Polynomiography

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Solving polynomial equations still plays a major role in the secondary mathematics curriculum. Over the years how one goes about solving them has changed dramatically. Before the arrival of powerful computers and calculators, one learned how to use the quadratic formula, various approximation routines such as the bisection method and Newton's Method, and if one was really advanced the cubic and quartic formulas. Now with the arrival of graphing calculators, easily accessible computer algebra systems such as WolframAlpha, one can get quick solutions of any polynomial equation of any degree.

For me one of the gems of secondary mathematics has always been the Fundamental Theorem of Algebra that says that any nth degree polynomial equation with real coefficients has exactly n roots and that the complex roots occur in pairs. Recently, I have had the good fortune to be working on an NSF sponsored project in Computational Thinking with computer scientist Bahman Kalantari[1], who has developed a wonderful piece of software, *Polynomiography*[2], that allows one to both solve any polynomial equation using a variety of iterative methods, in particular Newton's Method, but also to create visual images that give you new and exciting insights into the nature of the solutions. The images can be stunning.

Since any polynomial equation with real coefficients can have both real and complex roots, and the real numbers are a subset of the complex numbers, it makes sense to work in the complex plane. So imagine you are in the complex plane and you are going to use Newton's Method to solve the quadratic equation $x^2 - 2x - 8 = 0$. If you type in $z^2 - 2z - 8$ into the polynomial entry window in *Polynomiography* and click on the green arrow key you get the image in Figure 1. The colored circular regions indicate how fast a guess in that region will converge to a solution. Figure 1 shows that an initial guess of $10 - 4i$ will converge to the solution $z = 4$ in 4 iterations. Any other guess in the same circular region will also converge to $z = 4$ in 4 iterations. Also, any guess in the green region will converge to $z = 4$ and any guess in the red region converges to $z = -2$

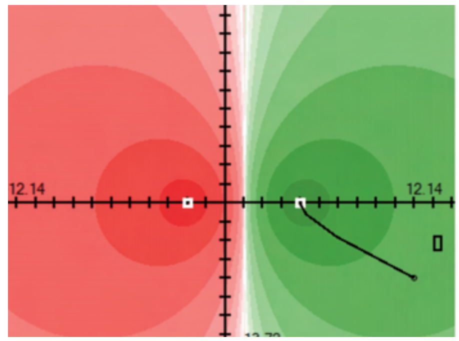
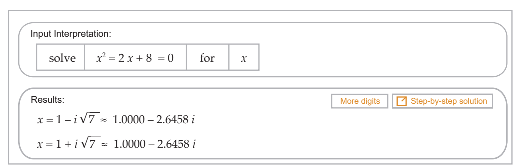


Figure 1. Solutions of $z^2 - 2z - 8 = 0$ using Polynomiography with an initial guess of $10 - 4i$.

If you use *Polynomiography* to solve a quadratic equation that has complex solutions you get a different picture. For example, If you solve the equation $z^2 - 2z + 8 = 0$ using *WolframAlpha* you get $z = 1 + \sqrt{7}i$ or $z = 1 - \sqrt{7}i$.



In *Polynomiography*, solving $z^2 - 2z + 8 = 0$ with an initial guess of $10 - 4i$ gives you the result in Figure 2

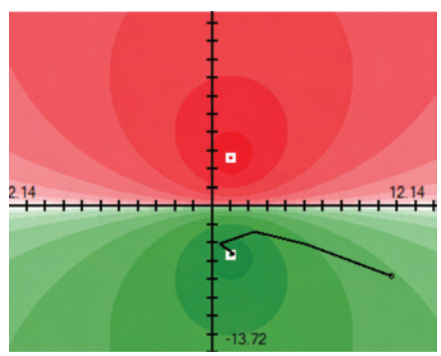


Figure 2. Solutions of $z^2 - 2z + 8 = 0$ using Polynomiography with an initial guess of $10 - 4i$.

Again, the colored circular regions indicate how fast a given guess converges to the solution.

Notice that when there are two real solutions as is the case in solving $z^2 - 2z - 8 = 0$, *Polynomiography* creates an image that is symmetric with respect to the line $x = 1$. The significance of the coloring is that any guess to the right of $x = 1$ will create a sequence that converges to the solution

$z = 4$. This region is often referred to as the Basin of Attraction for the root $z = 4$. Similarly, the region to the left is the Basin of Attraction for the root $z = -2$.

A similar result applies when there are two complex solutions but now the image is symmetric with respect to the x-axis. The region above the axis is the Basin of Attraction for the root $z = 1 + \sqrt{7}i$ and the region below is the Basin of Attraction for the root $z = 1 - \sqrt{7}i$.

What *Polynomiography* gives you is an image that shows the roots of the polynomial and the Basins of Attraction for each when Newton's Method or any other iterative method is used.

There is one other possibility for a solution to a quadratic equation and that is a double root. Figure 3 shows the image that *Polynomiograhy* gives for solutions to $(z - 2)^2 = 0$.

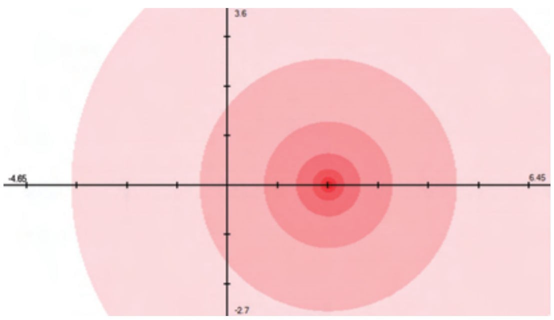


FIGURE 3. SOLUTIONS TO $(z - 2)^2 = 0$

Things begin to get really interesting when you start examining solutions of cubics with *Polynomiography*. The Fundamental Theorem of Algebra tells us that there are three possibilities for solutions: three real roots, a real root and a double real root, and a real root and conjugate complex roots. Figures 4–6 show the images that *Polynomiography* creates for each case.

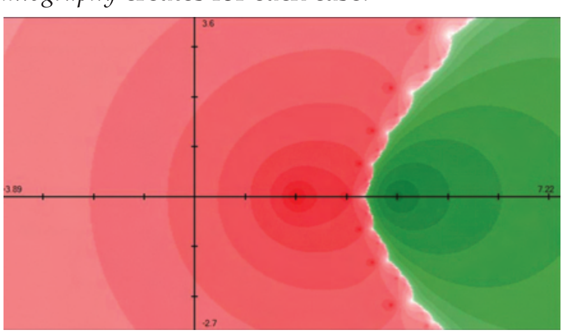


FIGURE 4. A DOUBLE REAL ROOT AND A SINGLE REAL ROOT

There has been a dramatic change in the images. The Basins of Attractions are no longer separated by straight lines as was the case with quadratics, but now consist of very intricate figures that turn out to be fractal like in the following sense. If you zoom in on the boundary, any time

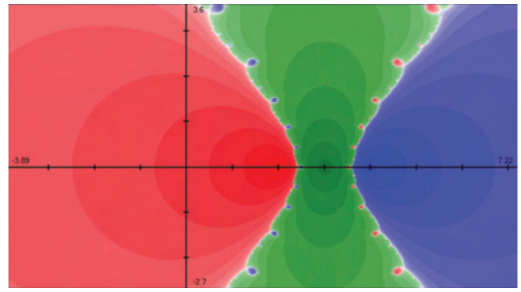


FIGURE 5. THREE REAL ROOTS

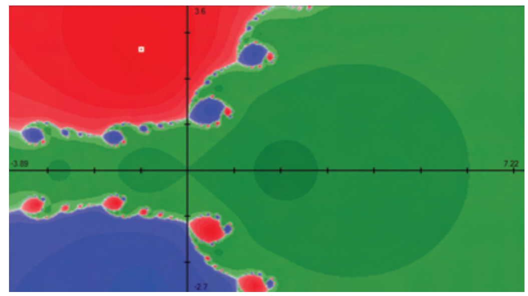


FIGURE 6. A PAIR OF COMPLEX CONJUGATE ROOTS AND A SINGLE REAL ROOT.

you have an image that contains two of the colors there will also be the third. Or put another way as you zoom in the image keeps repeating itself. It turns out that the images you get for finding the roots of any polynomial of degree greater than 2 will have Basins of Attraction whose boundaries are fractal.

Another feature of these images is that they are symmetric. Many of them will be symmetric with respect to the x – axis due to the fact that complex solutions come in conjugate pairs.

Moving up to quartics with real coefficients, there are now the following cases to consider, and each case produces a different type of image:

- All four real roots are distinct.
- A triple real root and a distinct second real root.
- A double real root and two distinct real roots.
- Two distinct double real roots.
- A pair of conjugate complex roots and two distinct real roots.
- A pair of conjugate complex roots and a double real root
- A double pair of conjugate complex roots.
- Two distinct pairs of complex roots.

Solving a given quartic with *Polynomiography* shows right away what the nature of the solutions is. For example, solving $z^4 - 3z^2 + 2z - 4 = 0$ shows that there is a pair of conjugate complex roots and two distinct real roots.

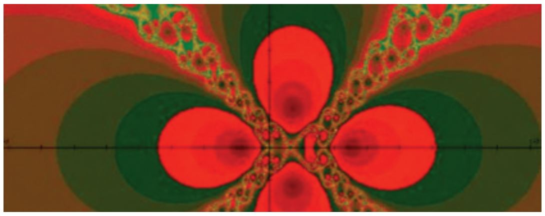


FIGURE 7. SOLUTIONS FOR $z^4 - 3z^2 + 2z - 4 = 0$.

One can use *Polynomiography* to examine and create images for any of the possible types of solutions for any nth–degree polynomial.

There are many ways one can use the software to create interesting images. Here is one of my favorites. Start with a quadratic function such as $f(z) = z^2 - 2z + 8$ and then create images of the solutions of $f(z^n) = z^{2n} - 2z^n + 8$ for increasing values of n. Figures 8, 9, and 10 show the images for $f(z^2) = 0$, $f(z^3) = 0$ and $f(z^4) = 0$.

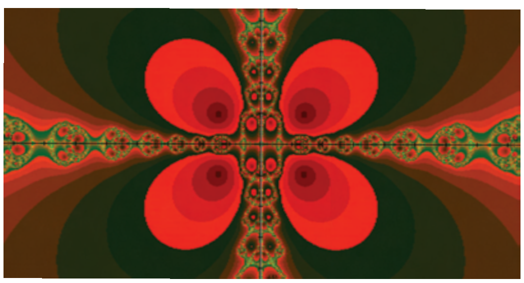


FIGURE 8. SOLUTIONS FOR $f(z^2) = 0$.

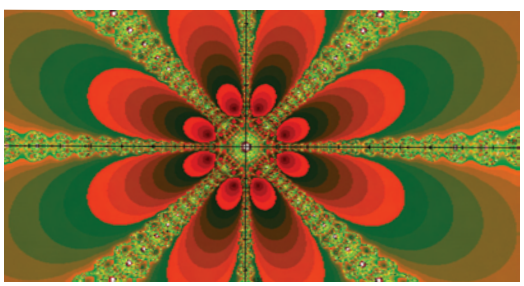


FIGURE 9. SOLUTIONS FOR $f(z^3) = 0$.

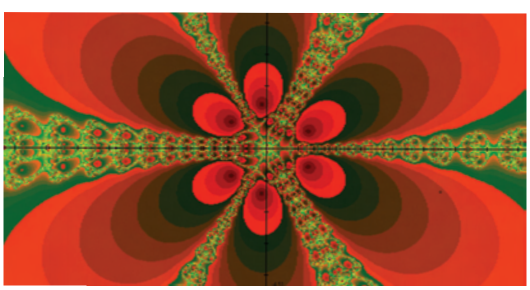


FIGURE 10. SOLUTIONS FOR $f(z^4) = 0$.

The software really encourages you to play. All you need are some polynomials and a willingness to experiment. Figure 11 shows my latest favorite that I created just by playing around. I used the polynomial $z^{12} + z^2 - 4z + 54 = 0$.

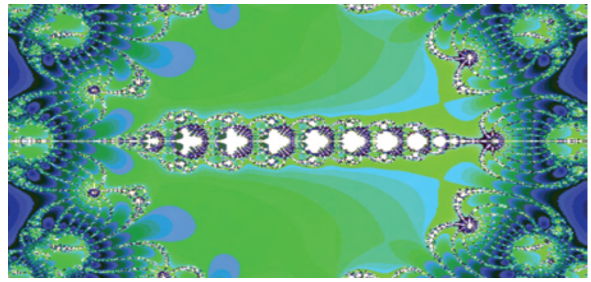


FIGURE 11. POLYNOMIOGRAPH FOR $z^{12} + z^2 - 4z + 54 = 0$.

Polynomiography can also be used to help students visualize complex nth roots. For example, if one compares the Polynomiograph for $z^4 - 1$ in Figure 12 to the Polynomiograph for $z^4 - (1 + i)$ in Figure 13, one can see that the roots still lie on the vertices of a square but the square has been enlarged and rotated.

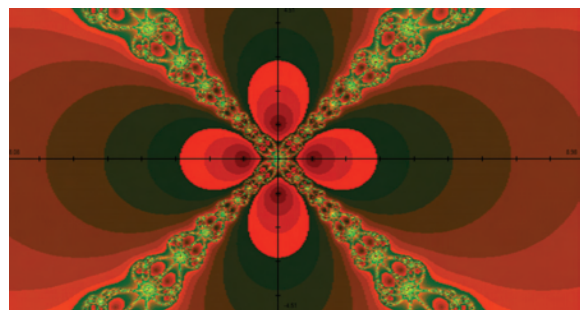


FIGURE 12. POLYNOMIOGRAPH FOR $z^4 - 1$.

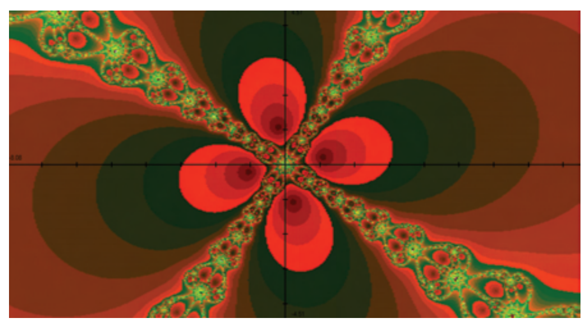


FIGURE 13. POLYNOMIOGRAPH FOR $z^4 - (1 + i)$.

References

[1] Professor Bahman Kalantari is a member of the Computer Science Department at Rutgers University. More information about Polynomiography can be found in: Kalantari, Bahman. 2008. Polynomial Root-Finding and Polynomiography. Hackensack, NJ: World Scientific Publishing.

Kalantari, Bahman and Bruce Torrence. 2013. "The Fundamental Theorem of Algebra for Artists." Math Horizons, 20 (4), 26-29.

[2] Polynomiography: www.polynomiography.com

Send solutions to old problems and any new ideas to the Geometer's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450. jchoate@groton.org