end{cases}$$

The function $t_i(s_j)$ is buyer i's payment conditional on getting the object. We interpret $t_i(s_j)$ as a *personalized price* at which the object is available to buyer i. Condition (1), the requirement for ex post incentive compatibility, may be rewritten

[5] Ex post incentive compatibility is the same as uniform equilibrium of D'Aspremont and Gerard-Varet (1979) and uniform incentive compatibility of Holmstrom and Myerson (1983).

[6] By adding to $\hat{t}_i(s_i, s_j)$ a lump sum payment $\phi_i(s_j)$, one can get ex post implementable mechanisms where this restriction does not hold.

as follows for mechanisms in which losing buyers pay nothing. For $i = 1, 2$, $i \neq j$,

$$\Big[V_i(s_i, s_j) - t_i(s_j)\Big] 1_{\{h(s_i,s_j)=a_i\}} \;\geq\; \Big[V_i(s_i, s_j) - t_i(s_j)\Big] 1_{\{h(s_i',s_j)=a_i\}}, \qquad \forall s_i,\, \forall s_i',\, \forall s_j. \tag{2}$$

A pair of personalized price functions $t_i(s_j)$, $t_j(s_i)$ is *admissible* if

$$V_i(s_i, s_j) \;>\; t_i(s_j) \;\Longrightarrow\; V_j(s_i, s_j) \;\leq\; t_j(s_i), \qquad \forall s_i, s_j. \tag{3}$$

That is, in each information state at most one buyer's value exceeds his personalized price. An allocation rule *implements* an admissible pair of prices t_1, t_2 if the rule assigns the object to a buyer if and only if the buyer's value exceeds his personalized price. That is, the allocation rule

$$h(s_1, s_2) \;\equiv\; \begin{cases} a_1, & \text{if } V_1(s_1, s_2) > t_1(s_2) \\ a_2, & \text{if } V_2(s_2, s_1) > t_2(s_1) \\ a_0, & \text{otherwise.} \end{cases} \tag{4}$$

implements the admissible pair t_1, t_2. Clearly, h is a feasible allocation rule in that it does not allocate more than one object. Observe that a buyer cannot change his personalized price by lying about his private signal, as each buyer's price depends only on the other buyer's (reported) signal. At each information state (s_i, s_j), the mechanism (h, t) allocates the object to buyer i for a payment of $t_i(s_j)$ if and only if $V_i(s_i, s_j) > t_i(s_j)$. Suppose that the information state is (s_i, s_j) and buyer i reports $s_i' \neq s_i$. If he gets the same allocation at (s_i, s_j) and (s_i', s_j) then the lie is not profitable. Therefore, suppose that $h(s_i, s_j) \neq a_i$ and $h(s_i', s_j) = a_i$. But then $V_i(s_i, s_j) \leq t_i(s_j)$ and with a report of s_i' buyer i buys at a price at least as large as his value for the object. Similarly, if $h(s_i, s_j) = a_i$ and $h(s_i', s_j) \neq a_i$ then with a report of s_i' he ends up not buying the object at a price strictly less than his value. Thus, (h, t) satisfies (2) (and 1) and is ex post incentive compatible. We have

Lemma 1: *If an allocation rule h implements an admissible pair of personalized prices $t = (t_1, t_2)$ then the mechanism (h, t) is ex post incentive compatible.*

A mechanism is *non-trivial* if there exist two distinct outcomes, each of which is implemented at a positive (Lebesgue) measure of information states by the mechanism.

It is possible to satisfy (3) by choosing personalized prices so high that each buyer's valuation is always less than his personalized price. Such prices lead to the trivial ex post incentive compatible mechanism in which $h(s) = a_0$, $\forall s$. In Section 4, we show that under reasonable conditions on buyers' information, there exists an admissible pair of personalized prices which is implemented by a non-trivial ex post incentive compatible mechanism. In particular, each of the outcomes a_0, a_1, and a_2 occur at a positive measure of information states. First, we illustrate a non-trivial mechanism in an example.

3 An example

The following example is from Jehiel, Meyer-ter-Vehn, and Moldovanu (2004). Contrary to their claim, this example has a (continuum of) non-trivial ex post incentive compatible mechanism(s).

Example 1: Two buyers compete for a single indivisible object. Each gets a pair of signals (p_i, c_i), $i = 1, 2$. Buyer i's valuation for the object is $V_i(p_i, c_i, p_j, c_j) = p_i + c_i c_j$, $j \neq i$. Further, each buyer's signal lies in the unit square: $(p_i, c_i) \in [0, 1]^2$, $i = 1, 2$.

Consider personalized prices $t_i(p_j, c_j) \equiv p_j + c_j^2$, $t_j(p_i, c_i) \equiv p_i + c_i^2$. Suppose that

$$V_i(p_i, c_i, p_j, c_j) \;=\; p_i + c_i c_j \;>\; p_j + c_j^2 \;=\; t_i(p_j, c_j).$$

Then

$$p_i - p_j \;>\; c_j^2 - c_i c_j \;\geq\; c_i c_j - c_i^2,$$

where we use the fact that $c_i^2 + c_j^2 - 2c_i c_j \geq 0$. Therefore,

$$V_j(p_i, c_i, p_j, c_j) \;=\; p_j + c_i c_j \;<\; p_i + c_i^2 \;=\; t_j(p_i, c_i).$$

Consequently, personalized prices $p_2 + c_2^2$ for buyer 1 and $p_1 + c_1^2$ for buyer 2 are admissible, that is they satisfy (3).

Using (4), define an allocation rule which implements these prices. In this mechanism, the buyers report their private signals to the mechanism designer. The mechanism designer allocates the object to buyer i for a payment equal to his personalized price $t_i(p_j, c_j) = p_j + c_j^2$ if and only if $V_i(p_i, c_i, p_j, c_j) = p_i + c_i c_j$ exceeds $t_i(p_j, c_j)$. By Lemma 1, this mechanism is ex post incentive compatible.

Let $h^{-1}(a_i)$ be the set of information states which are mapped on to a_i by this allocation mechanism. Each of the sets

$$\begin{aligned} h^{-1}(a_i) &= \{(p_i, c_i, p_j, c_j) \in [0, 1]^4 \,|\, p_i - p_j > c_j^2 - c_i c_j\}, \quad i = 1, 2 \\ h^{-1}(a_0) &= \{(p_i, c_i, p_j, c_j) \in [0, 1]^4 \,|\, c_i c_j - c_i^2 \leq p_i - p_j \leq c_j^2 - c_i c_j\} \end{aligned}$$

is of positive measure. Hence, the mechanism is non-trivial.[7]

[7] In fact, for any $(p_i, c_i) \neq (0, 0)$ and $(p_i, c_i) \neq (1, 1)$, each of the outcomes a_0, a_1, and a_2 is implemented for a positive measure of buyer j signals.

The boundary between the sets $h^{-1}(a_1)$ and $h^{-1}(a_2)$ is:

$$\overline{h^{-1}(a_1)} \cap \overline{h^{-1}(a_2)} \;=\; \{(p_1, c_1, p_2, c_2) \in [0,1]^4 \,|\, p_1 = p_2,\ c_1 = c_2 = 0\}$$

where $\overline{A}$ is the closure of set A. This boundary is a one dimensional set and the projection (onto buyer i's signal space) of boundary points with a fixed value of buyer j signal $(p_j, c_j) = (p, 0)$ is the single point $(p_i, c_i) = (p, 0)$. We shall return to this below.

There exists a continuum of non-trivial ex post incentive compatible mechanisms in this example. Consider personalized prices $t'_i(p_j, c_j) = p_j + c_j^2 + \epsilon_i(p_j, c_j)$ where $\epsilon_i(p_j, c_j)$ is non-negative. Since $t_i(p_j, c_j) = p_j + c_j^2$ satisfy (3), so do the prices $t'_i(p_j, c_j)$. An allocation rule that implements t'_1, t'_2 is ex post incentive compatible. For small enough $\epsilon_i(p_j, c_j)$, this mechanism is non-trivial. △

We summarize the main result of Jehiel, Meyer-ter-Vehn, and Moldovanu (2004). Consider a setting with two agents, 1 and 2, and two outcomes, a_1 and a_2. Each agent i gets a d^i-dimensional signal. Define $\mu^i(s)$, $i = 1, 2$, to be the difference between i's utility for outcomes a_1 and a_2.[8] The domain of signals, S, is shown schematically in Figure 1a. Any allocation rule partitions S into two subsets, depending on whether $h(s) = a_1$ or $h(s) = a_2$. The boundary between these two sets is the broken line in Figure 1a. Jehiel et al. show that for any non-trivial allocation rule (i) the projection of points on this boundary with fixed value of s_j onto the domain of i's signals, S_i, $i = 1, 2$, is a $d^i - 1$ dimensional submanifold[9] and (ii) for any ex post incentive compatible allocation rule the gradients of $\mu^1(s)$ and $\mu^2(s)$ must be, roughly speaking, co-directional on this submanifold. For generic $\mu^i(s)$ it is impossible to satisfy (i) and (ii). Thus, when there are two agents and two outcomes, non-trivial ex post incentive compatible mechanisms do not exist for generic utilities.

Jehiel et al. then assert that the non-existence result "immediately generalizes as this 2 by 2 model is naturally embedded in every model with more agents and alternatives." While the two by two model is naturally embedded in every model the non-existence result does not generalize. Even when there are only two agents and two outcomes, the argument summarized in the preceding paragraph depends on the assumption that each agent is not indifferent between the two outcomes a_1 and a_2 (or to be exact, for utilities which are generic in the space they consider agents are not indifferent between these two outcomes). Suppose we add a third outcome, a_0, to Jehiel et al.'s model. For their argument to extend it must be case that each agent is not indifferent between any two of the three outcomes. However, this assumption is

[8]Thus, in the above example, we restrict attention to mechanisms which allocate the object to a buyer at every information state s and $\mu^1(s) = V_1(s)$, $\mu^2(s) = -V_2(s)$.

[9]Hereafter, this condition is referred to as the boundary is of full dimension.

not satisfied in the above example where at every information state buyer i, $i = 1, 2$, is indifferent between a_0 and a_j, $j \neq i$.[10]

Therefore, consider a setting with two agents and three outcomes a_0, a_1, and a_2. Suppose that $a_0(s) \sim_1 a_2(s)$, $\forall s$ (i.e., agent 1 is indifferent between a_0 and a_2 in every information state) and $a_0(s) \sim_2 a_1(s)$, $\forall s$. This condition is satisfied in the allocation of a single object to (at most) one of two buyers, provided that there are no consumption externalities. Now consider a non-trivial allocation rule h that yields each of the three outcomes a_0, a_1, and a_2. All that Jehiel et al.'s theorem implies is that if h is ex post incentive compatible then the boundary between the $h^{-1}(a_1)$ and $h^{-1}(a_2)$ has less than full dimension. Their theorem does not impose any restriction on the dimensionality of the boundary between a_0 and a_i, $i = 1, 2$. In particular, the possibility that h partitions S as shown in Figure 1b is not ruled out. In fact, this figure is a schematic representation of Example 1 where we demonstrated existence of an ex post incentive compatible mechanism in which the boundary between $h^{-1}(a_1)$ and $h^{-1}(a_2)$ is a one dimensional set in S and the projection onto S_i of points on the boundary with a fixed value of s_j is a single point.

More generally, suppose there are $i = 1, 2, ..., n$ agents and L outcomes labeled a_ℓ, $\ell = 1, 2, ..., L$. Suppose that for some outcome a_ℓ there exists an outcome a_k and an agent j (where a_k may depend on a_ℓ and j may depend on a_ℓ and a_k) such that

$$a_\ell(s) \sim_i a_k(s), \quad \forall s, \ \forall i \neq j. \tag{5}$$

Then, for any ex post incentive compatible allocation rule h, the Jehiel et al. theorem places no restriction on the boundary between $h^{-1}(a_\ell)$ and $h^{-1}(a_k)$. Generic existence of a non-trivial ex post incentive compatible mechanism with outcomes a_ℓ and a_k is not precluded by the Jehiel et al. theorem when (5) is satisfied.

Consider the allocation of a bundle of private goods to n buyers. Each outcome is an assignment of objects among the n buyers, where we allow the possibility that not all objects are allocated to the buyers. Let a_ℓ be any assignment and let a_k be another assignment which differs from a_ℓ only in the allocation that buyer j receives.[11] Condition (5) is satisfied for each assignment a_ℓ. A full range ex post incentive compatible mechanism is a possibility.[12]

[10]In Example 1, apart from mechanisms with three outcomes described above, there also exist ex post incentive compatible mechanisms with two outcomes a_0 and, say, a_1.

[11]Thus, not all the objects are allocated in at least one of the two assignments a_ℓ, a_k.

[12]A mechanism has full range if each outcome is implemented at a positive measure of information states.

4 The main result

We prove an existence theorem for non-trivial ex post incentive compatible mechanisms for the allocation of a single object when buyers have multi-dimensional signals. The starting point is the model described in Section 2. First, we assume that higher signals correspond to better news. That is, players' reservation value do not decreases with buyer signals.[13] In order to simplify the proofs, we also assume that buyers' reservation values are continuous.

Assumption 1a: *$V_i(s)$ is non-decreasing in s, $i = 1, 2$.*
1b: *$V_i(\cdot)$ is continuous in all its arguments.*

The next assumption is a generalization of the single crossing property.

Assumption 2: *For any s_j we have*[14]

$$V_i(s'_i, s_j) - V_i(s_i, s_j) \quad \geq \quad V_j(s_j, s'_i) - V_j(s_j, s_i), \qquad \forall s'_i > s_i.$$

As buyer i's signal increases from s_i to s'_i, the increase in i's value is greater than the increase in buyer j's value. This is a requirement that buyer i's value is more sensitive than buyer j's value to changes in buyer i's signal. In a model with one dimensional signals, Assumption 2 is the single crossing property, which is a sufficient condition for existence of an efficient mechanism in such models (see Maskin 1992).

The next assumption avoids trivialities. If this assumption is violated, then there is a buyer i whose valuation is always (weakly) greater than the other buyer j's valuation; therefore the efficient rule (which is to always allocate the object to buyer i) is both trivial and ex post incentive compatible.

Assumption 3: *For each buyer, there exists a positive measure of information states at which this buyer's valuation is strictly greater than the other buyer's valuation.*

With Assumptions 1 and 2, we construct a personalized price function for each buyer. This pair of price functions is shown to satisfy (3) and is used to define an ex post incentive compatible mechanism. Assumption 3 will imply that this ex post incentive compatible mechanism is non-trivial.

Fix buyer j's signal at some level s_j. The domain of s_i, $i \neq j$, is the unit cube in $\Re^{d_i}_+$ and each buyer's valuation is non-decreasing in s_i. Therefore, with buyer j's signal

[13] The following terminology regarding monotonicity of a function $f : \Re^n \to R$ is adopted. For $x,\ x' \in \Re^n$, $x' > x$ denotes that x' is at least as large as x in every co-ordinate and $x' \neq x$. If $f(x') \geq f(x)$ whenever $x' > x$ then f is *non-decreasing*.

[14] An equivalent assumption is that for each s_j, $V_i(s_i, s_j) - V_j(s_j, s_i)$ is a non-decreasing function of s_i.

at s_j, the maximum of either buyer's reservation value as a function of buyer i's signal is attained when $s_i = \mathbf{1}$, where $\mathbf{1}$ denotes the point $(1, 1, ..., 1)$ in $\Re_+^{d_i}$. Similarly, the minimum of either buyer's value as a function of s_i is attained at $s_i = \mathbf{0} \equiv (0, 0, ..., 0)$. Define the set of signals of buyer i which lead to the same reservation value for buyer i as signal $\hat{s}_i = \lambda\mathbf{1}$:

$$S_i(\lambda, s_j) \;\equiv\; \{s_i \in S_i \,|\, V_i(s_i, s_j) = V_i(\lambda\mathbf{1}, s_j)\,\}, \qquad 0 \leq \lambda \leq 1.$$

Thus, for a fixed s_j, buyer i's signal space partitions into equivalence classes or "indifference" curves, $S_i(\lambda, s_j)$, one for each $\lambda \in [0, 1]$. For fixed s_j, equivalence classes are naturally ordered by λ as larger λ leads to larger buyer i reservation values.

While buyer i's value (as a function of s_i) is constant on $S_i(\lambda, s_j)$, buyer j's value will, in general, not be constant on this set. The maximum of buyer j's value on buyer i's equivalence class $S_i(\lambda, s_j)$ is

$$V_{ij}^m(\lambda, s_j) \;\equiv\; \max_{s_i \in S_i(\lambda, s_j)} V_j(s_j, s_i).$$

As S_i is compact and $V_i(\cdot, s_j)$ is continuous, $S_i(\lambda, s_j)$ is compact. The continuity of $V_i(\cdot, s_j)$ and $V_j(s_j, \cdot)$ implies that $V_{ij}^m(\lambda, s_j)$ exists and is continuous in λ and s_j. Let $s_{ij}^m(\lambda, s_j)$ be

$$s_{ij}^m(\lambda, s_j) \;\in\; \arg \max_{s_i \in S_i(\lambda, s_j)} V_j(s_j, s_i).$$

Thus, $V_{ij}^m(\lambda, s_j) = V_j(s_j, s_{ij}^m(\lambda, s_j))$ and $V_i(s_{ij}^m(\lambda, s_j), s_j) = V_i(\lambda\mathbf{1}, s_j)$. Observe that

$$V_j(s_j, s_i) \;\leq\; V_{ij}^m(\lambda, s_j), \qquad \forall s_i \in S_i(\lambda, s_j). \tag{6}$$

For a fixed realization of s_j, Figure 2 depicts indifference curves (i.e., equivalence classes) of buyers i and j in buyer i's (two dimensional) signal space. By Assumption 1a, indifference curves will be negatively sloped. However, they need not be convex and, as would be case in Example 1, they may touch the axes. $V_{ij}^m(\lambda, s_j)$, the maximum value of buyer j in buyer i's equivalence class $S_i(\lambda, s_j)$, may be attained at more than one value.

Ex post incentive compatibility imposes the following necessary condition. If buyer 1, say, is allocated the object at information state (s_1, s_2), then he should also be allocated the object at any information state (s_1', s_2) such that $V_1(s_1', s_2) > V_1(s_1, s_2)$. Otherwise, buyer 1 would have an incentive to report s_1 instead of s_1' at the information state (s_1', s_2). That is, an incentive compatible allocation rule must be non-decreasing in the marginal utility (NDMU) of the buyers.[15] Or in the terminology

[15] See Bikhchandani, Chatterji, and Sen (2003) for conditions under which NDMU is also sufficient for incentive compatibility.

of equivalence classes, if $s_1 \in S_1(\lambda, s_2)$ and buyer 1 is allocated the object at (s_1, s_2), then buyer 1 must be allocated the object at all $s_1' \in S_1(\lambda', s_2)$ where $\lambda' > \lambda$.

We construct an ex post incentive compatible mechanism in which, for each value of s_j, there exists a $\lambda_{ij}^*(s_j) \in [0,1]$ such that buyer i wins if his signal is in an equivalence class greater than $\lambda_{ij}^*(s_j)$, and buyer i loses otherwise. Clearly, this allocation rule satisfies NDMU. Call $S_i(\lambda_{ij}^*(s_j), s_j)$ the *pivotal* equivalence class for buyer i at s_j. (Any s_i in the pivotal equivalence class is a pivotal signal for buyer i.) Buyer i's personalized price is defined to be his value on the pivotal equivalence class. The next lemma is used to prove that personalized prices defined in this manner are admissible. It states that as buyer i's signal increases from equivalence class λ to equivalence class λ', the increase in i's value is at least as great as the increase in the maximum of buyer j's value on these two equivalence classes.

Lemma 2: *For any s_j and $1 \geq \lambda' > \lambda > 0$,*

$$V_i(\lambda'\mathbf{1}, s_j) - V_{ij}^m(\lambda', s_j) \geq V_i(\lambda\mathbf{1}, s_j) - V_{ij}^m(\lambda, s_j).$$

Proof: To simplify notation, we write $s_{ij}^m(\lambda')$, $s_{ij}^m(\lambda)$ for $s_{ij}^m(\lambda', s_j)$, $s_{ij}^m(\lambda, s_j)$. Figure 2 illustrates the proof.

Let $\lambda^m \in [0,1]$ be such that $V_i(\lambda^m s_{ij}^m(\lambda'), s_j) = V_i(\lambda\mathbf{1}, s_j)$. To see that λ^m exists, define $f(x) \equiv V_i(x s_{ij}^m(\lambda'), s_j)$, where $x \in [0,1]$ and note that $f(1) = V_i(s_{ij}^m(\lambda'), s_j) = V_i(\lambda'\mathbf{1}, s_j) \geq V_i(\lambda\mathbf{1}, s_j) \geq V(0, s_j) = f(0)$. By Assumption 1b, $f(x)$ is a continuous function of x, and therefore there exists λ^m such that $f(\lambda^m) = V_i(\lambda^m s_{ij}^m(\lambda'), s_j) = V_i(\lambda\mathbf{1}, s_j)$. Hence,

$$\begin{aligned}
V_i(\lambda\mathbf{1}, s_j) - V_{ij}^m(\lambda, s_j) &= V_i(\lambda^m s_{ij}^m(\lambda'), s_j) - V_{ij}^m(\lambda, s_j) \\
&\leq V_i(\lambda^m s_{ij}^m(\lambda'), s_j) - V_j(s_j, \lambda^m s_{ij}^m(\lambda')) \\
&\leq V_i(s_{ij}^m(\lambda'), s_j) - V_j(s_j, s_{ij}^m(\lambda')) \\
&= V_i(\lambda'\mathbf{1}, s_j) - V_{ij}^m(\lambda', s_j)
\end{aligned}$$

where the first inequality follows from the fact that $\lambda^m s_{ij}^m(\lambda') \in S_i(\lambda, s_j)$ and (6), and the second inequality from Assumption 2. ∎

For $\lambda \in [0,1]$, define

$$g_{ij}(\lambda; s_j) \equiv V_i(\lambda\mathbf{1}, s_j) - V_{ij}^m(\lambda\mathbf{1}, s_j).$$

Lemma 2 implies that $g_{ij}(\lambda; s_j)$ is a non-decreasing function of λ. The continuity of V_i and of V_{ij}^m implies that $g_{ij}(\lambda; s_j)$ is a continuous function. Thus, the following is well-defined:

$$\lambda^*_{ij}(s_j) \quad \equiv \quad \begin{cases} 1, & \text{if } g_{ij}(1; s_j) < 0, \\ \max\{\lambda \in [0,1] \,|\, g_{ij}(\lambda; s_j) = 0\}, & \text{if } g_{ij}(1; s_j) \geq 0 \geq g_{ij}(0; s_j), \\ 0, & \text{if } g_{ij}(0; s_j) > 0. \end{cases}$$

If $1 \geq \lambda > \lambda^*_{ij}$ then $V_i(\lambda\mathbf{1}, s_j) > V^m_{ij}(\lambda\mathbf{1}, s_j)$, and if $\lambda^*_{ij} > \lambda \geq 0$ then $V_i(\lambda\mathbf{1}, s_j) \leq V^m_{ij}(\lambda\mathbf{1}, s_j)$.[16] Let

$$t^*_i(s_j) \quad \equiv \quad V_i(\lambda^*_{ij}\mathbf{1}, s_j) \quad = \quad V^m_{ij}(\lambda^*_{ij}, s_j) \tag{7}$$

be buyer i's personalized price as a function of s_j.

Let $\lambda_i(s_i, s_j)$ be the index of the equivalence class that s_i belongs to at s_j. That is, $s_i \in S_i(\lambda_i(s_i, s_j), s_j)$. The main result shows that the following allocation rule is non-trivial and ex post incentive compatible: buyer i wins if and only if $V_i(s_i, s_j) > V^m_{ij}(\lambda_i(s_i, s_j), s_j)$. This rule is implemented through the personalized prices defined above.

Theorem: *The personalized prices $t^* = (t^*_1, t^*_2)$ defined in (7) are admissible. The mechanism (h^*, t^*), where h^* implements t^*, is non-trivial and ex post incentive compatible.*

Proof: Suppose that the information state is (s_1, s_2). We write λ_i instead of $\lambda_i(s_i, s_j)$ to simplify the notation. That is, $V_i(s_i, s_j) = V_i(\lambda_i\mathbf{1}, s_j)$. Note that (6) implies

$$V_2(s_2, s_1) \;\leq\; V^m_{12}(\lambda_1, s_2), \qquad V_1(s_1, s_2) \;\leq\; V^m_{21}(\lambda_2, s_1). \tag{8}$$

Suppose that $V_1(s_1, s_2) > t^*_1(s_2) = V_1(\lambda^*_{12}\mathbf{1}, s_2)$. Thus, $V_1(s_1, s_2) = V_1(\lambda_1\mathbf{1}, s_2) > V_1(\lambda^*_{12}\mathbf{1}, s_2)$. In other words, $\lambda_1 > \lambda^*_{12}$. The definition of λ^*_{12} and Lemma 2 imply that $V_1(s_1, s_2) = V_1(\lambda_1\mathbf{1}, s_2) > V^m_{12}(\lambda_1, s_2)$. Hence, (8) implies that $V^m_{21}(\lambda_2, s_1) > V_2(s_2, s_1) = V_2(\lambda_2\mathbf{1}, s_1)$. Hence, $\lambda^*_{21} > \lambda_2$ and $V_2(s_2, s_1) = V_2(\lambda_2\mathbf{1}, s_1) \leq V_2(\lambda^*_{21}\mathbf{1}, s_1) = t^*_2(s_1)$.

If instead $V_2(s_1, s_1) > t^*_2(s_1)$, a similar argument implies that $V_1(s_1, s_2) \leq t^*_1(s_2)$. Thus, the personalized prices satisfy (3). Therefore, by Lemma 1, the allocation rule

$$h^*(s_1, s_2) \quad \equiv \quad \begin{cases} a_1, & \text{if } V_1(s_1, s_2) > t^*_1(s_2) \\ a_2, & \text{if } V_2(s_2, s_1) > t^*_2(s_1) \\ a_0, & \text{otherwise} \end{cases}$$

which implements admissible prices $t^*_1(s_2)$, $t^*_2(s_1)$ is feasible and ex post incentive compatible.

To complete the proof, we show that the mechanism is non-trivial. Let information state $s^1 = (s^1_1, s^1_2)$ be such that $V_1(s^1_1, s^1_2) > V_2(s^1_1, s^1_2)$. Assumption 3 guarantees that

[16] Hereafter, the dependence of λ^*_{ij} on s_j is usually suppressed to simplify the notation.

a positive measure of such information states exist. By Assumption 2, $V_1(\mathbf{1}, s_2^1) > V_2(s_2^1, \mathbf{1})$ and by Assumption 1a, $V_2(s_2^1, \mathbf{1}) = V_{12}^m(1, s_2^1)$. Thus, $V_1(\mathbf{1}, s_2^1) > V_{12}^m(1, s_2^1)$ and $\lambda_{12}^*(s_2^1) < 1$. Hence buyer 1 gets the object at (s_1, s_2^1) for all $s_1 \in S_1(\lambda, s_2^1)$, $\lambda \in (\lambda_{12}^*(s_2^1), 1]$. As there are a positive measure of such information states (s_1^1, s_2^1) at which buyer 1's value is strictly greater than than buyer 2's value, there is a positive of information states at which buyer 1 is allocated the object. A similar argument establishes that buyer 2 is allocated the object at a positive measure of information states. Hence, the mechanism is non-trivial. ∎

This mechanism is weakly efficient in the sense that if the object is assigned to a buyer then this buyer must have the highest reservation value. To see this, suppose that buyer i is allocated the object at information state $s = (s_i, s_j)$. Let $\lambda_i(s_i, s_j)$ and $\lambda_{ij}^*(s_j)$ be defined at this state in the usual manner. Then, from the proof of the theorem it is clear that $\lambda_i(s_i, s_j) > \lambda_{ij}^*(s_j)$ and therefore $V_i(s_i, s_j) > V_{ij}^m(\lambda_i(s_i, s_j), s_j) \geq V_j(s_j, s_i)$.

From Jehiel and Moldovanu (2001) we know that this mechanism cannot be efficient. This may also be verified directly: neither buyer is allocated the object at information states $s = (s_i, s_j)$ such that $\lambda_i(s) \leq \lambda_{ij}^*(s_j)$ and $\lambda_j(s) \leq \lambda_{ji}^*(s_i)$. Such information states have positive measure (provided, of course, that Assumption 3 is satisfied).

Recall that any signal s_i in the pivotal equivalence class $S_i(\lambda_{ij}^*, s_j)$ is a pivotal signal for buyer i at s_j. With bidder j's signal fixed at s_j, buyer i wins (loses) at signals greater (less) than a pivotal signal. Thus a pivotal signal is an infimum of winning signals. The price paid by a winning buyer i equals the valuation of this buyer at a pivotal signal. This is similar to the efficient mechanisms in Ausubel (1999) and Dasgupta and Maskin (2000), where buyers have one dimensional signals.[17] However, unlike in these models, in the mechanism of the above theorem the valuations of buyers i and j need not be equal at a pivotal signal of buyer i; at a pivotal signal for buyer i, buyer i's valuation equals the most that buyer j's valuation can be in the pivotal equivalence class of buyer i. The difference arises because in one dimensional models, equivalence classes (or indifference curves) of buyer i signals are singletons and hence for a given realization of s_j there can be only one pivotal signal for bidder i. A second difference is the role of the single crossing property or Assumption 2. With one dimensional signals, the single crossing property is sufficient for existence of an efficient ex post incentive compatible mechanism whereas with multi-dimensional signals Assumption 2 is sufficient for the existence of an ex post

[17]There is one difference in Dasgupta and Maskin (2000). The mechanism designer (auctioneer) does not know the mapping from buyer signals to valuations. Hence, buyers submit contingent bids rather than report their private signals.

incentive compatible mechanism mechanism (which, as already noted, satisfies only a weak form of efficiency).

4.1 Extension to many buyers

We outline the minor changes in notation, assumptions, and analysis required to extend the existence theorem to $n > 2$ buyers. Each buyer's valuation depends on the (possibly multi-dimensional) signals of all n buyers. The information states are denoted $s = (s_1, s_2, ..., s_n) = (s_i, s_{-i})$. Change s_j to s_{-i} in Assumption 2, and require the assumption to hold for every V_i and V_j. Assumption 3 is required to hold for two distinct buyers, i.e., there exist two sets of information states, A^i and A^j, each set of positive measure, such that buyer i's [j's] value is strictly greater than all other buyers' values on the set A^i [A^j]. We write $V_i(s_i, s_{-i})$, $V_{ij}^m(\lambda, s_{-i})$, $g_{ij}(\lambda; s_{-i})$ instead of $V_i(s_i, s_j)$, $V_{ij}^m(\lambda, s_j)$, $g_{ij}(\lambda; s_j)$, etc. The definition of $\lambda_{ij}^*(s_{-i})$ is:

$$\lambda_{ij}^*(s_{-i}) \quad \equiv \quad \begin{cases} 1, & \text{if } g_{ij}(1; s_{-i}) < 0, \\ \max\{\lambda \in [0,1] \,|\, g_{ij}(\lambda; s_{-i}) = 0\}, & \text{if } g_{ij}(1; s_{-i}) \geq 0 \geq g_{ij}(0; s_{-i}), \\ 0, & \text{if } g_{ij}(0; s_{-i}) > 0. \end{cases}$$

where $g_{ij}(\lambda; s_{-i}) \equiv V_i(\lambda \mathbf{1}, s_{-i}) - V_{ij}^m(\lambda \mathbf{1}, s_{-i})$. Let $\lambda_i^* \equiv \max_{j \neq i} \lambda_{ij}^*$. Buyer i's personalized price is

$$t_i^*(s_j) \quad \equiv \quad V_i(\lambda_i^* \mathbf{1}, s_{-i}) \;=\; \max_{j \neq i} V_{ij}^m(\lambda_{ij}^*, s_{-i})$$

Once again, buyer i's personalized price equals his valuation at a pivotal signal which equals the maximum valuation of all other buyers on the pivotal equivalence class.

5 Concluding Remarks

If there are no consumption externalities, then Jehiel et al. (2004) does not rule out ex post implementation when buyers have multi-dimensional information and interdependent values. We established that in a simple private goods model in which an indivisible object is allocated to one among several buyers, an ex post incentive compatible mechanism exists when agents have multi-dimensional information. Existence was proved under the assumption that buyers' information satisfies a generalization of the single crossing property. The mechanism shares the feature with the generalized Vickrey auction of single dimensional information models that the price paid by the winning buyer is equal to this buyer's value at the lowest possible signal (equivalence class) at which this buyer would just win.

Selfish preferences are a natural assumption for private goods models. At a tiny perturbation of preferences away from selfish preferences, (5) is not satisfied by any pair of outcomes. Jehiel et al.'s theorem would then imply non-existence of non-trivial ex post incentive compatible mechanisms. However, as non-trivial mechanisms that are almost ex post incentive compatible still exist at these perturbed preferences, ex post incentive equilibrium is a robust equilibrium concept for private goods models. Under a small departure from the usual assumption of selfish preferences in private goods models, many results in economics would be only approximately true.

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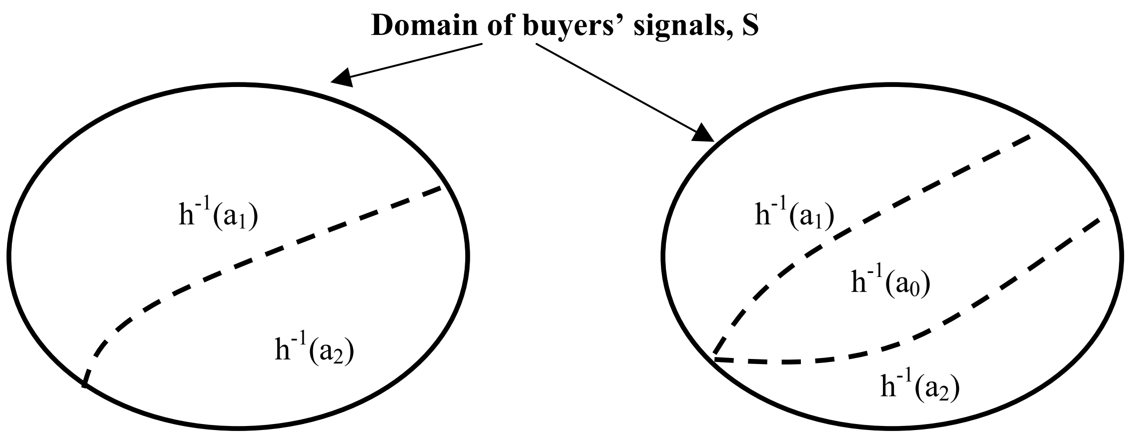

Two alternatives
Figure 1a

Three alternatives
Figure 1b

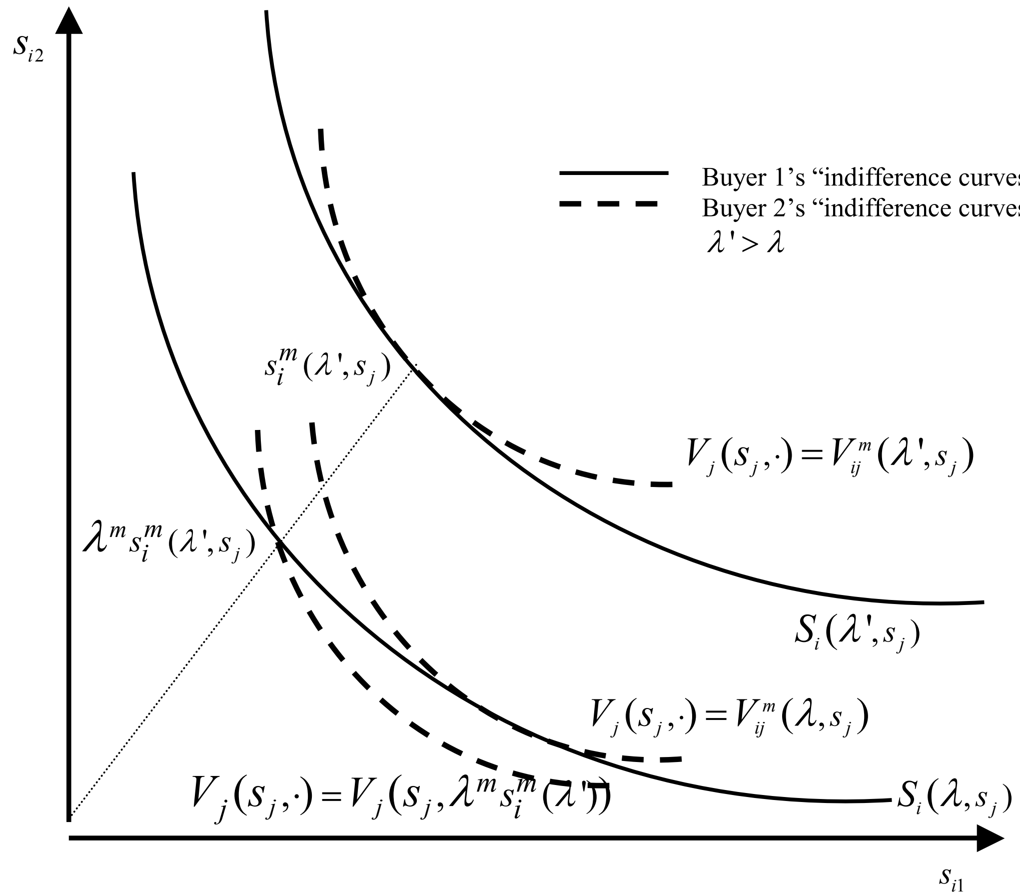
s_{i2}
Buyer 1's "indifference curves"
Buyer 2's "indifference curves"
$\lambda' > \lambda$
$s_i^m(\lambda', s_j)$
$V_j(s_j, \cdot) = V_{ij}^m(\lambda', s_j)$
$\lambda^m s_i^m(\lambda', s_j)$
$S_i(\lambda', s_j)$
$V_j(s_j, \cdot) = V_{ij}^m(\lambda, s_j)$
$V_j(s_j, \cdot) = V_j(s_j, \lambda^m s_i^m(\lambda'))$
$S_i(\lambda, s_j)$
s_{i1}

Figure 2