Combinatorics, Euler, and Toy Tracks

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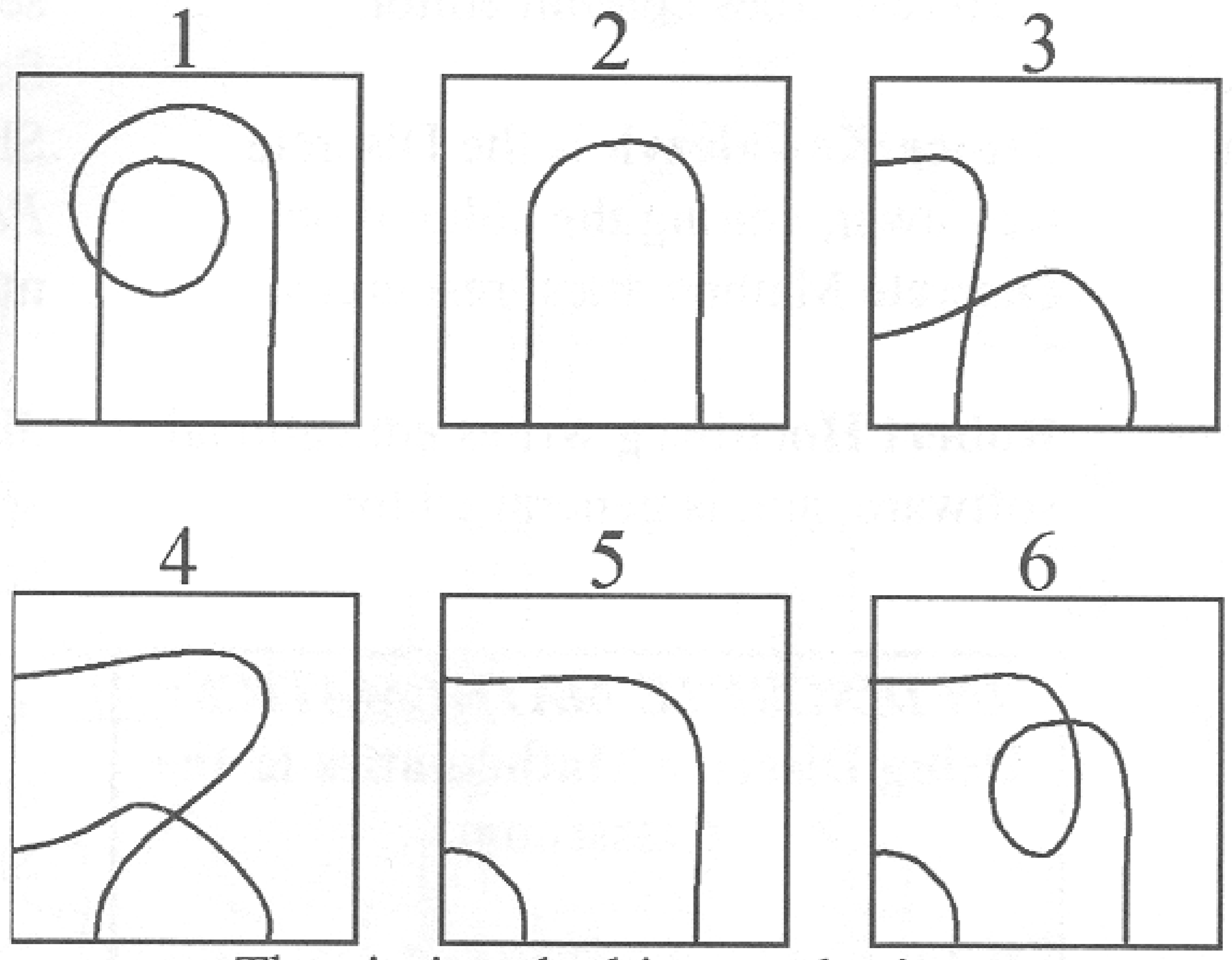
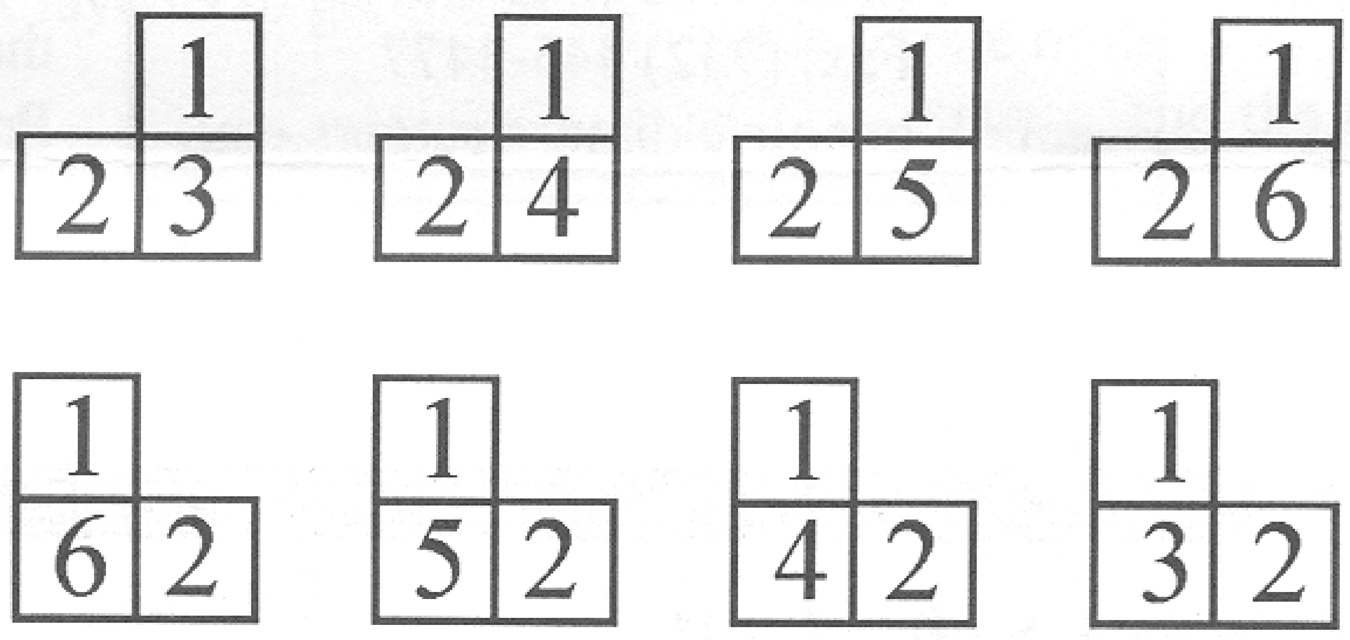
There is a cute toy made by PLASTOY called Puzzle Vehicle Set. It consists of six interlocking track pieces, a wind-up vehicle to run on the tracks, and three traffic signs. The box begs a mathematical question: "Six inter-changeable puzzle blocks for over fifty ... different layouts." Of course, in order to answer the question of "how many ways" you need to define what counts as different ways. Note that the pieces are so shaped that you can connect any two pieces along any edge where both have tracks.



The six interlocking track pieces

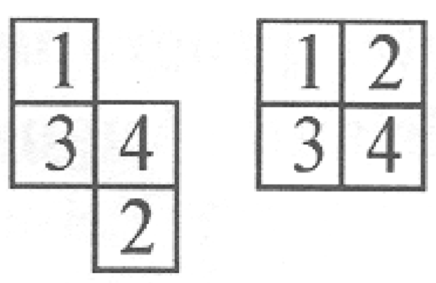
Number the pieces (as shown to the right) and pass the toys out to groups of students with instructions to decide if the statement on the box is correct. Are there really over fifty different ways to put the track together? Initially, instead of defining what constitutes putting the track together legally, let the students explore and record what they find. Some groups may start out trying to draw sketches of track assemblies, others might work more systematically to try to come up with all the ways to put three pieces together. Some examples are shown to the right.



Eight ways to put three pieces together

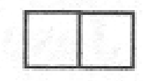
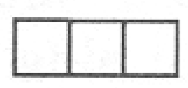
After some initial exploration, my students agreed that:

1. To count as a way to put the track together, the train has to be able to travel onto each piece without running off the track. These are similar to Euler circuits because it is possible to travel over the whole track ending up where the train started without going over any part of the track more than once.
2. There are different ways to put the same pieces together, for example, at the right are two ways to put pieces #1, #2, #3, and 4 together.



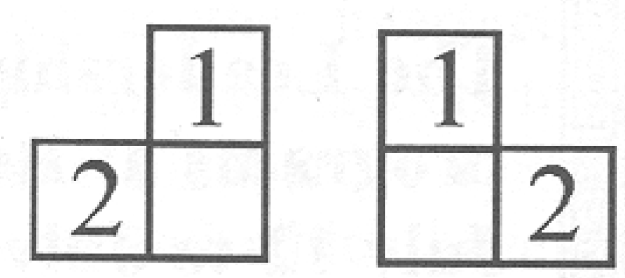
3. Rotating a completed track layout is not a new way.
4. If you use either piece #1 or #2, then you have to use the other one also.
5. The answer is not 6! (many students' original guess) because you can put together fewer than six pieces and still get a track that the train will stay on.

Because the track pieces can only be put together sharing a whole side, and all of the shapes are numbered, students are able to record their solutions using graph paper. Students were encouraged to discuss how they could systematically find all of the arrangements. They each had copies of the numbered track pieces to cut out and move around.

At first glance, the solutions resemble various polyominoes. However, some polyomino arrangements are impossible to build with the tracks while maintaining a circuit. For example, it is possible to put two pieces together (shapes #1 and #2)  in a straight line and still have a circuit, but not three, because pieces #3, #4, #5, and #6 involve a 90 degree turn when  connecting.

Finding all of the layouts which include pieces #1 and #2 involves first finding the number of possible track-design layouts starting with #1 and ending at #2 (see Figure A, soon to be discussed), and then counting the number of ways each of those layouts can be obtained using shapes #3, #4, #5 and #6.

For example, if you have a three-piece track which uses #1 and #2, you can have two basic designs, as shown below. The question becomes "How many ways can you put one piece from #3, #4, #5, and #6 between pieces #1 and #2. If you wish to place one piece between #1 and #2 (as in the figure below) then you have 4 choices (#3, #4, #5 or #6).



Refer now to Figure A, which depicts all the ways to lay out a connected track using the pieces #1 and #2. Notice how a tree diagram is used to systematically list all the ways, starting at the top with just the pieces #1 and #2. The arc in each square (except #1 and #2) indicates the two edges at which the tracks end in that square. Each row is obtained from the previous row by adding a new square in all possible ways. Then each row contains all track layouts which use a given number of pieces, including #1 and #2. For example, the bottom row shows 10 layouts using all six pieces.

There are 4 possible ways of having a four-piece track — see the third row of Figure A. In each, you have to place two pieces between #1 and #2. How many ways are there to do this? You have 4 choices for which piece comes first (meaning nearer to the end-piece #1) and then 3

(continued on page 7)