Voting Systems

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Decisions are made every day in every organization. The most famous decisions are public elections. Many people assume that the election results always indicate the will of the voters. You might be surprised to find out that the election results depend as much on the voting system or election method.

1 Plurality Voting

The voting system most people are familiar with is called plurality voting. Plurality voting is the voting system used in most state and local elections in the United States.

Plurality voting: the candidate with the most votes wins.

To introduce you to more kinds of voting systems, we will examine a vote in which plurality voting might not give what seems to be the fairest vote.

A math class is voting on when they would like to have a test review session. Lydia favors an evening session, so she can go after she gets out of work. An afternoon session would be okay with her, but she cannot get to a morning session. Her voting preference can be written EAM, where E (evening) is her first choice, A (afternoon) her second choice and M (morning) is her last choice. Five other people in the class have the same preference as Lydia. We can show this preference on a chart:

number of voters:	6
first choice	E
second choice	A
third choice	M

This means that six people (including Lydia) have the preference EAM.

Other people in the class do not share Lydia's preference for evenings. Ten people in the class would really like a morning review session, but an evening session would be okay. Their preference would be written MEA. Seven people in the class have the preference AEM. We can put all the voting preferences together in the table:

number of voters:	6	10	7
first choice	E	M	A
second choice	A	E	E
third choice	M	A	M

By plurality voting, the winner would be the morning session, since M has ten votes. However, if you look at the voting chart, you can see that $6 + 7 = 13$ people have morning as their *last* choice. You can also see that a good compromise might have been an evening session, since 6 people liked evening best, and $10 + 7 = 17$ people had evening as their second choice. Under plurality voting, only the students' first place choices were used.

Plurality voting does not take into account people's second or third place choices.

Thus, one of the consequences of using plurality voting for some offices is that voters' second and third choices are not used in determining the winner. If the two top candidates are very close, as with the morning and evening choices in our class vote, giving weight to second and third choices might help us reach a compromise that more people would like.

In this next example, we will again see what can happen when we use plurality voting, which ignores second and third choices.

A book club is voting on where to have the next meeting. Six people prefer Ryo's house (R), with the local coffee shop (C) as their second choice, and Malik's house (M) as their third choice. Their preference, RCM, is the first column in the table below.

number of voters:	6	2	7	1	4
first choice	R	M	M	R	C
second choice	C	R	C	M	R
third choice	M	C	R	C	M

The second column shows us that two people prefer Malik's house, with Ryo's as the next choice and the coffee shop as the last choice.

However, instead of looking at all three choices of the voters, if we look only at first place votes, we see that 9 people prefer Malik's house (M). We get this by counting the 2 people with preference MRC and the 7 people with preference MCR, since all these people have M as their first choice. Only 4 people prefer the coffee shop (C), and 7 people prefer Ryo's house (the 6 with preference RCM added to the 1 with preference RMC). Thus, Malik's house wins the plurality vote. However, a look at the third place shows that ten people have Malik's house as their least favorite. In this vote, C (the local coffee shop) might have been a good compromise, since 17 of the 20 people had C as their first or second choice.

Try it out:
In the following election for mayor, who would be the plurality winner, Dan Dirkman (D), Jermael Johnson (J), or Candace Carlin (C)?

number of voters:	5	2	3	1	4
first choice	J	D	C	J	C
second choice	C	J	D	D	J
third choice	D	C	J	C	D

Hint: count only the first place votes in each preference column. For example, since you see J (Jermael Johnson) in first place in two columns, you should count both those columns.

2 Plurality with Elimination

One way to take into account people's other preferences is called plurality with elimination, also known as instant runoff voting.

Plurality with elimination (or instant runoff): the candidate with the least first place votes is eliminated, and his votes are transferred to the next candidate in the ranking.

Using our previous example of the students voting for a test review time, if Lydia has the preference EAM and E loses, her vote goes to A, her second choice.

number of voters:	6	10	7
first choice	~~E~~	M	A
second choice	A	~~E~~	~~E~~
third choice	M	A	M

Now A gets $6 + 7 = 13$ votes and wins, so the session is held in the afternoon.

Now let's see what happens to the book club vote from the previous section if plurality with elimination is used.

In the first round of voting, choice C (the local coffee shop) gets the least votes and so is eliminated. We show this in our voting table by crossing out the letter C everywhere.

number of voters:	6	2	7	1	4
first choice	R	M	M	R	~~C~~
second choice	~~C~~	R	~~C~~	M	R
third choice	M	~~C~~	R	~~C~~	M

The 4 first-choice votes that went to C now go to R (Ryo's house). R now has those 4 votes, plus 6 votes for RCM and 1 vote for RMC, which gives Ryo a total of 11 votes, while M (Malik's house) only has 9 votes. Thus, Malik's house, which would have won under a strict plurality vote, comes in second when we use plurality with elimination.

This vote might still be considered problematic, since it does not take into account the 7 people who really did not want Ryo's house, or the fact that 17 people had C as a first or second choice.

Plurality with elimination is used in some elections in Australia and other places. In America, it was used in party primaries in several states in the first half of the 20th Century. However, it fell out of favor and was dropped. Currently, instant runoff is gaining popularity.

We will now look at another example where all possible voting preferences are shown. Suppose the fifteen member student senate wants to elect one of three nominees for the College's Professor of the Year award. The nominees are from the Art (A), Business (B), and Chemistry (C) departments. The preferences for each student senator are summarized below:

number of voters:	2	5	0	1	4	3
first choice	A	A	C	C	B	B
second choice	B	C	A	B	C	A
third choice	C	B	B	A	A	C

Notice that nobody has the preference CAB.

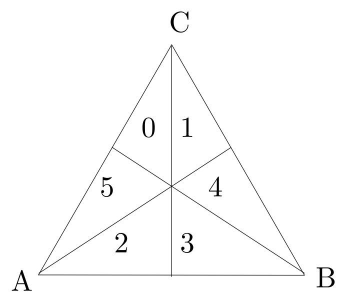
Using the plurality method and counting the first place votes, we have a tie: A gets 7 votes, B gets 7 votes, and C gets 1 vote.

The senate would like a clear majority for its selection, so it decides to use plurality with elimination. Candidate C is eliminated. The one person whose first choice was C now votes for B. Candidate B gets $1 + 4 + 3 = 8$ votes and wins with a majority of the votes against 7 for A.

3 The Saari Triangle

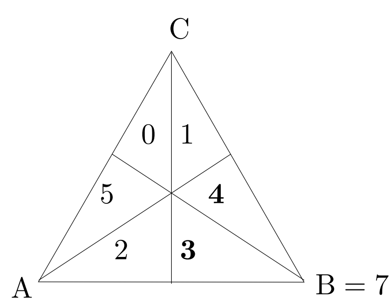
When there are many voting preferences, another way to illustrate them is called a *Saari representation triangle* (after Donald G. Saari, who created them in the 1990's). The triangle can make tallying total votes much easier, especially when other voting systems are used. *The representation triangle shows all possible voter preferences.*

We begin with a triangle with our three choices, one on each vertex, and lines drawn from each vertex to the middle of the opposite side. In each region, we put the number of votes for each voting preference:



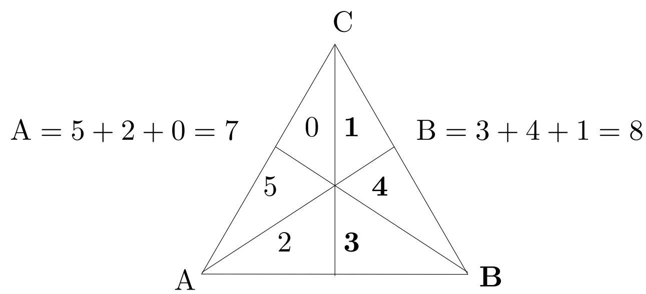
To tell which region corresponds to each voting preference, look at which vertex is closest to that region. For example, the region with 4 votes is closest to vertex B. Vertex C is the next closest and vertex A is farthest away, so the 4 corresponds to the preference BCA. Without looking at the table, try to figure out which voter preference corresponds to the number 5. Then check the table to see if you are correct.

Using the Saari triangle, we can get the *plurality totals* by *counting the two regions closest to each vertex.* For example, the total plurality vote for B is:



Verify that the total plurality vote for A is also 7, using the correct regions closest to A. What is the plurality total for C?

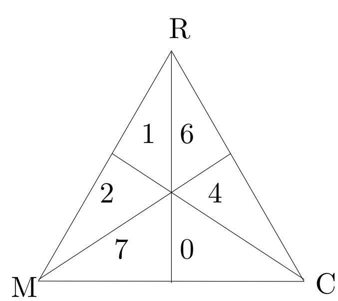
We can also use the triangle to help us find the total plurality with elimination tallies. Since C is eliminated, the tallies become the totals on the A side and the B side of the triangle:



In the Saari triangle, the plurality voting totals can be obtained by adding the two regions closest to each vertex. Plurality with elimination totals can be gotten by adding the three regions on each side of the two candidates left after eliminating the losing candidate.

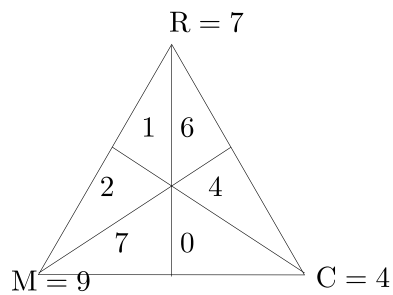
Let's take a look at how our book club preferences would be represented in the Saari Triangle. Since the book club was voting on choices C, R or M, those letters would be the vertices of the triangle. In our original table, the preference RCM had 6 votes, so the number 6 goes in the region closet to vertex R, with C the next closest vertex:

number of voters:	6	2	7	1	4
first choice	R	M	M	R	~~C~~
second choice	~~C~~	R	~~C~~	M	R
third choice	M	~~C~~	R	~~C~~	M



Similarly, since there were 2 votes for preference MRC, the number 2 goes in the region closest to the M, with R the next closest vertex. Notice the 0 in the triangle. This number corresponds to CMR, a preference not on the table, since it did not get any votes.

The plurality totals can again be found by counting the two regions closest to each vertex. By plurality vote, M wins with 9 votes.



To find the tallies for plurality with elimination, we first eliminate C, since it has the lowest plurality total. Next, we add the votes on the M and R sides. We get R = 6 + 4 + 1 = 11, and M = 7 + 2 + 0 = 9, so R wins the vote if plurality with elimination is used.

These results agree with what we got previously by looking at the table, but the Saari Triangle gives a visual image of the preferences that can be easier to take in quickly. We will also find that the Saari Triangle helps us to find the winner in many other types of voting systems.

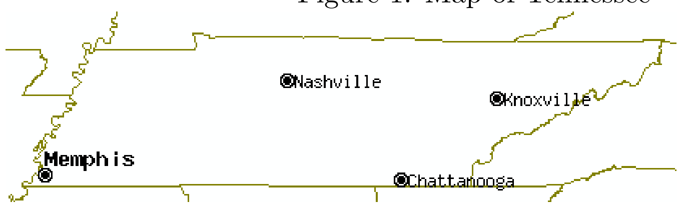
Try it out:

A group of 12 friends is trying to decide what type of food to have for dinner. Their choices of cuisine are: American (A), Brazilian (B) and Chinese (C). The group's preferences are summarized in the table below:

	2	3	3	2	2	0
first choice	A	A	C	C	B	B
second choice	B	C	A	B	C	A
third choice	C	B	B	A	A	C

Use the table to create a preference triangle. Then use the triangle to determine the plurality and plurality with elimination winners.

Figure 1: Map of Tennessee



4 Sequential Pairwise Voting

Sequential Pairwise voting: choose two candidates for a head-to-head contest. The winner moves on to face a third candidate in a head-to-head contest, with the winner of that contest moving on to face a fourth candidate. Continue until only one candidate is left.

With three or more candidates, a sequential pairwise election can occur in several ways, depending on the order in which candidates are to face-off.

Example: (Source: `www.census.gov`)
Imagine an election for the capital of Tennessee (Figure 1). Let's say the candidates for the capital are Memphis (M), Nashville (N) and Knoxville (K). Here's the population breakdown by metro area (surrounding county) as of the July, 2007 census estimates:

Memphis (Shelby County):	910,100
Nashville (Davidson County):	619,626
Knoxville (Knox County):	423,874

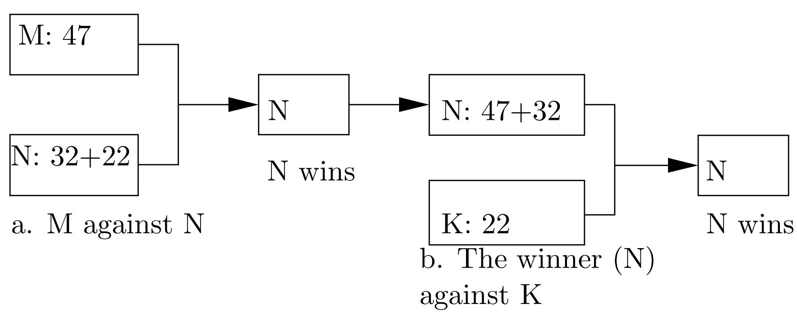
Suppose the voters vote based on geographic proximity. In other words, voters prefer the capital to be as close as possible to them. If we assume that everyone in the state lives in the three metro areas, then the preferences will be as follows:

Percent of voters:	47%	32%	22%
first choice	M	N	K
second choice	N	K	N
third choice	K	M	M

We can now use sequential pairwise voting to decide where the capital should be. There are three possible ways to do this: We can match up the voters who prefer Memphis (M) against the ones who prefer Nashville (N), then match the winner of that contest against the remaining area, Knoxville (K). Or we could pit N against K and whoever wins that voting round would be put in a contest with M. Or, finally, we could see who wins a contest between M and K, and pit the winner of that round against N. You may notice a similarity to playoff elimination rounds in sports.

The following diagrams illustrate the three ways to find the sequential pairwise winner.

1. M versus N and the winner against K:



The flowchart above shows how to find the winner by pitting M against N and the winner of that pair against N. For those who prefer to read a description, here's how to find the winner:

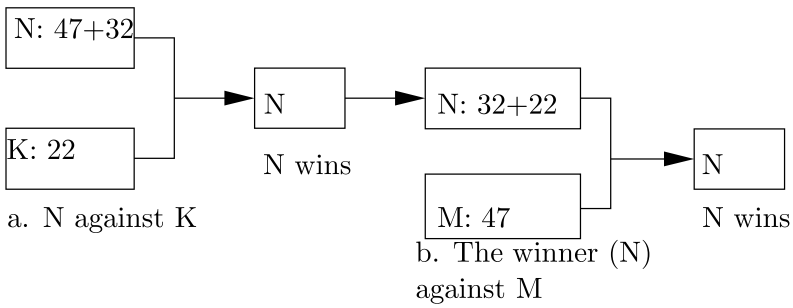
(a) To find the winner between M and N, we look at how many total voters prefer M over N as opposed to how many prefer N over M.

In the first column, 47% of the voters prefer M over N, because the preference ranking is MNK. In the second column, the ranking NKM shows that 32% of voters prefer N over M. The third column, with preference KNM, shows N preferred over M by 22% of the voters. Since the second and third columns both show N preferred to M, we add up the votes to get a total of 54%. Hence N wins.

(b) Next we pair N (the winner of the first round) with K, the remaining county, and do the process again for these two.

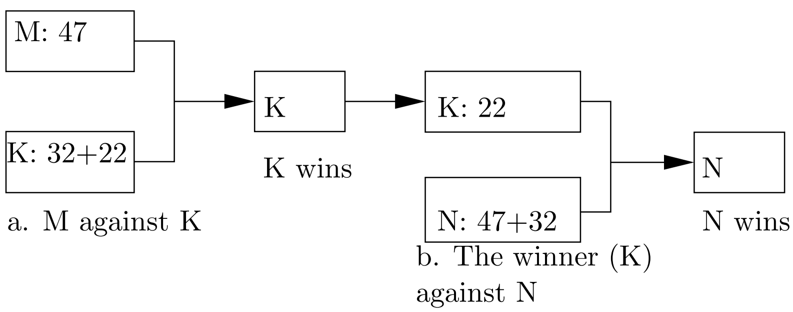
We again look at the preferences: 47% of voters have preference MNK, so they prefer N over K. The 32% in the next column of voters also prefer N over K, which gives us 79% of the voters preferring Nashville (N). In the third column only 22% prefer Knoxville (K) over Nashville (N). Hence N wins.

2. N versus K and the winner against M:

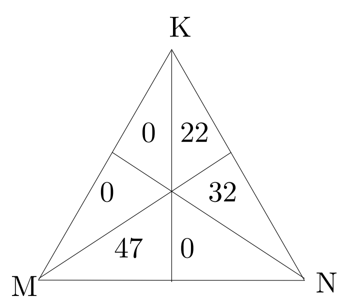


This time, we show the flowchart only and leave the description to the reader.

3. M versus K and the winner against N:



Another way to find a pairwise winner is to use the Saari Triangle.

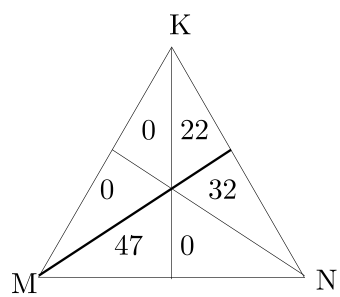


We will look at M versus N, then at the winner of that match versus K. Remember that any order of match ups will do.

In the contest M versus N, we first look at the three regions that represent the rankings that prefer M over N: MNK (47), MKN (0) and KMN (0). This gives us a total of 47 that prefer M over N. Notice that this total can be found by adding the numbers on the left side of the vertical line that separates M from N.

Next, we look at the rankings that prefer N over M: KNM (22), NKM (32) and NMK (0). This gives us a total of 54. This total can be found by adding the numbers to the right of the vertical line that separates M from N.

Since the N side of the triangle has more votes than the M side, N wins against N, and we continue by matching N against K, by dividing the triangle into two parts.

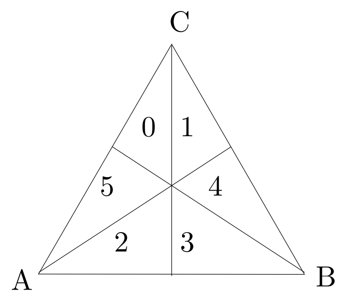


If we look at the N side of the triangle, we see 47+0+32=79 votes; while the K side has only 0+0+22= 22 votes, so K wins.

To find the sequential pairwise winner using the Saari Triangle, first divide the triangle along one of its lines and add the votes on each side. (This is equivalent to pitting two of the candidates against each other.) Then pit the winner of that vote against the remaining candidate by dividing the triangle along the line separating the first match winner and the remaining candidate.

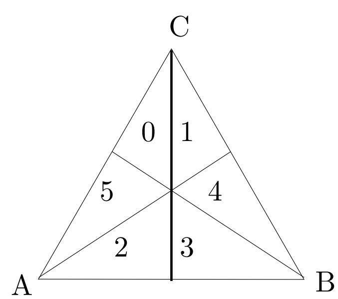
For an additional example, let's look at who would have won the Professor of the Year award from the previous section if we had used pairwise sequential voting. We'll use the Saari Triangle, and show how it can quickly give a visual of the pairwise winner.

number of voters:	2	5	0	1	4	3
first choice	A	A	C	C	B	B
second choice	B	C	A	B	C	A
third choice	C	B	B	A	A	C

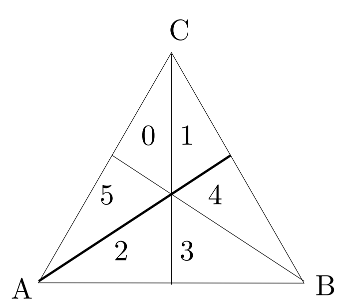


You may recall that plurality voting gave us a tie, while plurality with elimination gave us candidate B (from Business) as the winner. These results can be seen in the triangle as follows: adding the two numbers closest to A gives us 7, while adding the two closest to B also gives us 7, showing the plurality tie. Eliminating C gives us plurality with elimination totals of 7 for A and 8 for B, gotten by adding the entire side of the triangle closest to that vertex.

The Saari Triangle can give us a quick visual of the pairwise winners as follows: Let's start by looking at A against B. The rankings in which A is preferred over B are: ABC (2), ACB (5) and CAB (0). These are the voters on the left side of the triangle:



Professor B wins against A in the first match up. Then B is matched against C and B is the winner.



Thus, both sequential pairwise voting and plurality with elimination give us B as the winner.

Try it out:

1. (a) Using sequential pairwise voting on the same Professor of the Year Award example as above, show that pitting B against C and the winner against A will still result in B as the winner.

   (b) Show that pitting C against A and the winner against B will also still result in B as the winner.

2. In a survey, 8 people were asked to rank Pepsi (P), Cola (C) and Dr. Pepper (D) according to their preferences. The following is the distribution of their rankings.

3 voters	3 voters	2 voters
P	D	C
C	P	D
D	C	P

   (a) Begin by creating a Saari triangle to show the voting preferences.

   (b) Using the Saari triangle, find the winner if you start with the pair (P, C), then pit the winner against D.

   (c) Using the Saari triangle, find the winner if you start with the pair (P, D), then pit the winner against C.

   (d) Using the Saari triangle, find the winner if you start with the pair (D, C), then pit the winner against P.

   (e) Is there a clear winner? Explain.

5 Condorcet's method of pairwise comparison

The only "perfect" election (one that truly reflects the desires of the population) occurs when there are two candidates. The winner is the candidate who receives a majority of the votes and there is no doubt who beat whom. The 18th-century French mathematician and philosopher Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet, proposed a method which uses the idea of a two-candidate election to determine the winner when there are three or more candidates. His idea reduces down to looking at a series of head-to-head matchups between the candidates where the overall winner is the candidate who wins the majority of these sub-contests. To allow for tied matchups, we will give one point to a candidate who wins a matchup outright and 1/2-point to each candidate if there is a tie. The winner then will be the person with the most points.

In the Method of Pairwise Comparison, each candidate is matched with every other candidate in a one-on-one election. The winner of this sub-contest earns one point; if there is a tie, both candidates receive 1/2 point. The winner is the candidate with the most points.

Example:
Start with the preference table below for an election of Dogcatcher for Catscratch County, PA. Here we have three candidates: Smith, Jones, and Firkaly.

number of voters:	20	10	9	8
first choice	Smith	Firkaly	Firkaly	Jones
second choice	Firkaly	Smith	Jones	Firkaly
third choice	Jones	Jones	Smith	Smith

We will have three head-to-head matchups: Smith vs. Jones, Smith vs. Firkaly, and Jones vs. Firkaly. Let us look at the first one. In this instance, the third candidate (Firkaly) is ignored and the chart looks like this:

number of voters:	20	10	9	8
first choice	Smith			Jones
second choice		Smith	Jones	
third choice	Jones	Jones	Smith	Smith

Now Smith is above Jones in the first two columns only, giving 30 total votes for Smith while 17 voters preferred Jones. Thus, Smith gets 1 point.

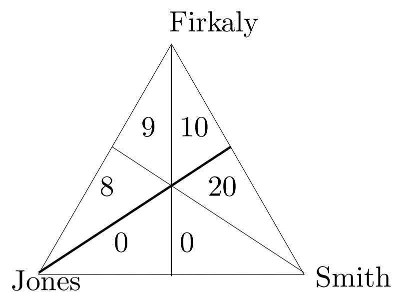
Next, we match Firkaly and Smith head-to-head, ignoring all other candidates:

number of voters:	20	10	9	8
first choice	Smith	Firkaly	Firkaly	
second choice	Firkaly	Smith		Firkaly
third choice			Smith	Smith

We see that Firkaly beats Smith 27 to 20, so Firkaly gets 1 point. Similarly, Firkaly beats Jones (39 to 8) so gets another point. Thus, Firkaly has

2 points, Smith 1, and Jones 0. Notice that Firkaly has won the election without having a plurality of first-place votes: Firkaly has only 19 first place votes, out of a total of 47 cast.

Again, the Saari triangle gives a quick visual of which candidate wins in a head-to-head, by drawing a dividing line down the middle of the triangle between the two candidates. In the figure below, the ballots which rank Firkaly above Smith are those on the right side of the line, while those who prefer Smith to Firkaly on the left. Thus it is 20 + 0 + 0 = 20 votes for Smith, and 10 + 9 + 8 = 27 votes for Firkaly, so Firkaly earns a point.



The simplicity of this idea leads to what is now called the Condorcet Criterion for a fair election. Basically if the winner of an election is anyone other than a Condorcet winner, then the election must be flawed.

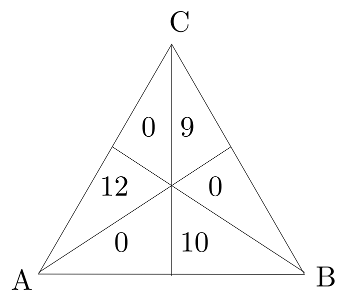
Condorcet Criterion: For an election to be fair, the winner of the election should be the candidate who wins the most of the pairwise elections.

For a real example of this, we turn to the 2000 Presidential Election between Al Gore, George W. Bush, and Ralph Nader. All data indicates that if it was a two-candidate election Gore would have beaten Bush and Gore would have won against Nader. So Bush was the electoral winner, but a Condorcet loser.

Are there drawbacks to the system? Yes, there are three worth commenting on.

First, this method can degenerate into a cycle. An example of one is below.

number of voters:	12	10	9
first choice	A	B	C
second choice	C	A	B
third choice	B	C	A



To see who wins in a match of A against B, draw a dividing line between the two sides of the triangle. On the left side, A has 12 + 0 + 0 = 12 points; on the right, B has 9 + 0 + 10 = 19 points, so B wins and gets a point. In a match of B against C, draw a dividing line between B and C to see that B gets 0 + 10 + 0 = 10 points while C gets 12 + 0 + 9 = 21 points, so C wins and gets a point. Finally, in a match of C against A, draw a dividing line between C and A to see that C gets 0 + 9 + 0 = 9 points while A gets 12 + 0 + 10 = 22 points, so A wins and gets a point. Thus each candidate has gotten just one point — so all candidates tie and no winner is found.

This problem with pairwise comparison can easily be overcome by putting in a contingency plan for how to break a cycle. For example, we can use plurality voting to determine the winner among the candidates in the cycle.

The second problem is not so easily solved since it lies not with the system, but with the voters. Condorcet methods are vulnerable to what is called "burying." In burying, a voter places a candidate lower than he/she sincerely feels in hopes of making a more preferred candidate able to have more head-to-head wins than the falsely lowered one.

Lastly, some experts in voting theory believe that Condorcet methods discourage candidates from revealing hard-line stances on some issues. This is due to the fact that the least offensive candidate will end up being ranked higher than the hard-liners and thus be able to win. The result of this is a less well-informed electorate.

Examples of variations on the Condorcet method include the Black Method (named after Duncan Black), the Minimax method, and the Schulze Method. The Software in the Public Interest Corporation uses Schulze to elect member of its board of directors as does the Debian Project (which runs a free computer operating system service).

6 Borda Count

Borda count: each voter submits his/her voting preferences ranking of the alternatives to be considered. The candidate in the last place receives 0 points; the next-to-last place 1 point; the next highest receives 2 points and so on. The number of points for each alternate is then totaled and the candidate with highest total is the winner.

For example, in the case of three candidates, a voter's first choice will receive 2 points, 2nd choice 1 point and the last choice 0 point. In this method the candidates that are not a voter's first choice are not totally neglected as they are in the plurality method.

Example:
The 1912 U. S. Presidential race was between Theodore Roosevelt (Bull Moose Party), William Taft (Democrat) and Woodrow Wilson (Republican). In the election, Wilson received about 45% of the vote, Roosevelt received about 30% of the vote and Taft received about 25% of the vote. Wilson won the Presidency. According to political historians, voters for Wilson generally preferred Roosevelt over Taft. Roosevelt voters supported Taft over Wilson and Taft voters supported Roosevelt instead of Wilson.

percent of voters:	45%	30%	25%
first choice	W	R	T
second choice	R	T	R
third choice	T	W	W

Determining the winner using the Borda count:
Wilson received 45% of the votes as the first choice candidate and 55% as last choice candidate. His Borda count is

$$\mathrm{W} = 45(2) + 0(1) + 55(0) = 90.$$

Roosevelt received 30% of the votes as the first choice candidate, 70% as second choice candidate and 0% as last choice candidate, so his Borda count is

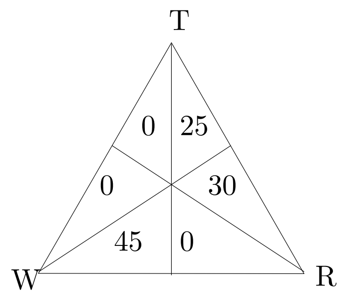
$$\mathrm{R} = 30(2) + 70(1) + 0(0) = 130.$$

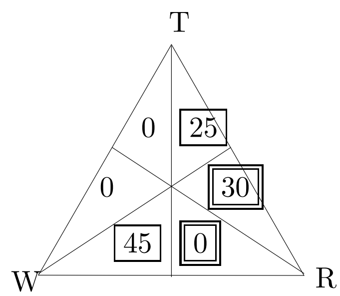
Likewise, Taft received 25% of the votes as the first choice candidate, 30% as second choice candidate and 45% as last choice candidate. His Borda count is therefore

$$T = 25(2) + 30(1) + 45(0) = 80.$$

Conclusion: Roosevelt wins using the Borda count method.

In the case of three candidates one can also use the Saari triangle to calculate the Borda count. For the above example, we get the following triangles:





Let's compute the Borda count for each candidate beginning with Roosevelt (R).

From the triangle above the numbers enclosed by double boxes indicate Roosevelt as first choice (2 points each vote), the ones enclosed by single boxes as second choice (1 point each vote) and the unboxed numbers as last choice (0 points each) candidate. Thus the Borda count for Roosevelt is

$$R = (30 + 0)(2) + (25 + 45)(1) + (0 + 0)(0) = 130.$$

Similarly, the Borda count for Wilson is

$$W = (45 + 0)(2) + (0 + 0)(1) + (25 + 30)(0) = 90,$$

and for Taft is

$$T = (25 + 0)(2) + (30 + 0)(1) + (45 + 0)(0) = 80.$$

Under the Borda count Roosevelt is the winner while Wilson won the plurality vote (and the presidency).

Try it out:
In a survey, 100 people were asked to rank Pepsi (P), Coke (C) and Dr. Pepper (D) according to their preferences. The following is the distribution of their rankings. Determine which beverage people prefer using the Borda count first by hand and then using Saari triangle.

number of voters:	32	35	25	8
first choice	P	C	D	C
second choice	D	P	C	D
third choice	C	D	P	P

7 Desirable properties in voting systems

In the previous sections we discussed several different voting systems and found that the results depend on the election method, confusing what could be called the will of the voters. Since this can happen we begin to wonder about our methods and then question arises "How do we evaluate the voting system under consideration?" To answer this, we first need to make up how we want it to be. In other words, what are the desirable properties of voting systems?

In two party system (two alternatives) the plurality (which is really the majority of votes) method is most commonly used method as it treats all the votes equally. This is the best system for choosing the winner in that situation. But what if we have more than two alternatives? As we saw in previous section situation gets complicated if there are more than two choices.

With three or more alternatives (multi-party system) an individual can be the plurality winner even though he/she may get less than half of the total population vote. In this case many of the voters may not be satisfied with the decision. An example is the 2000 U.S. Presidential election in which there were three major candidates; George W. Bush, Al Gore, and Ralph Nader. Bush won the election although he did not get the majority of the votes. For more information see the web site `http://www.uselectionatlas.org/`. Satisfying a majority will be a part of one of our criteria.

If we agree that plurality does not satisfy many of the voters, then one can think of using the Borda count as it gives the weight to the lower rankings. This system also has a drawback. Consider an election among three candidates (A, B, C) where A is ranked over B, yet when the election is done again and not a single voter switched the order of the preference for A and B, B now wins. This is shown in the example below.

Prior to an election a (very small) poll is taken. It has preference table

number of voters:	5	3	3
first choice	A	B	C
second choice	C	A	B
third choice	B	C	A

Using Borda Count candidate A has 24 points to B's 20 (and C's 22). Then, in a second poll right before election day, the five voters in the first column reverse the order of B and C (thus preserving A over B). The table now reads

number of voters:	5	3	3
first choice	A	B	C
second choice	B	A	B
third choice	C	C	A

Now A still has 24 points, but B has 25 (not only beating A, but this time winning the election).

So it seems that a voting system should have the property of picking the same winner independent of what happens to candidates ranked below the winner. This will, in turn, disallow strategic nominations of the candidates. Again, this will be a part of one of our properties.

The following are the most commonly considered criteria for a "satisfying" election:

Non-dictatorship: The decision-making should not simply follow the preference order of a single individual while ignoring all others.

Majority: If a majority of voters strictly prefers a given candidate to every other candidate and the voting is sincere, then that candidate should win.

Monotonicity: If an alternative X loses, and, in a second election, the only change to ballots is placing X in lower positions, then X must still lose. (It is generally considered a good thing if a voting system is monotonic since non-monotonic systems are vulnerable to tactical voting, where voters might try to elect their candidate by purposely lowering that candidate.)

Condorcet criterion: If a choice beats the majority of other choices in pairwise comparison, it should win.

Independence of irrelevant alternatives (I.I.A): If A is preferred to B in an election, then introducing a third, irrelevant, alternative X into a re-election should not make B preferred to A. In other words, whether A or B is better should not be changed by the availability of X.

Consistency:If the electorate is divided in two and choice A wins in both parts, then choice A should win the overall election.

Besides these, there are more extensive lists of criteria on the voting systems. We refer the interested reader to "*Electoral Systems: A Comparative Introduction*," by David M. Farrell for further information.

Now we look at those voting systems that we discussed in previous section satisfy above properties.

	Plurality	Instant Runoff	Pairwise Comparison	Borda Count
Non-dictatorship	P	P	P	P
Majority	P	P	P	**F**
Monotonicity	P	**F**	P	P
Condorcet	**F**	**F**	P	**F**
I.I.A.	**F**	**F**	**F**	**F**
Consistency	P	**F**	**F**	P

P: pass; **F**: fail

For the interested reader, some other desirable properties of voting system include simplicity, ambiguity of the method, speed of counting ballots, reduction for fraud and dispute.

As we see in the above table, no voting system is perfect. This can be discouraging and raises the new question, "Is there a voting system that satisfies all the desirable properties indicated above?" It is known that it is impossible for one voting system to pass all criteria. One such result, determined by Kenneth Arrow and called Arrow's Impossibility Theorem, showed that non-dictatorship, monotonicity, and independence of irrelevant alternatives are mutually contradictory for ranked voting systems. This means we cannot make our systems perfect. Now he is not saying that these criteria will always be violated, but that examples will exist (and can exist in real life) which violate these ideas for a "fair" election. For this Arrow won a 1972 Nobel Prize.

Some argue that Arrow required strong assumptions about what makes a voting method fair, but there is no inherent reason that his criteria should be considered fair. For example, Arrow requires implicitly that the voting method uses a ranked ballot, but it would be narrow-minded to state that any voting method that does not use a ranked ballot is unfair. If a method requires a random decision to break a tie, Arrow's criteria bring the decision down to an unfair one. They also point out that the independence of irrelevant alternatives (I.I.A.) is a flawed criterion. For example, if a population slightly preferred candidate B to candidate A, but candidate A supporters were far more loyal, then an introduction of a third candidate could split B's support far more than A's, leading to a win by A. In cases where one candidate's supporters feel they are compromising far more than the other candidate's supporters do, failing I.I.A. may not be a flaw.

Despite all the disagreements over Arrow's theorem, it is still considered useful in the sense that it provided a mathematical way of looking at the voting systems and their limitations.

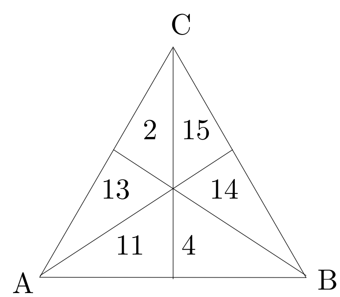
Anyone implementing a voting system has to decide which criteria are important for their election, keeping in mind both the fairness and the practicality of the system — and now Arrow has given us some tools for doing that. How will *you* decide what method of voting is the most fair?

8 Exercises and Activities

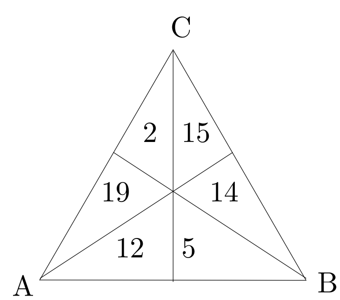
Exercises:

1. In Plurality with Elimination, there is more than one way to decide how to eliminate a candidate. For example, you can eliminate the person with the *least first place votes*, or you can eliminate the candidate with the *most last place votes*. For each of the Saari representation triangles below, determine (i) the Plurality winner, (ii) the Plurality with Elimination winner when eliminating by least first place votes, (iii) the Plurality with Elimination winner when eliminating by most last place votes, (iv) the Condorcet winner, and (v) the Borda count winner.

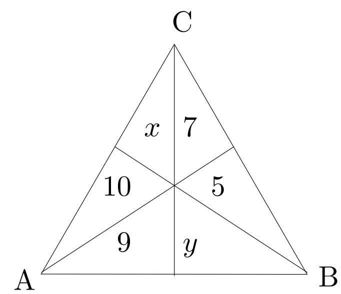
   (a) For the triangle



   (b) For the triangle



2. For the triangle below, determine the values of x and y so that simultaneously (1) A is the plurality winner and (2) B wins plurality with elimination. Who wins your Borda count?



3. Determine the Condorcet winner from the following preference table:

number of voters:	5	4	4	3
first choice	A	B	C	C
second choice	B	C	B	A
third choice	C	A	A	B

4. Given the table, construct a preference ballot to create a Condorcet cycle.

number of voters:	10	9	6
first choice	A	B	
second choice	B	C	
third choice	C	A	

5. How many head-to-head matchups are there for 4 candidates? 5? N?

6. Explain why this table cannot lead to a Condorcet cycle regardless of the order in the last column.

number of voters:	7	6	5
first choice	A	B	
second choice	B	A	
third choice	C	C	

7. Must a candidate who wins a *majority* of the first place votes be the Condorcet winner? Explain your answer.

8. The selection of Major League Baseball's (MLB) American League (AL) Most Valuable Player is done via a Borda count where two sportswriters from each city with an AL team make up the voters. For the 1999 season the top five candidates were as below.

Player	Team	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
I. Rodriguez	Rangers	7	6	7	0	5	2	1	0	0	0
P. Martinez	Red Sox	8	6	4	1	2	2	3	0	0	0
R. Alomar	Indians	4	7	6	4	4	0	1	1	0	0
M. Ramirez	Indians	4	4	5	9	1	5	0	0	0	0
R. Palmeiro	Rangers	4	1	2	4	8	4	5	1	0	1

(a) A variation of the Borda count is used by MLB, giving 14 points for first place, 9 for second, 8 for third, ..., 1 point for last. Find the rankings of these five players using this method.

(b) Find the rankings of these five players using the usual Borda count.

(c) Find the rankings if we only look at the first three players and their first, second, and third place votes using the usual Borda count.

(d) Find the rankings if we only look at the first four players and their first, second, third, and fourth place votes using the usual Borda count.

(e) Find the rankings for all five players if we only use their first through fifth place votes using the usual Borda count.

9. During a classroom evaluation, certain aspects are rated A = excellent through E = horrible. To find an "average," a university assigns points to each level: A = 4, B = 3, C = 2, D = 1, and E = 0. Dividing the Borda total by the number of responses gives this average. For example,

The manner in which this class was taught was ______________.

response	A	B	C	D	E
number	20	10	5	2	3

The average for this question would be

$$\frac{20(4) + 10(3) + 5(2) + 3(0)}{20 + 10 + 5 + 2 + 3} = \frac{122}{40} = 3.05.$$

(a) One professor decided to make her own averages using A = 5, B = 3, C = 2, D = 1, and E = 0. Why will this enable her to look better than she really is in the eyes of the administration?

(b) Faculty complain that E = 0 is unfair because multiplying by zero is just another zero. If the administration agrees and changes to A = 5, B = 4, C = 3, D = 2, and E = 1, by how much will that raise the scores? Does it really make a difference?

10. The table below shows an extrapolation of the data from the 1991 Louisiana Gubernatorial race, pitting moderate Republican incumbent Buddy Roemer against Democrat Edwin Edwards and conservative Republican David Duke. In this problem, other candidates are ignored.

Voter Group:	1	2	3	4	5
Preferences:	E	R	R	D	D
	R	E	D	R	E
	D	D	E	E	R
Proportion of voters	34	23	4	30	2

(a) Who would have won a plurality contest?

(b) Who would have been the Borda count winner?

(c) What results would have been reached if we used a plurality with elimination of the candidate with the fewest first place votes?

(d) A traditional primary method would have pitted Roemer against Duke (both being Republicans), with the winner going against Edwards. Who would have won in this system? (A traditional primary method is not used in Louisiana.)

(e) In your opinion, who should have won and why?

11. Consider the following preferences on four competing bills (A, B, C, and D) in the U.S. Senate:

Voter group:	1	2	3
Preferences:	A	B	C
	B	C	D
	C	D	A
	D	A	B
Number of voters:	40	32	28

(a) In your opinion, which bill should be passed? Why?

(b) Create a sequential pairwise agenda that contains all candidates and produces bill D as the winner.

Activity: Conduct a classroom election.

Have everyone in your class rank their preference of Coca Cola, Pepsi and Dr. Pepper.

a. Before looking at the data, decide on an election method to be used.

b. Show the class data in a Saari representation triangle.

c. Analyze the election using your election method.

d. Analyze the election using the other election methods you read about. Are there differences?

Activity: Does my vote count?

For this part of the module you will explore three different situations relatively common during elections. They refer to cases when the voters decide not to cast their votes. Using the Saari representation triangle, you will compare the results for the plurality, Borda count, and pairwise voting. Remember that when we write ABC it means that candidate A is the first choice, B is the second choice, and C is the third choice. These are the three common situations to consider:

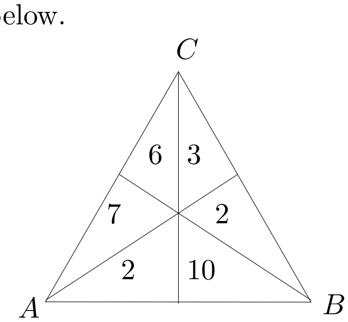
1. Your parents have different preferences to vote for candidates A, B, and C. Both have candidate C as the second choice, but they disagree on the first and last choice. After discussing their preferences they decide to refrain from voting.

2. Your parents disagree on the first and second choices, but agree on the last choice.

3. Your father believes his vote will not affect the result of the election and for that reason he decides not to vote.

SITUATION 0

The preferences of your parents are included in the tally given in the Saari representation triangle below.

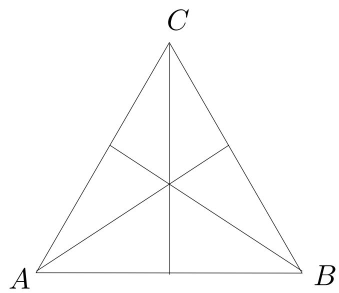


1. Determine the preference of the voters using the plurality vote.
2. Determine the preference of the voters using the Borda count vote.
3. Who would win each of the different pairwise races?

SITUATION 1

Your Mom and Dad have ACB and BCA, respectively, as their preferences. Since they have the same second choice but different first choices, they decided not to vote.

1. Record the new tally in the triangle below:



2. Refer to the triangle to complete the following table:

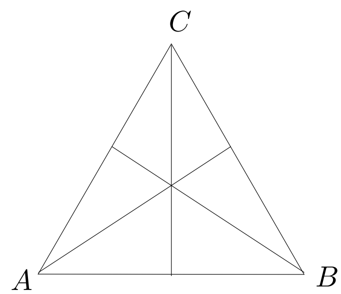
Election Results	Plurality	Borda Count	Pairwise		
			A vs. B	A vs. C	B vs. C
SITUATION 0					
SITUATION 1					

3. Have the voting outcomes changed? Write a short paragraph explaining your answer. Make sure you refer to the triangle for your explanation.
4. Under which voting system does candidate C benefit from your parents not voting?
5. Other couples, friends of your parents, have the same preferences as your parents. If you convince them not to vote, in which of the three systems would candidate C win? How many couples do you need to convince not to vote?

SITUATION 2

Your Mom and Dad now have as preferences ABC and BAC respectively. Once again they feel their votes will not contribute to the final outcome, and therefore they do not vote.

1. Record the new tally in the triangle below:



2. Refer to the triangle to complete the following table:

Election Results	Plurality	Borda Count	Pairwise		
			A vs. B	A vs. C	B vs. C
SITUATION 0					
SITUATION 1					

3. Have the voting outcomes changed? Write a short paragraph explaining your answer. Make sure you refer to the triangle for your explanation.

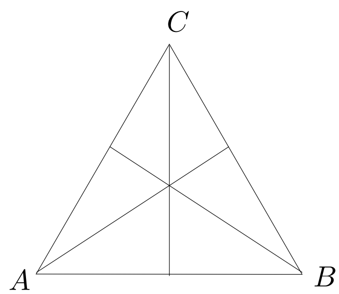
4. Under which voting system does candidate C benefit from your parents not voting? (Plurality, Borda, pairwise)

5. Other couples, friends of your parents, have the same preferences as your parents. If you convince them not to vote, in which of the three systems would candidate C win? How many couples do you need to convince not to vote?

SITUATION 3

Your father does not feel like voting since his candidate is "ahead" and a sure winner. His preference is BAC. He is convinced his vote will not change the final outcome.

1. Record the new tally in the triangle below:



2. Refer to the triangle to complete the following table:

Election Results	Plurality	Borda Count	Pairwise		
			A vs. B	A vs. C	B vs. C
SITUATION 0					
SITUATION 1					

3. Have the voting outcomes changed? Write a short paragraph explaining your answer. Make sure you refer to the triangle for your explanation.

4. Your father has friends with the same preferences as he. If three such friends refrain from voting, candidate B will loose the plurality voting. How many voters can refrain from voting so that he can still win using another voting technique? Explain.

Finally, write a paragraph to summarize your conclusions about the results you just obtained dealing with the three different situations.