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Acyclic, or totally tight, two-person game forms; characterization and main properties [1]

by

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ABSTRACT

It is known that a two-person game form g is Nash-solvable if and only if it is tight [12, 13]. We strengthen the concept of tightness as follows: game form is called *totally tight* if every its 2×2 subform is tight. (It is easy to show that in this case all, not only 2×2, subforms are tight.) We characterize totally tight game forms and derive from this characterization that they are tight, Nash-solvable, dominance-solvable, acyclic, and assignable. In particular, total tightness and acyclicity are equivalent properties of two-person game forms.

Keywords: game, game form, effectivity function, improvement cycle, acyclic, assignable, tight, totally tight, Nash-solvable, dominance-solvable

a_1	a_2
a_2	a_1

g

a_1	a_1
a_2	a_3

g'

a_1	a_1	a_3
a_1	a_2	a_2
a_3	a_2	a_3

g''

Figure 1: Tight and not tight game forms.

1 Introduction

We consider the following six classes of two-person game forms: tight (T), totally tight (TT), Nash-solvable (NS), dominance-solvable (DS), acyclic (AC), and assignable (AS) ones, and prove the following implications:

$$AS \Leftarrow TT \Leftrightarrow AC \Rightarrow DS \Rightarrow NS \Leftrightarrow T. \tag{1}$$

Some of them are known, while others follow from a characterization of the TT game forms obtained in this paper. We also give examples showing that no other implication holds for the considered six properties.

1.1 Game forms and games

A (two-person) *game form* is a mapping $g : X_1 \times X_2 \to A$, where X_1 (rows) and X_2 (columns) are the strategies of players 1 and 2, while A is a set of outcomes. In this paper we restrict ourselves by finite two-person game forms, that is, the above three sets, X_1, X_2, and A are finite. Three examples are given in Figure 1. Furthermore, let $u : \{1, 2\} \times A \to \mathbb{R}$ be a *utility (or payoff)* function. Given a player $i \in \{1, 2\}$ and an outcome $a \in A$, the value $u(i, a)$ is interpreted as the profit of the player i in case when the outcome a is realized. The pair (g, u) defines a normal form (bimatrix) game. A payoff u is called *zero-sum* if $u(1, a) + u(2, a) = 0$ for each $a \in A$. In this case (g, u) is a *matrix game.*

1.2 Nash equilibrium and Nash-solvability

The elements of the direct product $X = X_1 \times X_2$ are called *situations.* Given a game (g, u), a situation $x = (x_1, x_2) \in X_1 \times X_2 = X$ is called a *Nash equilibrium (NE)* if

$$u(1, g(x_1, x_2)) \geq u(1, g(x_1', x_2)) \; \forall \; x_1' \in X_1 \text{ and } u(2, g(x_1, x_2)) \geq u(2, g(x_1, x_2')) \; \forall \; x_2' \in X_2;$$

in other words, if no player can profit until the opponent keeps the strategy unchanged.

A NE of a zero-sum game is called a *saddle point.*

Theorem 1. *(Shapley (1964), [23]). A zero-sum game has a saddle point whenever each of its 2×2 subgames has one.* □

However, in general (for non-zero-sum games), the similar statement does not hold; see, for example, [15] or [5].

A game form g is called *Nash-solvable (NS)* if for each payoff u the obtained game (g, u) has a NE. Respectively, g is called *zero-sum-solvable* if for each zero-sum payoff u the obtained zero-sum game (g, u) has a saddle point.

1.3 Effectivity functions, game forms, and criteria of solvability

Given a game form $g : X_1 \times X_2 \to A$, we say that a player $i \in \{1, 2\}$ is effective for a subset of outcomes $B \subseteq A$ if i has a strategy $x_i \in X_i$ such that $g(x_i, x_{3-i}) \in B$ for every strategy $x_{3-i} \in X_{3-i}$ of the opponent. In this case we set $E_i(B) = 1$ and $E_i(B) = 0$ otherwise. Thus, two Boolean functions $E_i^g : 2^A \to \{0, 1\}$, $i = 1, 2$, are associated with every game form g. The pair (E_1^g, E_2^g) is called the *effectivity function (EFF)* of g; see [20, 19, 21] for more detail.

Obviously, equalities $E_1^g(B) = E_2^g(A \setminus B) = 1$ hold for no g, since every row and column in $X_1 \times X_2$ intersect. In contrast, $E_1^g(B) = E_2^g(A \setminus B) = 0$ might hold. For example, let us consider game form g in Figure 1 and set $B = \{a_1\}$ (or $B = \{a_2\}$). Then $E_1^g(B) = E_2^g(A \setminus B) = 0$, since all rows and columns contain both a_1 and a_2.

A game form g is called *tight* if $E_1^g(B) = 1 \Leftrightarrow E_2^g(A \setminus B) = 0$, or in other words, if

$$E_1^g(B) + E_2^g(A \setminus B) \equiv 1 \ \ \forall B \subseteq A. \tag{2}$$

For example, game forms g' and g'' in Figure 1 are tight, while g is not.

Given a game form g, let us assign to each outcome $a \in A$ a Boolean variable and denote it for simplicity by the same symbol a. Then, rows and columns of g naturally define two monotone disjunctive normal forms (DNFs) that represent, respectively, E_1^g and E_2^g:

$$D_1^g = \bigvee_{x_1 \in X_1} \bigwedge_{x_2 \in X_2} g(x_1, x_2), \quad D_2^g = \bigvee_{x_2 \in X_2} \bigwedge_{x_1 \in X_1} g(x_1, x_2). \tag{3}$$

It is not difficult to verify that a game form g is tight if and only if its two DNFs D_1^g and D_2^g are dual, $D_1^g = (D_2^g)^d$. This equation is just a reformulation of (2).

For example, for the three game forms g, g' and g'' in Figure 1 we have:

$D_1^g = D_2^g = a_1 a_2;\ D_1^g \neq (D_2^g)^d = a_1 \vee a_2;$
$D_1^{g'} = a_1 \vee a_2 a_3,\ D_2^{g'} = a_1 a_2 \vee a_1 a_3,\ D_1^{g'} = (D_2^{g'})^d;$
$D_1^{g''} = D_2^{g''} = (D_1^{g''})^d = (D_2^{g''})^d = a_1 a_2 \vee a_2 a_3 \vee a_3 a_1.$

Hence, g' and g'' are tight, while g is not.

Theorem 2. *([12], see also [13] and [4]). The following three properties of a game form are equivalent: tightness, Nash-solvability, and zero-sum-solvability.* □

For the zero-sum case this claim was proved earlier by Edmonds and Fulkerson [7] and independently in [11].

To verify tightness of a game form is an exciting open problem of complexity theory, so-called *dualization*. No polynomial algorithm is still known. However, it is very unlikely that dualization is NP-hard, since there is a quasi-polynomial recognition algorithm suggested by Fredman and Khachiyan [8]. Its complexity, $N^{o(\log N)} = 2^{o(N^2)}$ is closer to polynomials $2^{c \log N}$ than to exponents 2^{cN}, where c is a constant and N is the input complexity.

1.4 Totally tight and irreducible game forms; main theorem

We will call a game form g *totally tight (TT)* if each of its 2×2 subforms is tight.

Proposition 3. *Totally tight game forms are tight.*

Proof. Let g be a TT game form and g' be an arbitrary its 2×2 subform. By definition, g' is tight and, by Theorem 2, it is zero-sum-solvable. Then, by Theorem 1, g is zero-sum-solvable and, by Theorem 2, g is tight. □

By definition, total tightness of a game form can be verified in polynomial time.

Given a game form $g : X_1 \times X_2 \to A$, a strategy $x_1 \in X_1$ and the corresponding row (respectively, $x_2 \in X_2$ and the corresponding column) is called *constant* if there is an outcome $a \in A$ such that $g(x_1, x_2) \equiv a$ for all $x_2 \in X_2$ (respectively, for all $x_1 \in X_1$).

A game form g is called *reducible* if it has a constant line, row or column.

It is easy to verify that a 2×2 game form is reducible if and only if it is tight.

For example, in Figure 1, game form g' is tight and reducible (its first row is constant), while g is not tight and not reducible.

Let us remark that, by the above definition, an $m \times n$ game form is reducible whenever $m = 1$ or $n = 1$. Indeed, in this case each column or, respectively, row is constant. Moreover, formally, even a 1×1 game form is reducible, although there is no game form to reduce it to. By convention, let us say that it is reduced to the empty game form.

By definition, the reducibility of a game form can be verified in linear time.

A game form will be called *totally reducible* if it can be reduced to the empty one by successive elimination of constant lines. In [17] these game forms are called semi-dictatorial. For example, g' in Figure 1 is such a game form.

Proposition 4. *Totally reducible game forms are totally tight.*

Proof. The induction by $m + n$ is obvious. □

More generally, given a game form g, let us eliminate successively its constant lines until we obtain an irreducible game form g' which might be empty or not.

Proposition 5. *Game form g' is well-defined, that is, unique. Moreover, g' is TT if and only if g is TT.*

Proof. Again, it is obvious. □

In Section 2, we will prove that all such (non-empty irreducible TT) game forms have the same effectivity function. This, so-called 2-majority, EFF $E = E\binom{3}{2}$ is defined as follows: three exist three outcomes $a_1, a_2, a_3 \in A$ such that each player $i \in \{1,2\}$ is effective for any two of them, $E_i(\{a_1,a_2\}) = E_i(\{a_1,a_3\}) = E_i(\{a_2,a_3\}) = 1$, and, of course, for every superset of such a subset of cardinality 2, as well.

Theorem 6. *Every non-empty irreducible TT game form g has a 2-majority effectivity function, that is, there are outcomes $a_1, a_2, a_3 \in A$ such that $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$.*

This result clarifies the structure of a TT game form g "almost completely": g is either totally reducible, or it is reduced to an irreducible game form g' with a 2-majority EFF.

Somewhat surprisingly, even under this (very strong) restriction it appears not that easy to characterize the TT game forms explicitly. However, in Section 3 a characterization of the following type is obtained: we construct recursively an infinite family of TT game forms and show that each TT game form is a subform of a game form from this family.

Furthermore, in Section 4 we prove that TT game forms are (i) acyclic, (ii) dominance-solvable, and (iii) assignable; see the next three subsections for the definitions. Recently, (i) was proved, while (ii) and (iii)) conjectured by Kukushkin, [17].

Results (i) and (ii) are significantly strengthened and generalized in [3], see also [2].

1.5 Acyclic game forms

Given positive integral m, n and k such that $2 \le k \le \min(m,n)$, a $m \times n$ bimatrix game (g,u), and k distinct strategies of each player, $x_1^1, \dots, x_1^k \in X_1$ and $x_2^1, \dots, x_2^k \in X_2$, we say that these strategies form a *(strict improvement) cycle* C_k if

$$
\begin{aligned}
&u(2, g(x_1^1, x_2^1)) < u(2, g(x_1^1, x_2^2)),\ u(1, g(x_1^1, x_2^2)) < u(1, g(x_1^2, x_2^2)),\\
&u(2, g(x_1^2, x_2^2)) < u(2, g(x_1^2, x_2^3)), \dots,\\
&u(2, g(x_1^{n-1}, x_2^{n-1})) < u(2, g(x_1^{n-1}, x_2^n)),\ u(1, g(x_1^{n-1}, x_2^n)) < u(1, g(x_1^n, x_2^n)),\\
&u(2, g(x_1^n, x_2^n)) < u(2, g(x_1^n, x_2^1)),\ u(1, g(x_1^n, x_2^1)) < u(1, g(x_1^1, x_2^1));
\end{aligned}
$$

or in words, if two players alternating can strictly improve their payoffs (k times each), so that they begin and end with the same pair of strategies (x_1^1, x_2^1).

A game that have no cycles is called *acyclic*. It is both obvious and well-known that every acyclic game has a NE.

A game form g will be called *acyclic* (AC) if for any payoff u the obtained game (g,u) is acyclic. It is clear that each acyclic game form is Nash-solvable and, hence, it is tight.

It is an easy exercise to verify that a 2×2 game form is tight if and only if it is acyclic. Hence, acyclic game forms are TT. Recently, it was shown that the inverse holds, too.

Proposition 7. *(Kukushkin (2007), [17]). A game form is totally tight if and only if it is acyclic.*

In Section 4 we derive this claim from Theorem 6; see also [1] for an independent proof.

a_1	a_1	a_1
a_1	a_1	a_2
a_1	a_2	a_1

g

a_1	a_1
a_1	a_2
a_2	a_1

g'

a_1	a_2
a_2	a_1

g''

a_1	a_1
a_2	a_2
a_1	a_2
a_2	a_1

g'''

Figure 2: Adding and eliminating constant lines, rows and/or columns; g is NS (tight) but not DS; g' is not tight; g'' is assignable but not tight; g''' is the A-extension of g''.

1.6 Dominance-solvable game forms

Given a game (g, u) and two strategies $x_i, x'_i \in X_i$ of a player $i \in \{1, 2\}$, we say that x'_i is dominated by x_i if $u(i, g(x_i, x_{3-i})) \geq u(i, g(x'_i, x_{3-i}))$ for every strategy $x_{3-i} \in X_{3-i}$ of the opponent; in other words, if player i cannot profit by substituting x'_i for x_i until the opponent keeps the same (arbitrary) strategy.

Let us eliminate successively dominated strategies of players. Game (g, u) is called *dominance-solvable* if this procedure results in a 1×1 terminal subgame. The obtained situation is called *domination equilibrium (DE)*. (In the literature, it is also called *sophisticated equilibrium.*) It is well-known and easy to see that each DE is a NE; see, for example, [18], [19] Chapter 5, or [9].

Although, in general, the result might depend on the order in which dominated strategies are eliminated, yet, there are simple conditions under which the above procedure and concept of domination are well-defined; namely, when utility functions $u_i : A \to \mathbb{R}$ of both players are injective; in other words, when $u(1, a) = u(1, a')$ if and only if $u(2, a) = u(2, a')$ for all $a, a' \in A$; see [18], [19] Chapter 5, or [9] again.

A game form g is called *dominance-solvable* (DS) if for any payoff u the obtained game (g, u) is DS. Obviously, $DS \Rightarrow NS$, since, as we already mentioned, each DE is a NE. Yet, the inverse implication does not hold. For example, game form g in Figure 2 is tight and, hence, NS but it is not DS; there is no DE if both players prefer a_2 to a_1.

Proposition 8. *Totally tight game forms are dominance-solvable.*

In Section 4, we derive this implication from Theorem 6; see [1] for an independent proof.

Up to our knowledge, the complexity of verifying if a given game form is DS is open.

1.7 Assignable game forms

Let us call a game form $g : X_1 \times X_2 \to A$ *assignable* (AS) if there are mappings $g_1 : X_1 \to A$ and $g_2 : X_2 \to A$ such that $g(x_1, x_2)$ equals $g_1(x_1)$ or $g_2(x_2)$ for all $x_1 \in X_1,\ x_2 \in X_2$.

It is easy to verify that all seven game forms in Figures 1 and 2 and even g' in Figure 3 are assignable, while g is not.

The concept of assignability was suggested by Kukushkin (private communications); he conjectured that the following implication holds.

a_1	a_1	a_2	a_2
a_3	a_4	a_3	a_4

g

a_1	a_2
a_4	a_3

g'

Figure 3: Game form g is tight and DS but not TT and not AS.
Tightness and dominance-solvability are not hereditary properties

Proposition 9. *Totally tight game forms are assignable.*

In Section 4 we will derive this statement from Theorem 6.

It is easy to see that all 2×2 game forms, as well game forms with only two outcomes, are assignable. In particular, g'' in Figure 2 is AS, yet, it is not tight. On the contrary, game form g in Figure 3 is tight and DS but not AS.

Verifying whether a given game form $g : X_1 \times X_2 \to A$ is assignable can be executed in polynomial time, since this problem is polynomially reduced, for example, to 2-satisfyability.
Indeed, let us consider g and two more mappings $g_1 : X_1 \to A$ and $g_2 : X_2 \to A$. Given $i \in I = \{1, 2\}$, a strategy $x_i \in X_i$, and an outcome $a \in A$, let us define a Boolean variable $y = y(x_i, a)$ as follows: $y = 1$ if $g_i(x_i) = a$ and $y = 0$ otherwise. Then, let us consider a 2-CNF

$$C(g) = \bigwedge_{a,a' \in A \mid a \neq a';\, x_i \in X_i,\, i \in \{1,2\}} (\bar{y}(x_i, a) \vee \bar{y}(x_i, a')) \bigwedge_{x_1 \in X_1,\, x_2 \in X_2,\, a \in A} (y(x_1, a) \vee y(x_2, a)). \quad (4)$$

It is easily seen that this CNF $C(g)$ is satisfiable if and only if the corresponding game form g is assignable. Indeed, in CNF (4) the first conjunction is equal to 1 if and only if at most one outcome $a \in A$ is assigned by a mapping g_i to each strategy $x_i \in X_i$ for $i \in \{1, 2\}$; respectively, the second conjunction of (4) equals 1 if and only if $g(x_1, x_2) = g_1(x_1)$ or $g(x_1, x_2) = g_2(x_1)$ for every situation $(x_1, x_2) \in X_1 \times X_2)$.
Let us remark, however, that the above arguments hold only for two-person game forms.

As we already mentioned, all 2×2 game forms are assignable. Moreover, for 2×2 game forms the following six properties are equivalent: T, TT, DS, NS, AC, and reducibility.

1.8 Hereditary properties

Given a game form $g : X_1 \times X_2 \to A$ (respectively, a game (g, u)) and a pair of subsets $X'_1 \subseteq X_1$, $X'_2 \subseteq X_2$, standardly a subform g' of g and subgame (g', u) of (g, u) is defined by the restriction of g to $X'_1 \times X'_2 \subseteq X'_1 \times X'_2$.
A property P of a game (g, u) (game form g) is called *hereditary* if P holds for any subgame (g', u) of (g, u)) (subform g' of g) whenever P holds for (g, u) (for g) itself.
By definitions, TT, AC, and AS are hereditary properties of game forms. In contrast, properties T, NS, and DS can disappear even after eliminating a constant line, row or column.

For example, game form g''' in Figure 2 is DS; hence, it is NS and tight, too. Yet, eliminating its second (constant) row we obtain game form g' that has none of these three properties; for example, it is not tight, since its Boolean functions $E_1^{g'} = a_1$ and $E_2^{g'} = a_1 a_2$ are not dual.

1.9 Adding and eliminating constant lines; A-extensions

Given a game form $g : X_1 \times X_2 \to A$, let us define its row A-extension $g_1^A : X_1^A \times X_2 \to A$ by setting $X_1^A = X_1 \cup \{x_1^a,\ a \in A\}$ and $g_1^A(x_1^a, x_2) \equiv a$ for every $x_2 \in X_2$ and $a \in A$. In other words, we extend X_1 by adding $p = |A|$ constant strategies x_a for all ooutcomes $a \in A$. For example, in Figure 2 game form g''' is the row A-extension of g''. Similarly, we introduce the column A-extension $g_2^A : X_1 \times X_2^A \to A$ of a game form $g : X_1 \times X_2 \to A$.

It is easy to verify that for an arbitrary game form g both its A-extensions are tight, NS, and DS; furthermore they are TT, AC, or AS if and only if g has the corresponding property.

Let us consider three transformations of game forms: A-extension, eliminating and adding a constant line. (For example, A-extension itself was defined as adding $p = |A|$ constant lines, one for each outcome $a \in A$.)

The following meta-language will simplify our statements. We say that a property $\mathcal{P}$ is treated by a transformation $\mathcal{T}$ and consider three transformations defined above, our standard six properties partitioned in two triplets, $\mathcal{X} = \{T, NS, DS\}$ and $\mathcal{Y} = \{TT, AC, AS\}$, and the following four types of treatment. We apply $\mathcal{T}$ to a game form g, obtain a transformed game form g', and say that:

$\mathcal{P}$ is *encouraged* by $\mathcal{T}$ if $\mathcal{P}$ cannot disappear (but, maybe, it can appear);

$\mathcal{P}$ is *discouraged* by $\mathcal{T}$ if $\mathcal{P}$ cannot appear (but, maybe, it can disappear);

$\mathcal{P}$ is *respected* by $\mathcal{T}$ if $\mathcal{P}$ can neither appear, nor disappear;

$\mathcal{P}$ is *enforced* by $\mathcal{T}$ if $\mathcal{P}$ cannot disappear and must appear.

$\mathcal{P}$ is *denied* by $\mathcal{T}$ if $\mathcal{P}$ cannot appear and must disappear.

Theorem 10. *(i) Eliminating constant lines discourage properties of $\mathcal{X} = \{T, NS, DS\}$ and encourage properties of $\mathcal{Y} = \{TT, AC, AS\}$; moreover the latter properties are hereditary;*

(ii) Adding constant lines encourage $\mathcal{X}$ and respect $\mathcal{Y}$;

(iii) A-extensions enforce $\mathcal{X}$ and respect $\mathcal{Y}$.

Proof. It is tedious, since there are very many cases, but simple.

For example, let us notice that Nash- or dominance-solvability of a game form g cannot disappear after g is extended by a constant strategy x_i^0 of a player $i = 1$ or $i = 2$. Indeed, although x_i^0 might "kill" a NE or DE in the game (g, u), yet obviously, in this case a new one (related to x_i^0) must appear in the transformed game.

We leave the analysis of numerous remaining cases to the careful reader. □

All cases of Theorem 10 are summarized in two tables given in Figure 4.

∃NE, ∃DE, T, NS, DS	eliminate	add	A-extend
can disappear	YES	NO	NO
can appear	NO	YES	YES
must appear	NO	NO	YES

TT, AC, AS	eliminate	add	A-extend
can disappear	NO	NO	NO
can appear	YES	NO	NO
must appear	NO	NO	NO

Figure 4: Eliminating and adding constant rows and columns

Remark 1. *The set of properties $\mathcal{X} = \{T, NS, DS\}$ can be extended to $\mathcal{X}' = \{T, NS, DS, \exists NE, \exists DE\}$, where the last two properties are related to games rather than to game forms and mean that a game has a NE or, respectively, DE. If we substitute $\mathcal{X}'$ for $\mathcal{X}$ the modified Theorem 10 will still hold.*

Let us also note that all claims extend the case of n-person game forms.

1.10 Equivalent definitions and main corollaries of total tightness

Let us summarize some of the above observations.

Theorem 11. *The following twelve properties of a game form g are equivalent:*
every 2×2 subform of g is (1) tight, (2) Nash-solvable, (3) zero-sum-solvable, (4) dominance-solvable, (5) acyclic; furthermore, every subform g' of g is (1′) tight, (2′) Nash-solvable, (3′) zero-sum-solvable, (4′) dominance-solvable, (5′) acyclic; finally, g itself is (6) acyclic, and (7) totally tight.

In particular, total tightness and acyclicity are equivalent. In Section 4, we will prove that total tightness implies acyclicity, assignability, and dominance-solvability.

Furthermore, it is well-known that dominance-solvability implies Nash-solvability, see, for example, [19, 9], and that Nash-solvability is equivalent to tightness [12, 13]. Let us also recall that total tightness implies tightness, by Proposition 3.

Relations between main classes of two-person game forms are summarized by (1).

Let us underline that no other implications hold. Indeed, in Figure 2, game form g is tight but not DS, while g'' is AS but not tight; furthermore, g in Figure 3 is DS but not TT and not AS.

Remark 2. *The last example is just a representative of a large family. It is well-known that a game form g is DS whenever it is obtained from a positional game form with perfect* *information Gale (1953); see also Chapter 5 of [19]. However, in this case, g is acyclic (or equivalently, TT) if and only if all positions of each player belong to a single play in the corresponding tree. This result was obtained in 2002 by Kukushkin; see Theorem 1 of [16]. (Both results hold for n-person case, not only for $n = 2$.)*

Another large family of DS but not TT game forms is related to veto-voting; see manuscript [10] and also [1].

Let us recall that a game form g is tight if and only if the corresponding monotone Boolean functions E_1^g and E_2^g are dual. In Section 2, we will prove Theorem 6: if g is TT then $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$. However, the inverse does not hold and it is not easy to characterize TT game forms explicitly. In Section 3 we obtain recursively an infinite family of them and show that each TT game form is a subform of a game form from this family.

Remark 3. *Let us notice that the above important necessary conditions for acyclicity (or equivalently, for total tightness) of a two-person game form are given in terms of its effectivity function. Somewhat surprisingly, many properties of game forms can be characterized in such terms. For example, a two-person game form g is Nash-solvable if and only if it is tight, that is, its effectivity function is self-dual. More example can be found in [14].*

2 Proof of Theorem 6

Let g be a totally tight game form. By Proposition 3, g is tight, that is, the corresponding two monotone Boolean functions E_1^g and E_2^g are dual. Yet, Theorem 6 claims much more, namely, all TT game forms generate the same self-dual pair: $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

2.1 Game correspondences and associated game forms

A *game correspondence* is defined as a mapping $G : X_1 \times X_2 \to 2^A$. In other words, to each situation $(x_1, x_2) \in X_1 \times X_2$ we assign a subset of outcomes $G(x_1, x_2) \subseteq A$. If $|G(x_1, x_2)| = 1$ for all situations $(x_1, x_2) \in X_1 \times X_2$, we obtain a game form.

In general, with a game correspondence G we associate $k = \prod_{(x_1,x_2)\in X_1\times X_2} |G(x_1, x_2)|$ game forms $g \in G$, by choosing for each situation $(x_1, x_2) \in X_1 \times X_2$ an outcome $g(x_1, x_2) \in G(x_1, x_2)$. Let us notice that $k = 0$ whenever $G(x_1, x_2) = \emptyset$ for at least one situation.

We will say that $g \in G$ is *associated with* G and call G *(totally) tight* if $k > 0$ and at least one $g \in G$ is (totally) tight.

2.2 Game correspondences associated with pairs of dual monotone DNFs or Boolean functions

First, let us recall the following two well-known properties of dual monotone Boolean functions that will be instrumental for our analysis.

a_1/a_3	a_1	a_3
a_1	a_2/a_1	a_2
a_3	a_2	a_3/a_2

Figure 5: $\binom{3}{2}$ majority voting game correspondence;
only 2 from 8 game forms associated with this game correspondence are TT; see, e.g., g'' in Figure 1.

Lemma 12. *(see, for example, [6], Part I, Chapter 4).*
(i) Every two dual implicants α of E and β of E^d have at least one variable in common.
(ii) Given a prime implicant α of E and a variable x of α, there is a prime implicant β of E^d such that x is the only common variable of α and β. □

Given arbitrary monotone (that is, negation-free) DNFs $D_1 = \bigvee_{x_1 \in X_1} B_{x_1}$ and $D_2 = \bigvee_{x_2 \in X_2} B_{x_2}$ over the set of variables A, let us define a game correspondence $G = G^{D_1,D_2} : X_1 \times X_2 \to 2^A$ by setting $G(x_1, x_2) = B_{x_1} \cap B_{x_2}$ for every situation $(x_1, x_2) \in X_1 \times X_2$; see, for example, G^{D_1,D_2} in Figure 5, where $D_1 = D_2 = a_1a_2 \vee a_2a_3 \vee a_3a_1$.

Lemma 13. *([13], see also [22]). If D_1 and D_2 are dual then game correspondence $G(D_1, D_2)$ is tight. In particular, in this case $G(x_1, x_2) \neq \emptyset$ for all $(x_1, x_2) \in X_1 \times X_2$; moreover, all associated game forms $g \in G$ have the same Boolean functions E_1^g and E_2^g defined by DNFs D_1 and D_2, respectively. Conversely, if at least one game form $g \in G^{D_1,D_2}$ is tight then DNFs D_1 and D_2 are dual.*

Proof. It follows immediately from Lemma 12 (i) and (ii). □

Let us recall that, by definition, G is TT if at least one $g \in G$ is TT. However, in contrast with tightness, this does not mean that *all* $g \in G$ are TT. Let us consider, for example, game correspondence G in Figure 5. Only two game forms associated with G are TT (one of them is g'' in Figure 1, while it is easy to verify that the remaining six are not TT.

Given a DNF D, let D^0 denote the corresponding irredundant DNF, that is, disjunction of all prime (irreducible) implicants of D.

Lemma 14. *Game correspondence G^{D_1,D_2} is TT if and only if $G^{D_1^0,D_2^0}$ is TT.*

Proof. The "only if part" immediately follows, since total tightness is a hereditary property of game forms and game correspondences.

Lemma 15. *A subcorrespondence G' of G is TT whenever G is TT.* □

Let us prove the "if part". By assumption, there is a TT game form $g^0 \in G^0 = G^{D_1^0,D_2^0}$. Let us extend it to a TT game form $g \in G = G^{D_1,D_2}$ as follows. For $i = 1, 2$ to each strategy $x_i \in X_i$ in G assign a strategy $x_i^0 \in X_i$ in G^0 such that $B_{x_i^0} \subseteq B_{x_i}$. Then for each situation $x = (x_1, x_2)$ of G choose the same outcome as for $x^0 = (x_1^0, x_2^0)$ in g^0. Obviously, the obtained extension g of g^0 is totally tight, too. □

a_1	a_2	a_2
a_3	a_2/a_3	a_2
a_3	a_3	a_4

G

a_1	a_2
a_3	a_4

g'

Figure 6: No TT game form is associated with this game correspondence.

2.3 Totally tight Boolean functions

Thus, we can restrict ourselves by dual pairs of irredundant DNFs. In other words, keeping in mind the characterization of TT game forms, we will take as the input a monotone Boolean function E rather than a game form g. Given E, we set $E_1 = E$ and $E_2 = E^d$, consider the corresponding irredundant DNFs D_1^0 and D_2^0 and game correspondence $G = G^E = G^{D_1^0, D_2^0}$. We will call E TT if G is TT, or in other words, if there is a TT $g \in G$. By construction, E is TT if and only if E^d is TT. Let us consider several examples.

If $E = E_1 = a_1a_2 \vee a_3a_4$ then $E^d = E_2 = a_1a_3 \vee a_1a_4 \vee a_2a_3 \vee a_2a_4$. It is easy to see that every two prime implicants, one of E and the other of E^d, have exactly one variable in common. (This is a characteristic property of so-called monotone *read-once* Boolean functions; see [6], Chapter 12.) In other words, game correspondence G^E is, in fact, a game form, since $|G^E(x_1, x_2)| = 1$ for every situation $(x_1, x_2) \in X_1 \times X_2$. This game form g is shown on Figure 3. However, this game form is not TT, since it has a 2×2 subform g' that is not tight, see Figure 3.

In general, G^E is a game form, $G^E = g^E$, if and only if E is read-once. It is not difficult to show that in this case E is TT if and only if g^E is totally reducible; see Proposition 4. (This is a characteristic property of so-called monotone *threshold* Boolean functions; see [6], Part II, Chapter 10.) However, we are looking for irreducible TT game forms.

As another example, let us consider

$E = E_1 = a_1a_2 \vee a_2a_3 \vee a_3a_4$ and $E^d = E_2 = a_1a_3 \vee a_3a_2 \vee a_2a_4$.

It is easy to check that G^E is not TT, since it contains a 2×2 subform g'; see Figure 6.

A case analysis might be needed for more difficult examples.

Let $E = E\binom{5}{3} := \bigvee_{\{i,j,k\} \subseteq \{1,2,3,4,5\}} a_i a_j a_k$, where i, j, and k are pairwise distinct triplets; in other words, $E = 1$ if and only if at least 3 out of its 5 variables are equal to 1. To show that G^E is not TT let us consider its 4×4 subcorrespondence G given in Figure 7 (where, to save space, we substitute only the subscript $j \in \{1, 2, 3, 4, 5\}$ for a_j). Let us choose an arbitrary game form $g \in G$. Due to obvious symmetry, we can choose a_1 from $\{a_1, a_2, a_3\}$, without any loss of generality. Yet, in this case G already contains a 2×2 subconfiguration G' that is clearly not TT; see Figure 7. Hence, g cannot be TT and, by Lemma 15, G and G^E are not TT, either.

The following Lemma is instrumental in characterizing TT Boolean functions.

Given E, let us choose two of its distinct prime implicants and denote by $B, B' \subseteq A$ the corresponding two set of variables. Obviously, $B \setminus B' \neq \emptyset$ and $B' \setminus B \neq \emptyset$.

	123	145	245	345
123	**1**23	1	2	**3**
145	1	145	45	45
245	**2**	45	245	**45**
345	3	45	45	345

G

1	**3**
2	**45**

G'

Figure 7: $\binom{5}{3}$ majority voting, a 4×4 subcorrespondence;
this subcorrespondence is not TT, since no TT game form is associated with it.

B	a_1	a_2	b_1	b_2
B'	b_3	b_4	a_3	a_4

a_1	a_2	b	b
b'	b'	a_3	a_4

$\mathbf{a_1}$	a_2	a_2	$\mathbf{a_2}$
$\mathbf{a_3}$	a_3	a_3	$\mathbf{a_4}$

Figure 8: $|B \setminus B'| = 1$ or $|B' \setminus B| = 1$.

Lemma 16. *If E is totally tight then $|B \setminus B'| = 1$ or $|B' \setminus B| = 1$.*

Proof. Let us assume indirectly that $|B \setminus B'| \geq 2$ and $|B' \setminus B| \geq 2$, say, $a_1, a_2 \in B \setminus B'$ and $a_3, a_4 \in B' \setminus B$, where $a_1, a_2, a_3, a_4 \in A$ are four pairwise distinct outcomes, yet, E is TT.

By Lemma 12 (ii), there are four prime implicants of E^d whose sets of variables B_1, B_2, B_3, B_4 are such that $B_1 \cap B = \{a_1\}$, $B_2 \cap B = \{a_2\}$, $B_3 \cap B' = \{a_3\}$, $B_4 \cap B' = \{a_4\}$.

Let us fix a game form $g \in G^E$ and consider the corresponding 2×4 subform g' in g; it is given in Figure 8, where the first (second) row is assigned to B (respectively, to B') and it contains a_1 and a_2 (respectively, a_3 and a_4). The remaining four outcomes $b_1, b_2, b_3, b_4 \in A$ are not necessarily pairwise distinct, yet, $\{b_1, b_2\} \cap \{a_3, a_4\} = \{b_3, b_4\} \cap \{a_1, a_2\} = \emptyset$, since $b_1, b_2 \in B$ and $b_3, b_4 \in B'$; see Figure 8.1.

By assumption, Boolean function E and game correspondence G^E is TT. Hence, we can assume that the associated game form $g \in G^E$, and its subform g' are TT, too. Then $b_1 = b_2$ and $b_3 = b_4$, since otherwise the first or the last two columns of g' form a not tight subform. Let us set $b_1 = b_2 = b$ and $b_3 = b_4 = b'$; see Figure 8.2. Yet, b (respectively, b') cannot be equal to both a_1 and a_2 (respectively, a_3 and a_4), since the letter are distinct. Without loss of generality, assume that $b \neq a_1$ and $b' \neq a_4$; see Figure 8.3. Then the first and last columns of g' form a not tight subform (even if $b = b'$) and we obtain a contradiction. □

2.4 Irreducible TT Boolean functions are self-dual

There is a simple characterization of reducibility of a game form in Boolean terms.

Lemma 17. *Game correspondence G^E contains a constant row (column) whose every entry is an outcome $a \in A$ if and only if $E = a \vee E'$ (respectively, $E^d = a \vee E''$). In both cases, every associated game form $g \in G^E$ is reducible.*

Proof. It follows immediately from the definitions. □

Thus, we can reformulate Theorem 6 as follows: If E is TT then either $E = a \vee E'$ or $E^d = a \vee E''$ or $E = E^d = a_1a_2 \vee a_2a_3 \vee a_3a_1$. In first two cases we will call E *reducible.*

Lemma 18. *If E is TT and irreducible then every two of its prime implicants have a variable in common.*

Proof. Let us assume indirectly that there are two prime implicants of E with disjoint set of variables $B, B' \subseteq A$. By Lemma 17, if E is TT then $|B| = 1$ or $|B'| = 1$, in other words, E is reducible and we get a contradiction. □

Lemma 19. *If E is TT and irreducible then it is self-dual, $E = E^d$.*

Proof. It is both obvious and well-known (see, for example, [6]) that E is dual-minor, $E \leq E^d$, if and only if every two prime implicants of E have a variable in common. Thus, by Lemma 18, if E is irreducible and TT then it is dual-minor, $E \leq E^d$. Furthermore, E is irreducible and TT if and only if E^d is irreducible and TT. To see this, it would suffice just to rename players 1 and 2. Hence, E and E^d are both dual-minor: $E \leq E^d$ and $E^d \leq (E^d)^d = E$. Hence, $E = E^d$, that is, E is self-dual. □

Furthermore, we will show that only one self-dual function is TT, all other are not. For example, let us consider the classical function associated with the Fano projective plane:

$E_F = a_1a_2a_3 \vee a_3a_4a_5 \vee a_5a_6a_1 \vee a_0a_1a_4 \vee a_0a_2a_5 \vee a_0a_3a_6 \vee a_2a_4a_6.$

It is well-known and not difficult to verify that E_F is self-dual, $E_F = E_F^d$. Yet, by Lemma 16, E_F is not TT. Indeed, rows $\{a_1, a_2, a_3\}$, $\{a_3, a_4, a_5\}$ and columns $\{a_0, a_1, a_4\}$, $\{a_0, a_2, a_5\}$ form a 2×2 game form that is not tight.

As another example, let us recall that the 3-majority EFF $E(\binom{5}{3})$ is self-dual but not TT; see Figure 7.

2.5 The only TT self-dual Boolean functions is the 2-wheel

Let us consider one more example. The so-called *k-wheel* is defined for $k \geq 2$ by formula

$$E_k = a_0a_1 \vee a_0a_2 \vee \ldots \vee a_0a_k \vee a_1a_2 \ldots a_k.$$

Again, it is well-known and easy to check that E_k is self-dual, $E_k = E_k^d$ for any $k \geq 2$. Game correspondences, G^{E_k} are given in Figure 9 for $k = 2, 3$, and in general. (Again, to save space we substitute for an outcome a_j only its subscript j.) Let us fix an arbitrary $g \in G^{E_k}$. Due to obvious symmetry, without loss of generality, we can choose a_k from $\{a_1, a_2, \ldots, a_k\}$. Yet, then a 2×2 not tight subform g' appears in g whenever $k \geq 3$; see Figure 9.

Yet, as we already know, 2-wheel E_2 is TT. There are two associated with G^{E_2} TT game forms; see Figure 5 (in which $i + 1$ is substituted for $i = 0, 1$ and 2).

Furthermore, we can strengthen Lemma 19 as follows.

Lemma 20. *If E is TT and irreducible then it is a 2-wheel.*

	01	02	12
01	01	0	1
02	0	02	2
12	1	2	12

	01	02	03	123
01	01	**0**	0	**1**
02	0	02	0	2
03	0	0	03	3
123	1	**2**	3	12**3**

	01	02	$\cdots$	$0k$	$12\ldots k$
01	01	**0**	$\cdots$	0	**1**
02	0	02		0	2
$\vdots$			$\ddots$		$\vdots$
$0k$	0	0		$0k$	k
12...k	1	**2**	$\cdots$	k	12...**k**

Figure 9: 2-wheel, 3-wheel, and k-wheel.

Proof. Let us fix a prime implicant of E with the largest set of variables, which we will denote, without loss of generality, by $B = \{a_1, \ldots, a_k\} \subseteq A$. Since E is irreducible, $k \geq 2$.

By Lemma 19, E is self-dual, $E = E^d$. Then, by Lemma 12 (ii), for every $j = 1, \ldots, k$ function E contains a prime implicant with the set of variables B_j such that $B \cap B_j = \{a_j\}$. Furthermore, by Lemma 16, $|B \setminus B_j| = 1$ or $|B_j \setminus B| = 1$.

Let us assume that $k \geq 3$. Then $|B \setminus B_j| \geq 2$. Hence, $|B_j \setminus B| = 1$, that is, $B_j = \{a_j, b_j\}$ for each $j = 1, \ldots, k$. Moreover, by Lemma 12 (i), all b_j must coincide, that is, $B_j = \{a_0, a_j\}$ for each $j = 1, \ldots, k$. In other words, E is a k-wheel with $k \geq 3$. Yet, as we already know, in this case E_k is not TT. Hence, $k = 2$, that is, every prime implicant of E has exactly two variables; in other words, $E = a_1a_2 \vee a_0a_1 \vee a_0a_2$ is the 2-wheel. □

Thus, all TT irreducible game forms have the same EFF, the 2-wheel. This completes the proof of Theorem 6. □

3 Characterizing totally tight game forms

3.1 Canonical partition of a totally tight game form

Let g be a TT game form. We know that $E_1^g = E_2^g = a_1a_2 \vee a_2a_3 \vee a_3a_1$. Yet, the corresponding DNFs $D_1 = D_1^g$ and $D_2 = D_2^g$ might be redundant. Let us consider partitions

$X_i = X_i^{12} \cup X_i^{13} \cup X_i^{23} \cup X_i^{123} \cup X_i^{1234}$ for $i \in \{1, 2\}$,

where the first four sets of lines, rows ($i = 1$) and columns ($i = 2$), consist of outomes $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$, and $\{a_1, a_2, a_3\}$, respectively, while X_i^{1234} is the set of lines that contain an outcome $a \notin \{a_1, a_2, a_3\}$. Let us notice that $X_i^{12} \neq \emptyset$, $X_i^{13} \neq \emptyset$, and $X_i^{23} \neq \emptyset$, while X_i^{123} and X_i^{1234} might be empty.

3.2 Subform $\{X_1^{12} \cup X_1^{13} \cup X_1^{23}\} \times \{X_2^{12} \cup X_2^{13} \cup X_2^{23}\}$

It is easy to see that

$g(x_1, x_2) = a_1$ when $x_1 \in X_1^{12}, x_2 \in X_2^{13}$ or $x_1 \in X_1^{13}, x_2 \in X_2^{12}$;

$g(x_1, x_2) = a_2$ when $x_1 \in X_1^{12}, x_2 \in X_2^{23}$ or $x_1 \in X_1^{23}, x_2 \in X_2^{12}$;

$g(x_1, x_2) = a_3$ when $x_1 \in X_1^{13}, x_2 \in X_2^{23}$ or $x_1 \in X_1^{23}, x_2 \in X_2^{13}$.

As we already mentioned in Section 2.2, only the following two assignments are feasible in the main diagonal, see Figure 5,

(i) $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{13}, x_2 \in X_2^{13}$, $g(x_1, x_2) = a_2$ when $x_1 \in X_1^{12}, x_2 \in X_2^{12}$, and $g(x_1, x_2) = a_3$ when $x_1 \in X_1^{23}, x_2 \in X_2^{23}$;

(ii) $g(x_1, x_2) = a_3$ when $x_1 \in X_1^{13}, x_2 \in X_2^{13}$, $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{12}, x_2 \in X_2^{12}$, and $g(x_1, x_2) = a_2$ when $x_1 \in X_1^{23}, x_2 \in X_2^{23}$.

It is not difficult to verify that any mixture of (i) and (ii) is in contradiction with total tightness of g. Due to symmetry, we can fix either (i) or (ii) without any loss of generality. From now on, we will assume that (i) holds, as in Figures 5, where we substitute only subscript j for a_j.

3.3 Subforms $X_1^{1234} \times \{X_2^{12} \cup X_2^{13} \cup X_2^{23}\}$ and $\{X_1^{12} \cup X_1^{13} \cup X_1^{23}\} \times X_2^{1234}$; Approximation I

Let us show that $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{13}$.

The last inclusion implies that $g(x_1, x_2)$ equals either a_1 or a_3. Let us assume indirectly that $g(x_1, x_2) = a_3$. Then, $g(x_1, x_2) = a_1$ when $x_1 \in X_1^{12} \cup X_1^{13}$ and $x_2 \in X_2^{1234}$, otherwise g is not TT; see Figure 10. Furthermore, from total tightness of g we also derive that equalities $g(x_1, x_2) = a_2$ and $g(x_1, x_2) = a_3$ hold simultaneously when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{23}$; see Figure 10 again. The obtained contradiction proves our claim.

By the same arguments, we show five similar claims and obtain that

$g(x_1, x_2) = a_1$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{13}$,
$g(x_1, x_2) = a_2$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{12}$,
$g(x_1, x_2) = a_3$ when $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{23}$;
$g(x_1, x_2) = a_1$ when $x_1 \in X_1^{13}$ and $x_2 \in X_2^{1234}$,
$g(x_1, x_2) = a_2$ when $x_1 \in X_1^{12}$ and $x_2 \in X_2^{1234}$,
$g(x_1, x_2) = a_3$ when $x_1 \in X_1^{23}$ and $x_2 \in X_2^{1234}$.

The results are summarized in Figure 11. Let us notice that lines X_1^{1234} and X_2^{1234} are filled in accordance with the majority rule, that is, each entry of the last line is the most frequent outcome in the corresponding orthogonal line. Yet, we have to identify equal lines before counting.

Let us also notice the following important corollary: if a line contains an outcome $a \notin \{a_1, a_2, a_3\}$ then this line must contain a_1, a_2, and a_3 too. For example, no line can consist of outcomes a_1, a_2, a_4 or a_1, a_2, a_4, a_5 only.

3.4 Further partition of sets X_1^{123} and X_2^{123}; Approximation II

From total tightness of g we can also derive the following implication. If $g(x_1, x_2) = a_3$ for some $x_1 \in X_1^{123}$ and $x_2 \in X_2^{13}$ then $g(x_1, x_2') = a_2$ (respectively, $g(x_1, x_2') = a_3$) for the

	X_2^{13}	X_2^{12}	X_2^{23}	X_2^{1234}
X_1^{13}	1	1	3	1
X_1^{12}	1	2	2	2
X_1^{23}	3	2	3	3
X_1^{1234}	1	2	3	4

1	1	3	1
1	2	2	1
3	2	3	
3		2/3	4

1	1	3	
1	2	2	2
3	2	3	2
1/3	1		4

1	1	3	3
1	2	2	
3	2	3	3
	1/2	2	4

Figure 10: Contradictions.

	X_2^{13}	X_2^{12}	X_2^{23}	X_2^{123}	X_2^{1234}
X_1^{13}	1	1	3	13	1
X_1^{12}	1	2	2	12	2
X_1^{23}	3	2	3	23	3
X_1^{123}	13	12	23	123	123
X_1^{1234}	1	2	3	123	1234

Figure 11: Structure of a TT game form; Approximation I.

same x_1 and arbitrary $x'_2 \in X_2^{12}$ (respectively, $x'_2 \in X_2^{23}$). Indeed, we already know that $g(x_1, x'_2)$ equals a_1 or a_2 (respectively, a_2 or a_3). Let us assume indirectly that $g(x_1, x'_2) = a_1$ (respectively, $g(x_1, x'_2) = a_2$) and choose an arbitrary $x'_1 \in X_1^{12}$. It is easy to verify that rows x_1, x'_1 and columns x_2, x'_2 result in a 2×2 game form that is not tight. Hence, g is not TT and we get a contradiction.

Let subset $X_1^{3123} \subseteq X_1^{123}$ be defined by the following property: for each $x_1 \in X_1^{3123}$ there is a $x_2 \in X_2^{13}$ such that $g(x_1, x_2) = a_3$. In other words, subform $g' : X_1^{3123} \times X_2^{13} \to A$ takes only two values a_1, a_3 and a_3 appears in every its row. (In the next section we will show that a_1 appears in every its row, too.)

Since g is TT, g' is also TT, that is, every 2×2 subform of g' is tight. Hence, by permutations of rows and columns we can transform g' so that in every its row outcomes a_3 go first, while a_1 (if any) follow; in contrast, for each column outcomes a_1 (if any) go first, while a_3 (if any) follow; see Figure 12, where standardly j substitutes for a_j.

Definitely, the considered subform has a column whose every entry is a_3 (we will call it an a_3-column). In contrast, a_1-columns might exist or not (or, more precisely, their existence is not proven, yet). The corresponding two cases are denoted in Figure 12 by the dashed and dottet lines, respectively.

By symmetry, applying the same arguments, we will obtain two partitions:

$$X_i^{123} = X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123} \quad \text{for rows, } (i = 1) \text{ and columns } (i = 2). \tag{5}$$

To do this, first we substitute $i = 2$ for $i = 1$ to define subset of columns $X_2^{3123} \subseteq X_2^{123}$. Then we introduce subsets X_i^{1223} and X_i^{1123} for $i \in \{1, 2\}$, similarly to X_i^{3123}, using the cyclic shift of outcomes: $a_3 \to a_2 \to a_1$.

Finally, we define $X_i^{0123} \subseteq X_i^{123}$ as the set of rows ($i = 1$) or columns ($i = 2$) such that $g(x_i, x_{3-i}) = a_1$ (respectively, a_2 and a_3) for every $x_i \in X_i^{0123}$ and $x_{3-i} \in X_{3-i}^{13}$ (respectively, $\in X_{3-i}^{12}$ and $\in X_{3-i}^{23}$). The above arguments show that each line of X_i^{123} belongs to exactly one of the four subsets $X_i^{1123}, X_i^{1223}, X_i^{3123}, X_i^{0123}$. The obtained two partitions

$$X_i = X_i^{12} \cup X_i^{13} \cup X_i^{23} \cup X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123} \cup X_i^{1234} \tag{6}$$

for rows ($i = 1$) and columns ($i = 2$) are given in Figure 12.

Let us remark that the last five sets might be empty, while the first three cannot.

Remark also that the next six subforms have pairwise disjoint sets of rows and columns:

$$X_i^{1123} \times X_{3-i}^{12}, \quad X_i^{1223} \times X_{3-i}^{23}, \quad X_i^{3123} \times X_{3-i}^{13}; \quad \text{where } i = 1, 2.$$

Hence, we can bring them simultaneously to the “staircase” form shown in Figures 12.

3.5 From Approximation II to Approximation III

3.5.1 Preliminary remarks

In this Section we analyse Figure 12 further to get the next approximation, III, whose table is given in Figure 13. Let us notice that it contains the table of the approximation I in

	X_2^{13}	X_2^{12}	X_2^{23}	X_2^{1123}	X_2^{1223}	X_2^{3123}	X_2^{0123}	X_2^{1234}
X_1^{13}	1	1	3	1	1	1 3	1	1
X_1^{12}	1	2	2	2 1	2	2	2	2
X_1^{23}	3	2	3	3	3 2	3	3	3
X_1^{1123}	1	1 2	3					
X_1^{1223}	1	2	2 3					
X_1^{3123}	3 1	2	3					
X_1^{0123}	1	2	3				1 2 3	1 2 3
X_1^{1234}	1	2	3				1 2 3	1 2 3 4

Figure 12: Structure of TT game forms; Approximation II

Figure 11 as a subtable; furthermore, the rest of it is uniquely defined. All these properties we will prove in this section.

The reader should pay attention that, although the tables in Figures 12 and 13 are of size 8×8, we take into account that in each of the considered two partitions (6) only three from 8 sets are definitely non-empty, $X_i^{13}, X_i^{12}, X_i^{23}$, for $i = 1, 2$, while some (or all) of the remaining five might be empty. Of course, analyzing a subform $X_1^a \times X_2^b$, we also assume that the considered parts X_1^a and X_2^b are non-empty-empty too.

3.5.2 On table in Figure 12

By definition, lines X_i^{13}, X_i^{12}, and X_i^{23} consist of outcomes $\{a_1, a_3\}$, $\{a_1, a_2\}$, and $\{a_2, a_3\}$, respectively, while any other line contains all three outcomes $\{a_1, a_2, a_3\}$. Indeed, in Section

	X_2^{13}	X_2^{12}	X_2^{23}	X_2^{1123}	X_2^{1223}	X_2^{3123}	X_2^{0123}	X_2^{1234}
X_1^{13}	1	1	3	1	1	3 / 1	1	1
X_1^{12}	1	2	2	1 / 2	2	2	2	2
X_1^{23}	3	2	3	3	2 / 3	3	3	3
X_1^{1123}	1	2 / 1	3	1	1	3	1 3	1
X_1^{1223}	1	2	3 / 2	1	2	2	1 2	2
X_1^{3123}	1 / 3	2	3	3	2	3	2 3	3
X_1^{0123}	1	2	3	1 3	1 2	2 3	1 2 3	1 2 3
X_1^{1234}	1	2	3	1	2	3	1 2 3	1 2 3 4

Figure 13: Structure of a TT game form; Approximation III contains I as a subtable.

3.3, we proved this for X_i^{1234}, while the lines of X_i^{123} consist of a_1, a_2, a_3, by definition. Also by definition, all lines of X_i^{1234} and no others contain an outcome $a \notin \{a_1, a_2, a_3\}$.

To summarize, in Section 3.4, we computed the entries of subforms

$$\{X_{3-i}^{13} \cup X_{3-i}^{12} \cup X_{3-i}^{23}\} \times \{X_i^{13} \cup X_i^{12} \cup X_i^{23} \cup X_i^{1123} \cup X_i^{1223} \cup X_i^{3123} \cup X_i^{0123} \cup X_i^{1234}\}.$$

for $i = 1$ and $i = 2$; see Figure 12. In particular, subform $X_{3-i}^{1234} \times X_i^{13}$ (respectively, $X_{3-i}^{1234} \times X_i^{12}$ and $X_{3-i}^{1234} \times X_i^{23}$) contains a unique outcome a_1 (respectively, a_2 and a_3).

A subform whose each entry is a_j will be called an a_j-subform.

	X_2^{1123}	X_2^{1223}	X_2^{3123}
X_1^{1123}	$\mathbf{1}$	$\mathbf{1}_2$	$\mathbf{3}_1$
X_1^{1223}	$\mathbf{1}_2$	$\mathbf{2}$	$\mathbf{2}_3$
X_1^{1223}	$\mathbf{3}_1$	$\mathbf{2}_3$	$\mathbf{3}$

Figure 14: On the central subform

3.5.3 Subforms $X_{3-i}^{1234} \times \{X_i^{1123} \cup X_i^{1223} \cup X_i^{3123}\}$

For example, let us consider rows $X_1^{13} \cup X_1^{1234}$ and columns $X_2^{3123} \cup X_2^{1234}$. By definition, in every row of the subform $X_1^{1234} \times X_2^{1234}$ there is an outcome $a \notin \{a_1, a_2, a_3\}$. Also by definition, the subform $X_1^{13} \times X_2^{3123}$ contains a row whose every entry is a_3 (so-called a_3-row). These two observations together with total tightness imply that $g(x_1, x_2) = a_3$ for all $x_1 \in X_1^{1234}$ and $x_2 \in X_2^{3123}$. By symmetry, we fill subforms $X_{3-i}^{1234} \times \{X_i^{1123} \cup X_i^{1223} \cup X_i^{3123}\}$ for $i = 1, 2$, as in Figure 13.

3.5.4 On the subforms $X_{3-i}^{0123} \times \{X_i^{1123} \cup X_i^{1223} \cup X_i^{3123}\}$; $i = 1, 2$

For example, let us consider rows $X_1^{13} \cup X_1^{0123}$ and columns $X_2^{12} \cup X_2^{3123}$. As we already mentioned, the subform $X_1^{13} \times X_2^{3123}$ contains a a_3-row. This observation together with total tightness imply that $g(x_1, x_2)$ equals a_2 or a_3 for all $x_1 \in X_1^{0123}$ and $x_2 \in X_2^{3123}$. By symmetry, we fill subforms $X_{3-i}^{0123} \times \{X_i^{3123} \cup X_i^{1223} \cup X_i^{1123}\}$. for $i = 1, 2$, as in Figure 13.

Recall also that the subform $X_{3-i}^{0123} \times X_i^{13}$ (respectively, $X_{3-i}^{0123} \times X_i^{12}$ and $X_{3-i}^{0123} \times X_i^{23}$) contains only outcome a_1 (respectively, a_2 and a_3), by definition of X_{3-i}^{0123}.

3.5.5 On the cental subform $\{X_1^{1123} \cup X_1^{1223} \cup X_1^{3123}\} \times \{X_2^{1123} \cup X_2^{1223} \cup X_2^{3123}\}$

Let us choose rows $X_1^{13} \cup X_1^{3123}$ and columns $X_2^{13} \cup X_2^{3123}$. By definition, subforms $X_1^{13} \times X_2^{3123}$ and $X_1^{3123} \times X_2^{13}$ contain respectively an a_3-row and a_3-column. This observation and total tightness imply that $X_1^{3123} \times X_2^{3123}$ is a a_3-subform (that is, each its entry is a_3). By symmetry, we conclude that subforms $X_1^{1223} \times X_2^{1223}$ and $X_1^{1123} \times X_2^{1123}$ are a_2- and a_1-subforms, respectively, as shown in Figure 13.

Now, let us consider rows $X_1^{12} \cup X_1^{3123}$ and columns $X_2^{13} \cup X_2^{1223}$. As we already mentioned, subform $X_1^{3123} \times X_2^{13}$ contains a a_3-column. This observation together with total tightness imply that subform $X_1^{3123} \times X_2^{1223}$ contains only outcomes a_2 and a_3. By symmetry we conclude that for $i = 1, 2$ the subforms $X_i^{1123} \times X_{3-i}^{1223}$, $X_i^{1123} \times X_{3-i}^{3123}$, and $X_i^{1223} \times X_{3-i}^{3123}$ contain only outcomes $\{a_1, a_2\}$, $\{a_1, a_3\}$, and $\{a_2, a_3\}$, respectively; see Figure 14.

3.5.6 The dashed lines takes place in Figure 12

By definition, subform $X_1^{3123} \times X_2^{13}$ contains an a_3-column x_2^3. Now we want to show that it contains an a_1-column x_2^1, too.

First let us notice that a_1 must appear in every row $x_1 \in X_1^{3123}$, since otherwise this row would belong to X_1^{23} rather than X_1^{3123}. Furthermore, from the obtained results it follows that $g(x_1, x_2) = a_1$ can hold only if $x_2 \in X_2^{13} \cup X_2^{1123}$ Assume indirectly that $x_1 = x_1^3 \in X_1^{3123}$ has no a_1, that is, x_1^3 is an a_3-row. Then if $g(x_1, x_2) = a_1$ then $x_2 = x_2^1 \in X_2^{1123}$ must hold.

Now, let us consider the subform $X_1^{12} \times X_2^{1123}$. If it has an a_2-row x_1^2 then four lines, x_1^2, x_1^3, x_2^3 and any $x_2 \in X_2^{1123}$ form a 2×2 subform that is not tight. Hence, there is no a_2-row in $X_1^{12} \times X_2^{1123}$. Yet, then there are a_1-columns. Let $X_2^{11231} \subseteq X_2^{1123}$ denote the set of these columns.

From the above observations and total tightness it is not difficult to derive that $x_2^1 \in X_2^{11231}$ and, moreover, column x_2^1 contains only a_1 and a_3. Yet, in this case it would belong to X_2^{13} rather than X_2^{1123}, a contradiction.

By symmetry, we conclude that the following six subforms

$X_i^{1123} \times X_{3-i}^{12}$, $X_i^{1223} \times X_{3-i}^{23}$, and $X_i^{3123} \times X_{3-i}^{13}$; i = 1,2,

contains a_2-, a_3-, and a_1-columns ($i = 1$) and -rows ($i = 2$), respectively.

In other words, dashed lines take place in Figure 12.

3.5.7 Finalizing the cental subform $\{X_1^{1123} \cup X_1^{1223} \cup X_1^{3123}\} \times \{X_2^{1123} \cup X_2^{1223} \cup X_2^{3123}\}$

In Section 3.5.6, we proved that subform $X_1^{3123} \times X_2^{1123}$ can contain only a_1 and a_3. Yet, let x_1^3 be an a_3-column in $X_1^{3123} \times X_2^{13}$ and x_2^2 be an a_2-row in $X_1^{12} \times X_2^{1123}$. By adding to these two an arbitrary row $x_1 \in X_1^{3123}$ and column $x_2 \in X_2^{1123}$, we conclude that the considered subform $X_1^{3123} \times X_2^{1123}$ can contain only a_2 and a_3. Hence, only a_3 can take place. By symmetry, we conclude that

$X_1^{3123} \times X_2^{1223}$ and $X_1^{1223} \times X_2^{3123}$ are a_2-subforms; $X_1^{3123} \times X_2^{1123}$ and $X_1^{1123} \times X_2^{3123}$ are a_3-subforms; $X_1^{1223} \times X_2^{1123}$ and $X_1^{1123} \times X_2^{1223}$ are a_1-subforms.

in Figure **??** the central 3×3 subtable must be exact copy of the 3×3 subtable in the upper left corner. In other words,

In other words the following two 3×3 subtables have exactly the same structure:
$X_1^{1123} \cup X_1^{1223} \cup X_1^{3123} \times X_2^{1123} \cup X_2^{1223} \cup X_2^{3123}$ and $X_1^{13} \cup X_1^{12} \cup X_1^{23} \times X_2^{13} \cup X_2^{12} \cup X_2^{23}$.

This is an important observation showing that the 3×3 blocks along the main diagonal repeat themselves. However, the size of these blocks might become less than 3×3, since as we already mentioned, some (or all) of the six sets X_i^{1123}, X_i^{1223}, and X_i^{3123}, for $i = 1, 2$, can be empty.

Still, TT game forms are not explicitly characterized, since Figure 13 contains subforms $X_{3-i} \times \{(X_i^{3123} \cup X_i^{1223} \cup X_i^{1123} \cup X_i^{0123} \cup X_i^{1234}\}$ for $i = 1, 2$, which are not well-defined, yet.

3.6 Recursive description of TT game forms; Approximation IV

The following two important properties of Approximation III form the base for a recursion.

(i) Every row of $X_1^{0123} \cup X_1^{1234}$ and column $X_2^{0123} \cup X_2^{1234}$ begins with a_1, a_2, a_3; see Figure 13. More precisely, $g(x_1, x_2) = a_1$, respectively, a_2 and a_3, whenever

$x_1 \in X_1^{0123} \cup X_1^{1234},\ x_2 \in X_2^{13}$ or $x_2 \in X_2^{0123} \cup X_2^{1234},\ x_1 \in X_1^{13}$;
$x_1 \in X_1^{0123} \cup X_1^{1234},\ x_2 \in X_2^{12}$ or $x_2 \in X_2^{0123} \cup X_2^{1234},\ x_1 \in X_1^{12}$;
$x_1 \in X_1^{0123} \cup X_1^{1234},\ x_2 \in X_2^{23}$ or $x_2 \in X_2^{0123} \cup X_2^{1234},\ x_1 \in X_1^{23}$.

(ii) Given a TT game form $g : X_1 \times X_2 \to A$, where X_1 and X_2 are partitioned as shown in Figure 13, let us delete rows $X_1^{13} \cup X_1^{12} \cup X_1^{23}$ from X_1, columns $X_2^{13} \cup X_2^{12} \cup X_2^{23}$ from X_2, and denote the obtained subform by $g' : X'_1 \times X'_2 \to A$. This reduction results exactly in Approximation I, as one can see by comparing Figures 13 and 11.

Let us partition the sets X_1^{0123} and X_2^{0123} in the same way as we partitioned X_1^{123} and X_2^{123} in Section 3.4, etc. The obtained table is given in Figure 15, where

$$X_i = \bigcup_{j=0,1,\dots} \{X_i^{j1} \cup X_i^{j2} \cup X_i^{j3}\};\ i = 1, 2. \tag{7}$$

Let us show that all the 3×3 blocks $\{X_1^{j1} \cup X_1^{j2} \cup X_1^{j3}\} \times \{X_2^{j1} \cup X_2^{j2} \cup X_2^{j3}\}$ are uniquely defined and have the same structure for all j. We already know this for $j = 0, 1$. Now, let $j = 2$. First, analyzing $X_1^{11} \cup X_1^{23} \times X_2^{11} \cup X_2^{23}$ we conclude that $X_1^{23} \times X_2^{23}$ is an a_3-subform. Analyzing in a similar way two subtables $X_1^{02} \cup X_1^{23} \times X_2^{11} \cup X_2^{22}$ and $X_1^{13} \cup X_1^{23} \times X_2^{01} \cup X_2^{22}$ we derive that $X_1^{23} \times X_2^{22}$ is an a_2-subform. Indeed, first we see that it can contain only outcomes a_2 and a_3, then that only a_1 and a_2.

The following remarks are important. If the a_3-subform $X_1^{23} \times X_2^{23}$ is not empty then, of course, $X_1^{23} \neq \emptyset$ and $X_2^{23} \neq \emptyset$. Moreover, $X_1^{11} \neq \emptyset$ and $X_2^{11} \neq \emptyset$, either. Indeed, if X_i^{11} is empty then the sets of strategies X_{3-i}^{23} and X_{3-i}^{13} would merge. Similar arguments hold for the a_2-subform: if $X_1^{23} \times X_2^{22} \neq \emptyset$ then, of course, $X_1^{23} \neq \emptyset$ and $X_2^{22} \neq \emptyset$; moreover, $X_2^{11} \neq \emptyset$ and $X_2^{22} \neq \emptyset$; finally, $X_2^{01} \neq \emptyset$ and $X_1^{02} \neq \emptyset$, by definition.

Now, by symmetry, the whole subtable (7) is uniquely defined, as shown in Figure 15, for $j = 2$. The same arguments work for all $j \geq 2$, too.

Moreover, we can repeat all arguments of Sections 3.4 and 3.5, except only one, of Section 3.5.6, where we proved that the subform $X_1^{02} \times X_2^{11}$ contains an a_2-row. However, recursion does not keep this property. For example, it is no longer the case with the next subform $X_1^{12} \times X_2^{21}$. Moreover, in general, the subforms $X_1^{k2} \times X_2^{(k+1)1}$ might contain no a_2-rows whenever $k \geq 1$. In general, the subforms

$$X_1^{k1} \times X_2^{(k+1)3},\ X_1^{k2} \times X_2^{(k+1)1},\ X_1^{k3} \times X_2^{(k+1)2};\ X_2^{k1} \times X_1^{(k+1)3},\ X_2^{k2} \times X_1^{(k+1)1},\ X_2^{k3} \times X_1^{(k+1)2}$$

contain, respectively, a_1-, a_2-, a_3-rows and a_1-, a_2-, a_3-columns if $k = 0$ but might not contain them when $k \geq 1$; see Figure 16.

	X_2^{01}	X_2^{02}	X_2^{03}	X_2^{11}	X_2^{12}	X_2^{13}	X_2^{21}	X_2^{22}	X_2^{23}	
X_1^{01}	1	1	3	1	1	1, 3	1	1	1	1
X_1^{02}	1	2	2	2, 1	2	2	2	2	2	2
X_1^{03}	3	2	3	3	3, 2	3	3	3	3	3
X_1^{11}	1	1, 2	3	1	1	3	1	1	1, 3	1
X_1^{12}	1	2	2, 3	1	2	2	2, 1	2	2	2
X_1^{13}	3, 1	2	3	3	2	3	3	3, 2	3	3
X_1^{21}	1	2	3	1	1, 2	3	1	1	3	1
X_1^{22}	1	2	3	1	2	2, 3	1	2	2	2
X_1^{23}	1	2	3	3, 1	2	3	3	2	3	3
	1	2	3	1	2	3	1	2	3	g'

Figure 15: Approximation IV and recursion

The above recursive procedure is shown in Figure 15, where the game form g' might appear after any number ℓ of recursive steps; for example, $\ell = 3$ in Figure 15. Recursion works whenever g' is TT.

Proposition 21. *Game form g is TT if and only if g' is TT.*

Proof. The "only if" part is obvious, since total tightness is a hereditary property. Let us

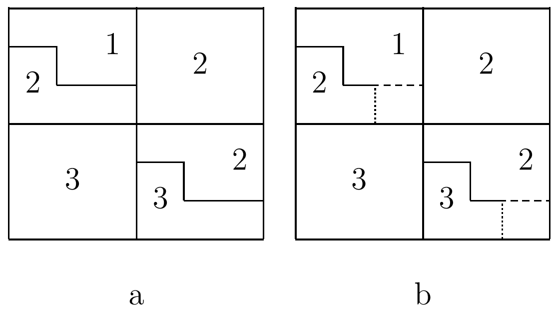

Figure 16: Two fragments of Approximation II

assume that g' is TT and show that g is TT, too, that is, every 2×2 subform g'' of g is tight, or which is the same, has a constant line. Let g' and g'' have k entries in common. Obviously, k might take values $0, 1, 2$ or 4. In the last case g'' is tight, since g' is TT. Furthermore, let us notice that for each line of g', a row or column, its extension to $g \setminus g'$ has a very simple structure: $(123123\ldots)$. From this observation it is easily seen that g'' is also tight when k equals 0 or 2; see Figure 15. Let $k = 1$ and $x = (x_1, x_2)$ be the only common situation of g' and g''. Although this case contains very many subcases, still it is not difficult to verify that the 2×2 game form g'' formed by two pairs of strategies x_1, x_1' and x_2, x_2' has a constant line for any choice of $x_1' \in X_1(g) \setminus X_1(g')$ and $x_2' \in X_2(g) \setminus X_2(g')$; see Figure 15. □

Let us underline that g' can be an arbitrary TT game form. In particular, it might contain constant lines and outcomes distinct from $\{a_1, a_2, a_3\}$. Let us also remark that Figure 15 represents the case when in each iteration all six sets X_i^{1123}, X_i^{1223}, X_i^{3123}; $i = 1, 2$ of approximation II in Figure 13 are not empty. Yet, some of them might be empty.

Thus, we cannot claim that all TT game forms are produced by the above recursive procedure. Yet, it is proven that every TT game form g is a subform of a game form g' produced by this procedure.

4 Totally tight game forms are dominance-solvable, acyclic, and assignable; proofs of Propositions 7, 8, and 9

These three claims easily follow from Approximations III and IV. Let us recall that, by definition, TT, AC, and AS are hereditary properties of game forms, while DS is not.

4.1 Proof of Proposition 8, $TT \Rightarrow DS$

Let us assume indirectly that a TT game form g is not DS. Then there is a payoff (or preference profile) u such that game (g, u) is not DS. Let us eliminate successively dominated strategies from (g, u) in an arbitrary order until we obtain a domination-free subgame (g', u). Yet, game form g' is TT, since g was TT. However, g' might be reducible. Then, let us successive eliminate constant lines, rows or columns, from g' until we obtain a (unique) irreducible game form g''.

Clearly, game (g'', u) is still domination-free, since elimination of a constant line respects this property. Since g'' is TT and irreducible, it must be of of type given in Figure 13, Approximation III. Let us recall that sets of rows X_i^{12}, X_i^{13}, and X_i^{23} are not empty for $i = 1, 2$; in contrast, sets X_i^{1123}, X_i^{1223}, X_i^{3123}, X_i^{0123}, and X_i^{1234} might be empty.

It is not difficult to verify that if a_1 (respectively, a_2 or a_3) is the worst outcome for player 1 among $\{a_1, a_2, a_3\}$ then every row from X_1^{13} (respectively, from X_1^{12} or X_1^{23}) is dominated by each row of X_1^{23} (respectively, of X_1^{13} or X_1^{12}); see Figure 13. Thus, game (g'', u) is not domination-free and we obtain a contradiction. □

4.2 Proof of Proposition 7, $TT \Rightarrow AC$

Given a TT game form g', assume indirectly that it is not acyclic, i.e., there is a payoff (or preference profile) u such that game (g', u) has a strict improvement n-cycle C_n. Let us consider the corresponding $n \times n$ subform g; obviously, it is TT, too. Moreover, in every line, row or column, of g there is exactly one arc of C_n. Since, a constant line cannot contain such an arc, we conclude that g is irreducible. Furthermore, being TT and irreducible, g is of of type given in Figure 13.

Then let us notice that every row (column) from $X_i^{12} \cup X_i^{13} \cup X_i^{23}$, where $i = 1$ (respectively, $i = 2$), contains exactly two outcomes: $\{a_1, a_2\}$, $\{a_1, a_3\}$, and $\{a_2, a_3\}$. Hence, $u(i, a_{j'}) \neq u(i, a_{j''})$ for all $i \in \{1, 2\}$ and distinct $j', j'' \in \{1, 2, 3\}$. Indeed, otherwise each line of the corresponding set $X_i^{j'j''}$ is constant and, hence, it contains no arc of C_n.

Obviously, the chain of inequalities $u(i, a_1) > u(i, a_2) > u(i, a_3) > u(i, a_1)$ cannot hold, by transitivity. Without loss of generality, let us assume that $u(1, a_1) > u(1, a_3)$. and prove that then $u(2, a_3) > u(2, a_2)$. Assume indirectly that $u(2, a_3) < u(2, a_2)$. Each column of X_2^{13} contains a (unique) arc of C_n. This arc goes from a_1 to a_3 and this a_3 is either in X_1^{23} or in X_1^{3123}; see Figure 13. Where the next arc of C_n can lead to? If a_3 is in X_1^{3123} then it can lead only to a_1 in a column of X_2^{13} again. This column also contain a (unique) arc of C_n that can lead only to a_3, etc. Thus, sooner or later, cycle C_n will come to a_3 in X_1^{23}. Then the next arc can only lead to a_2. Hence, $u(2, a_3) > u(2, a_2)$. Thus, we proved the implication: if $u(1, a_1) > u(1, a_3)$ then $u(2, a_3) > u(2, a_2)$. Exactly the same arguments prove the following chain of similar implications:

$u(1, a_1) > u(1, a_3) \Rightarrow u(2, a_3) > u(2, a_2) \Rightarrow u(1, a_2) > u(1, a_1) \Rightarrow u(2, a_1) > u(2, a_3) \Rightarrow u(1, a_3) > u(1, a_2) \Rightarrow u(2, a_2) > u(2, a_1) \Rightarrow u(2, a_2) > u(2, a_1)$.

Yet, it is easy to notice that they contradict transitivity of both $u(1, *)$ and $u(2, *)$; see inequalities $1, 3, 5$ and $2, 4, 6$, respectively. □

4.3 Proof of Proposition 9, $TT \Rightarrow AS$

Let us remark that our proofs for the acyclicity and dominance-solvability of a TT game form were based on Approximation III (Figure 13), while to derive the assignablity we will need Approximation IV. Yet, the proof itself is easier. Let us recall the recursion in Figure 15. Given an assignment for the subform g', we extend it to the whole game form g by assigning a_j to each strategy x_i^{kj}, where $j = 1, 2, 3,\ i = 1, 2$, and $k = 0, 1, \dots$

Since, as we know, both properties, TT and AS, are hereditary and each TT game form is a subform of a game form obtained by the above recursion, our claim follows. □

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