DIMACS
Center for Discrete Mathematics & Theoretical Computer Science

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MODULE 09-1
Geometry of Power Indicess
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Module Description Information

- **Title:**

  Geometry of Power Indices

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- **Abstract:**

  This module provides an overview of power indices for weighted voting systems with three voters, from a geometric viewpoint, using 2-dimensional simplices to represent both the domain and range of the power index function. Students learn to work with barycentric coordinates; to partition the domain into regions based on the combination of winning coalitions; and to partition the range into regions based on the relative power rankings of the voters. Power indices are also interpreted in terms of linear combinations of basis vectors so that a large family of power indices falls within the convex hull of the basis vectors projected onto the range. The geometry is then used to explore paradoxes and non-simple weighted games.

- **Informal Description:**

  Section 1 provides an overview of weighted voting systems and power indices. In sections 2 and 3, the power index is thought of as a function on coalitions. Section 2 investigates the geometry of the domain when examining a three person voting system, while section 3 investigates the geometry of the range (i.e. power index of each voter). The concept of a power index can be generalized to include situations where coalitions are not just winning or losing. This is introduced in section 4, where vectors and linear algebra plays an important role in the formulation of a power index in this generalized situation. Section 5 gives an example of a voting system where the ranking of the different players, based on their power, depends on the power index used. Potential student projects projects include investigating the paradoxes associated with power indices and analyzing the geometry of power indices of 4-person voting games. Exercises occur throughout the text, to encourage students to make connections among the new concepts as they are introduced. Selected solutions are included.

- **Target Audience:**

  This module is designed for students taking courses at a wide variety of levels. It could be used in courses such as linear algebra, finite mathematics, discrete mathematics, or any kind of "introduction to proof" course. The module becomes progressively more difficult with each section.

- **Prerequisites:**

  This module assumes a very basic knowledge of 3 dimensional rectangular coordinate system, including plotting points in 3 dimensions, and graphing lines and planes.

- **Mathematical Field:**

  Game Theory, Voting Theory

- **Application Areas:**

  Social Sciences

- **Mathematics Subject Classification:**

  Primary: 91B12, 91A12, Secondary: 91F10

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1 Introduction

A voting system in which voters choose between two alternatives (sometimes called a yes-no voting system) is a weighted voting system if we can specify numerical weights for the voters and a numerical quota, so that a group of voters is a winning coalition exactly when the sum of the weights of the voters in the group is greater than or equal to the quota. In a weighted voting system, the preferences of some voters can carry more weight than the preferences of others. For example, a shareholder who holds 30% of a particular stock has more "voting weight" than one who holds 10% of the stock.

Political scientists often use weighted voting systems to model institutions like the European Union, the U.S. Supreme Court, and the U.S. Senate, and study how the weight of each voter is connected to the voter's ability to influence decisions, that is, the voter's power. In this module we will investigate how to measure the way power is distributed among the voters in a weighted voting system. We will use tools from linear algebra and geometry to generalize measurements of power, and to gain intuition about weighted voting systems.

1.1 A review of weighted voting systems

A *weighted voting system* consists of a set of voters

$$v_1, v_2, \ldots, v_n$$

with weights $w_1, w_2, \ldots w_n$ and a quota q. Each voter casts either a "yes" vote or a "no" vote and the voter's weight shows how many votes he or she has. The quota is the total number of votes needed to pass a measure. Such a system is denoted:

$$[q : w_1, w_2, ..., w_n].$$

We will assume that $w/2 < q \leq w$ where $w = w_1 + \cdots + w_n$ is the total weight of the weighted voting system. For example, suppose a governing body is composed of people from three parties: A, B, C and there are 50 members from party A, 49 members from party B and one member from party C. We'll assume all members of the party vote the same way, and a simple majority is needed to pass a measure. Thus this weighted voting system consists of 3 voters, v_1 (members from party A), v_2 (members from party B), and v_3 (members from party C) and can be represented as:

$$[51 : 50, 49, 1].$$

Any group of voters is called a *coalition*; a coalition is a *winning coalition* if the sum of the weights of the voters in the coalition equals or exceeds the quota. If a coalition is not a winning coalition, then it is a *losing coalition.* A *blocking coalition* is a coalition that is not a winning coalition but has enough votes so that the *complementary* coalition (which is made up of all the voters not in the blocking coalition), is a losing coalition. Thus the members of a blocking coalition can prevent a measure from passing by voting against it. In the weighted voting system of 3 voters v_1, v_2 and v_3 that is represented as

$$[51 : 50, 49, 1],$$

there are three winning coalitions and four losing coalitions. The winning coalitions are $\{v_1, v_2, v_3\}, \{v_1, v_2\}$, and $\{v_1, v_3\}$ and the losing coalitions are $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_2, v_3\}$. The blocking coalitions are $\{v_1\}$ and $\{v_2, v_3\}$. We can also see that the weighted voting system $[3 : 2, 1, 1]$ has the same winning, losing and blocking coalitions as the system $[51 : 50, 49, 1]$.

Another example of a voting system that can be represented as a weighted voting system (although it's not obvious exactly how, just from the description) is the United Nations Security Council. The Council consists of fifteen members; five of the members are permanent (China, France, the Russia Federation, the United Kingdom, and the United States) and ten are elected for two-year periods on a rotating basis. For example, in 2008, the ten non-permanent members were Belgium, Burkino Faso, Costa Rica, Croatia, Indonesia, Italy, Libyan Arab Jamahiriya, Panama, South Africa, and Vietnam. In order to pass, a motion must be approved by all five permanent members and at least four of the non-permanent members. Thus each permanent member has veto power since it can keep a motion from passing by not voting for it. If we consider the first five voters in the system as the five permanent members and the others as the rotating members, this system can be represented as a weighted voting system as follows:

$$[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

The next three exercises explore this representation of the U.N. Security Council as a weighted voting system.

Exercise 1.1 *Describe all the winning coalitions of the United Nations Security Council weighted voting system. How can you tell that all the permanent members have veto power?*

Exercise 1.2 *Give an example of a blocking coalition in the United Nations Security Council.*

Exercise 1.3 *What is the minimum number of votes a coalition must have to be a blocking coalition in the United Nations Security Council?*

1.2 The Banzhaf Score and the Shapley-Shubik Score

Suppose five friends are trying to decide which of two restaurants to go to for lunch, each friend has one vote and majority rules. This situation can be described using the weighted voting system $[3 : 1, 1, 1, 1, 1]$ since at least three votes out of five are required to win. In this case, it makes sense that each friend has the same "power" to influence the decision. But what if one friend has three votes, (maybe because she holds the car keys)? Then at least four votes out of seven are required to win, changing the weighted voting system to $[4 : 3, 1, 1, 1, 1]$. Does the friend who has three times as many votes as the others have three times the "power" of the others as well? We'll look at two, specific, well-known power indices, the *Banzhaf power index* and the *Shapley-Shubik power index*, and then consider power indices more generally.

The Banzhaf power index was introduced by attorney John F. Banzhaf III in 1965 in conjunction with a law suit in Nassau County, New York [1]. The *Banzhaf power index*, denoted BPI, of an individual voter is based on that voter's Banzhaf score. The *Banzhaf score* is equal to the number of winning coalitions to which that voter belongs but which are no longer winning coalitions if the voter leaves. In such cases, the voter is called a *critical voter* in that coalition. The Banzhaf power index is obtained from the Banzhaf score and will be defined in Section 1.4 below.

Example 1.4 *Let's compute the Banzhaf score for each voter in the weighted voting system*

$$[3 : 2, 1, 1].$$

We'll call the three voters v_1, v_2, and v_3. We first list all the winning coalitions:

$$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}.$$

Then for each voter, we count the number of winning coalitions to which the voter belongs that become losing coalitions if the voter leaves:

Voter	*Winning Coalitions that become losing if voter leaves*	*Banzhaf Score*
v_1	$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}$	3
v_2	$\{v_1, v_2\}$	1
v_3	$\{v_1, v_3\}$	1

We denote the Banzhaf scores of this simple weighted voting system as $(3, 1, 1)$.

Exercise 1.5 *Compute the Banzhaf scores for the weighted voting systems*

$$[51 : 50, 49, 1] \text{ and } [6 : 4, 3, 2].$$

How do the Banzhaf scores for these systems compare with those for $[3 : 2, 1, 1]$*?*

To compute the *Shapley-Shubik power index* for each voter, we begin by listing all of the possible sequential coalitions of all voters. That is, we list all possible orderings of all voters. (If there are n voters then there are $n!$ sequential coalitions of all n voters.) We think of the coalition forming as we move from left to right in a particular sequential ordering. A *pivotal voter* for a sequential ordering is the voter who changes the coalition from a losing coalition to a winning one as we move from left to right in the sequential ordering. The *Shapley-Shubik score* for a particular voter is the number of sequential orderings for which the voter is pivotal. The Shapley-Shubik power index will be defined in Section 1.4 below.

Example 1.6 *Let's find the Shapley-Shubik score for each voter in the weighted voting system*

$$[3 : 2, 1, 1].$$

We first list all possible orderings of the three voters $v_1, v_2,$ *and* v_3 :

$$v_1v_2v_3,\ v_1v_3v_2,\ v_2v_1v_3,\ v_2v_3v_1,\ v_3v_1v_2,\ v_3v_2v_1.$$

Now in each sequential ordering, we start tallying the votes from left to right; we underline the voter in each ordering who tips the total number of votes to meet or exceed the quota.

$$\begin{array}{ccc} v_1 & \underline{v_2} & v_3 \\ v_1 & \underline{v_3} & v_2 \\ v_2 & \underline{v_1} & v_3 \\ v_2 & v_3 & \underline{v_1} \\ v_3 & \underline{v_1} & v_2 \\ v_3 & v_2 & \underline{v_1} \end{array}$$

Finally, we count the number of sequential orderings for which each voter is pivotal.

Voter	*Shapley-Shubik Score*
v_1	4
v_2	1
v_3	1

We denote the Shapley-Shubik scores of this simple weighted voting system as $(4, 1, 1)$.

Exercise 1.7 *Compute the Shapley-Shubik score of each voter for the weighted voting systems*

$$[51 : 50, 49, 1] \ \ \textit{and} \ \ [6 : 4, 3, 2].$$

How do they compare with the Shapley-Shubik scores for the $[3 : 2, 1, 1]$ *system in Example 1.6?*

Exercise 1.8 *Compute the Banzhaf and Shapley-Shubik scores of each voter for the weighted voting system*

$$[3 : 2, 2, 1].$$

How do they compare with the Banzhaf and Shapley-Shubik scores for the other examples and exercises?

Exercise 1.9 *The U.S. Supreme Court is often analyzed in terms of its conservative and liberal blocs. In 2009, the conservative bloc consisted of Chief Justice Renquist and Justices Alito, Scalia and Thomas. The liberal bloc consisted of Justices Breyer, Ginsberg, Stevens and Sotomayor. Justice Kennedy was considered a swing voter who voted sometimes with the conservative and sometimes with the liberal justices. Find the Banzhaf and Shapley-Shubik scores for Justice Kennedy, assuming the supreme court can be modeled as a* $[5 : 4, 4, 1]$ *weighted voting system.*

Exercise 1.10 *Compute the Banzhaf score of each voter for the weighted voting system that represents the five friends who want to decide on a place for lunch, with each friend having one vote and majority rules:*

$$[3 : 1, 1, 1, 1, 1].$$

Then compute the Banzhaf score of each voter for the system where one friend has three votes:

$$[4 : 3, 1, 1, 1, 1].$$

How do they compare?

Exercise 1.11 *Compute the Banzhaf and Shapley-Shubik scores for each voter in each of the following systems.*

1. $[4 : 2, 2, 1]$
2. $[5 : 2, 5, 2]$

3. $[2 : 1, 1, 1]$

4. $[7 : 1, 6, 5]$

What do you notice about the scores?

Exercise 1.12 *Experiment with several three-voter weighted voting systems, and find an example in which the weights are kept the same, but an increase in quota results in no change in the Banzhaf scores. Find another example in which the weights are kept the same, but an increase in the quota results in a change in the Banzhaf scores.*

A voter in a weighted voting system is a *dictator* if the coalition comprised of that voter alone is a winning coalition. A voter in a weighted voting system is a *dummy* if the voter is never a critical voter in a winning coalition and is never a critical voter in any blocking coalition.

Exercise 1.13 *What is the Shapley-Shubik score of a dummy voter in an $n-$voter system? What is the Shapley-Shubik score of a dictator in an $n-$voter system?*

Exercise 1.14 *Construct two different weighted voting systems so that two of the voters are dummy voters and one voter is a dictator.*

Exercise 1.15 *Construct an example of a weighted voting system, different from all of those above, such that the Banzhaf scores of voters v_2 and v_3 are the same, yet the Banzhaf score of voter v_1 is less than the Banzhaf scores of voters v_2 and v_3.*

1.3 Equivalent weighted voting systems

Two weighted voting systems with the same number of voters are *equivalent* if it is possible to rename the voters of one system with the names of the voters in the other system so that winning coalitions in the two systems are the same.

Example 1.16 *The weighted voting systems*

$$[51 : 50, 49, 1] \quad [6 : 4, 3, 2] \quad \textit{and} \quad [6 : 3, 2, 4]$$

are equivalent. The first two voting systems are equivalent because they both have winning coalitions

$$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1.v_2, v_3\}.$$

The third voting system is also equivalent since its voters 1, 2 and 3 can be renamed to correspond to the voters 2, 3 and 1 in the other weighted voting systems, in which case its winning coalitions will be the same.

Exercise 1.17 *Give another example of two different weighted voting systems that are equivalent.*

Exercise 1.18 *Is it possible to have the same winning coalitions but different losing coalitions in two different weighted voting systems?*

Exercise 1.19 *If two weighted voting systems are equivalent, must they have the same Banzhaf scores? How about the Shapley-Shubik scores?*

Exercise 1.20 *There are five non-equivalent three-voter weighted voting systems. All other three-voter weighted voting systems are equivalent to one of these five. List an example of each of the five, along with the Banzhaf and Shapley-Shubik scores of each. (The exercises above should provide plenty of examples.)*

1.4 The Banzhaf Power Index and The Shapley-Shubik Score

In this section, we look at how to convert Banzhaf and Shapley-Shubik scores to power indices. It is sometimes convenient to think of power indices as fractions of a whole. After all, the power of an individual voter is in some sense a measure of the fraction of total power that the voter possesses. We can obtain the *Banzhaf power index* (BPI) of a voter in a weighted voting system from the voter's Banzhaf score, and the *Shapley-Shubik power index* (SSPI) from the Shapley-Shubik score, by *normalizing*. To normalize, we divide each voter's score by the sum of all the voters' scores in the weighted voting system, as the following example illustrates.

Example 1.21 *Let's normalize the Banzhaf scores and the Shapley-Shubik scores to find the Banzhaf power indices and the Shapley-Shubik power indices for the simple weighted voting system*

$$[3 : 2, 1, 1].$$

Recall that we computed the Shapley-Shubik scores as $(4, 1, 1)$*. The sum of the scores is* $4+1+1 = 6$*. To normalize this score we simply divide each score by 6, obtaining the Shapley-Shubik power indices*

$$\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right).$$

Note that the sum of the Shapley-Shubik power indices of the three voters is 1. To normalize the Banzhaf scores of $(3, 1, 1)$*, we again divide each score by the sum of all scores, which is* $3+1+1 = 5$*, and obtain the Banzhaf power indices* $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$*.*

Exercise 1.22 *Normalize the Banzhaf scores in Exercise 1.10 above to find the Banzhaf power indices.*

While it is true that equivalent voting systems have the same set of power indices, the reverse is not true. That is, there are weighted voting systems which have the same power indices but are not equivalent.

Exercise 1.23 *Find two weighted voting systems with the same power indices that are not equivalent.*

Clearly, when the Banzhaf or Shapley-Shubik scores are normalized, some information is lost. However, there are a number of advantages to normalizing beyond that of identifying each voter's share of power. The most important of these is the ease with which weighted voting systems and their power indices can be visualized on a graph. This is the subject of the next several sections.

Exercise 1.24 *Use Exercise 1.20 to argue that in a weighted voting system with three voters, at least two of the voters must have the same BPI.*

2 The Geometry of Weighted Voting Systems for Three Voters

2.1 Graphing Weight Distributions in Weighted Voting Systems

In order to fully understand the relationship between weighted voting systems and power indices, it is helpful to develop a visual picture of each step in the process. Let's start with the distribution of weights itself. Suppose we have a three-person weighted voting system consisting of voters v_1, v_2, and v_3, with weights w_1, w_2, and w_3, and that the total weight of the system is fixed, say at 30, so that

$$w_1 + w_2 + w_3 = 30.$$

Just as we normalized the power indices in the previous section, we can normalize the weight distribution by dividing the equation by thirty to get

$$\tilde{w}_1 + \tilde{w}_2 + \tilde{w}_3 = 1,$$

where $\tilde{w}_1 = w_1/30$ is the fraction of the total weight held by v_1, and similarly for $\tilde{w}_2$ and $\tilde{w}_3$.

We will use the symbols $\tilde{w}_1, \tilde{w}_2$ and $\tilde{w}_3$ to represent the normalized weight distribution in everything that follows. Bear in mind, however, that the formula for obtaining these values will vary depending on the total weight of the weighted voting system. We can interpret this equation as representing a plane in $\mathbb{R}^3$, with $\tilde{w}_1, \tilde{w}_2$ and $\tilde{w}_3$ corresponding to the x, y and z axes respectively. Moreover, since each fraction lies between zero and one, the graph is restricted to the triangular portion of the plane lying in the positive octant, known as the unit simplex. Thus, each weight distribution can be represented by a specific point on the simplex. For example, the weighted voting system

$$[18 : 15, 10, 5]$$

has a total weight of thirty. Thus its weight distribution can be represented by the point

$$\left(\frac{15}{30}, \frac{10}{30}, \frac{5}{30}\right) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right).$$

Note that different weight distributions will correspond to the same point on the simplex if their normalized forms contain equivalent fractions. Thus each point on the simplex shown in Figure 1 represents an infinite number of different weight distributions.

To make visualizing and plotting points easier, let's redraw the simplex in two dimensions so that it appears as an equilateral triangle (See Figure 2.).

Vertex A has coordinates $(1, 0, 0)$ and corresponds to the weighted voting system when voter v_1 has all the weight. Vertices B and C have coordinates $(0, 1, 0)$ and $(0, 0, 1)$, respectively, and correspond to weighted voting systems when voter v_2 and voter v_3, respectively, have all the weight. Notice the placement of the point

$$P = \left(\frac{15}{30}, \frac{10}{30}, \frac{5}{30}\right) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$$

in the triangle. Since v_1 has the largest share of the votes, P is closest to vertex A. Similarly, since v_3 has the smallest share of the votes, P is farthest from vertex C. This illustrates a general principle when graphing weighted voting systems on the unit simplex: the larger the coordinate, the closer the vertex.

Another useful principle in graphing points is that when two coordinates are equal, the point must lie on the perpendicular bisector between the two corresponding vertices. (Recall that a perpendicular bisector in an equilateral triangle runs from a vertex to the opposite side, lying equidistant between the two remaining vertices.) These principles can be used to find the approximate locations of points before locating them exactly.

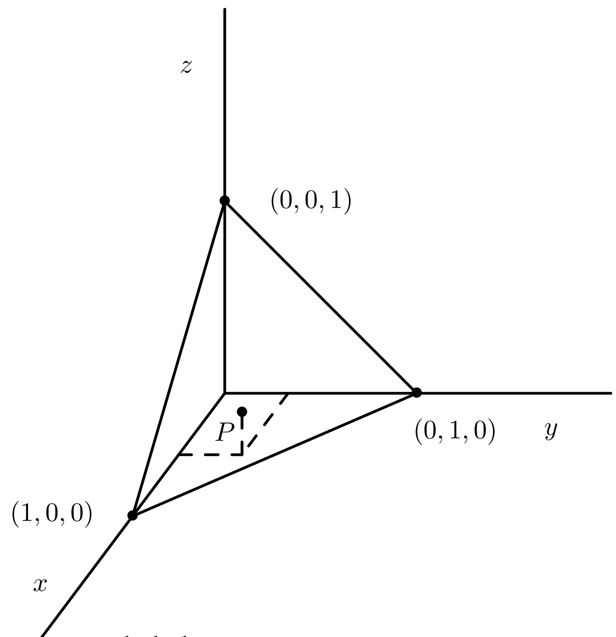


Figure 1: Point $P = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ plotted on the unit simplex $x + y + z = 1$.

Example 2.1 *Find the approximate placement of the point corresponding to the weighted voting system* $[15 : 12, 12, 6]$.

The normalized weight distribution is

$$\left(\frac{12}{30}, \frac{12}{30}, \frac{6}{30}\right).$$

Since $\tilde{w}_1 = \tilde{w}_2$, *P lies on the perpendicular bisector between A and B. Since,* $\tilde{w}_3$ *is less than either* $\tilde{w}_1$ *or* $\tilde{w}_2$, *P lies closer to side AB than to vertex C. See Figure 3.*

Example 2.2 *Find the approximate placement of the point corresponding to the weighted voting system* $[6 : 0, 4, 4]$.

The normalized distribution of weights is

$$\left(\frac{0}{8}, \frac{4}{8}, \frac{4}{8}\right).$$

Since $\tilde{w}_2 = \tilde{w}_3$, *P lies on the perpendicular bisector between B and C. In this case, the remaining coordinate* $\tilde{w}_1 = 0$ *places P at the farthest point from A on the line BC. See Figure 3.*

Exercise 2.3 *Find the approximate placement of the points corresponding to the following weighted voting systems*

1. $[14 : 6, 8, 6]$
2. $[3 : 4, 0, 0]$
3. $[3 : 2, 2, 0]$

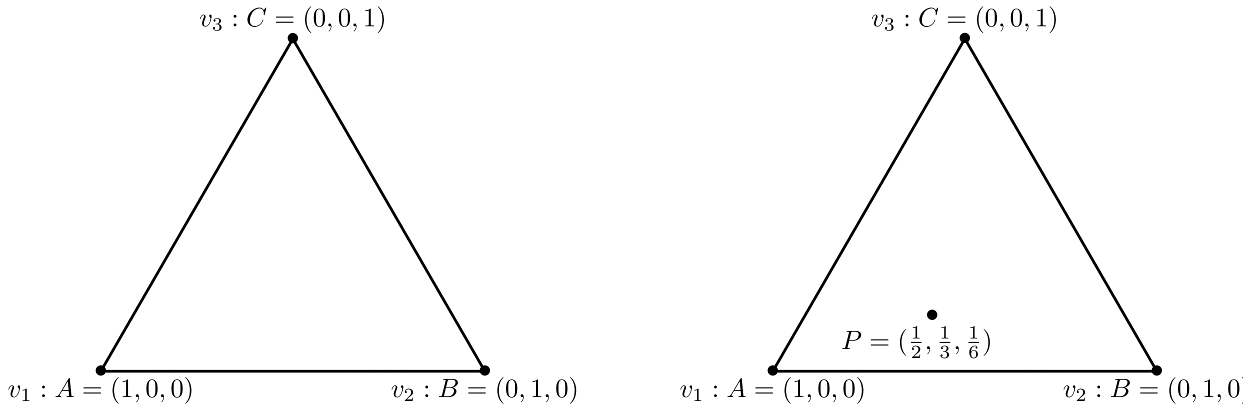


Figure 2: The simplex drawn as an equilateral triangle. Point $P = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ plotted on the triangle. Notice that P is closest to vertex A and farthest from vertex C.

4. $[3 : 1, 1, 1]$

Exercise 2.4 *What property must P satisfy for it to lie on one of the sides of the triangle?*

Exercise 2.5 *What property must P satisfy for it to lie at one of the vertices of the triangle?*

The linearity of the plane underlying the simplex can be used to determine the exact position of points on the simplex. Recall that in Example 2.1, $P = \left(\frac{12}{30}, \frac{12}{30}, \frac{6}{30}\right)$ lies on the perpendicular bisector between A and B, closer to the AB line than to vertex C. For reference, consider the coordinates of the bisector's endpoints,

$$Q = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \text{ and } C = (0,0,1),$$

as shown in Figure 4.

As we trace a line upwards from Q to C, the third coordinates increase while the first two coordinates decrease. Moreover, the distance from Q increases *linearly* as the third coordinate increases. Thus since $\frac{6}{30}$ is numerically $\frac{1}{5}$ of the way from 0 to 1, point P lies $\frac{1}{5}$ of the way from Q to C.

If no two coordinates are the same, it is easiest to graph the point as the intersection between two lines.

Example 2.6 *Find the exact location of $P = \left(\frac{4}{15}, \frac{6}{15}, \frac{5}{15}\right)$.*

We start by graphing the line $\tilde{w}_1 = \frac{4}{15}$. Since $\tilde{w}_1$ is fixed, it will lie parallel to the side of the triangle where $\tilde{w}_1$ is also fixed. i.e., where $\tilde{w}_1 = 0$. From Exercise 2.4, we know that points along BC have $v_1 = 0$ and thus satisfy $\tilde{w}_1 = 0$. Thus the line $\tilde{w}_1 = \frac{4}{15}$ is parallel to the side BC, and intersects AC and AB at points Q_1 and Q_2 as shown in Figure 5.

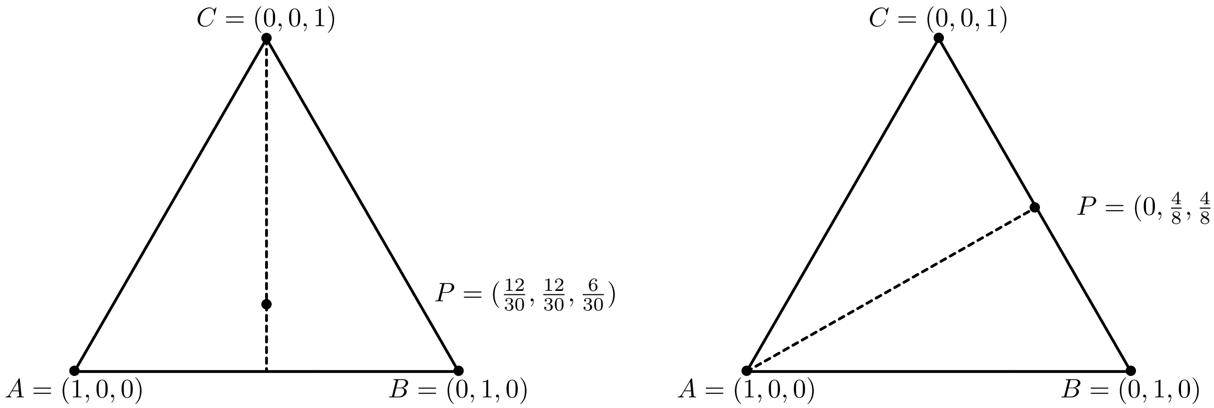


Figure 3: In the figure on the left, P is on the perpendicular bisector between A and B. It lies closer to side AB than vertex C. In the figure on the right, P is on the perpendicular bisector between B and C. Since $\tilde{w}_1 = 0$, P lies on side BC.

Both Q_1 and Q_2 have one coordinate equal to $\frac{4}{15}$, a second coordinate equal to 0 and a third coordinate equal to 11/15 (since they must sum to 1). Thus

$$Q_1 = \left(\frac{4}{15}, 0, \frac{11}{15}\right) \quad \textit{and} \quad Q_2 = \left(\frac{4}{15}, \frac{11}{15}, 0\right).$$

This allows us to locate the line $\tilde{w}_1 = \frac{4}{15}$ exactly. In the same way, we can graph the line $\tilde{w}_2 = \frac{6}{15}$. The point P now lies at their intersection. As a check, note that the P lies closest to B and farthest from A, corresponding to the inequalities $\tilde{w}_2 > \tilde{w}_3 > \tilde{w}_1$.

Although we used the lines $\tilde{w}_1 = \frac{4}{15}$ and $\tilde{w}_2 = \frac{6}{15}$ to plot P, we could equally have used $\tilde{w}_1 = \frac{4}{15}$ and $\tilde{w}_3 = \frac{5}{15}$, or $\tilde{w}_2 = \frac{6}{15}$ and $\tilde{w}_3 = \frac{5}{15}$.

Exercise 2.7 *Locate the following points using the method outlined above.*

1. $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2. $(\frac{3}{11}, \frac{4}{11}, \frac{4}{11})$
3. $(\frac{1}{2}, \frac{1}{5}, \frac{3}{10})$

2.2 Graphing Weighted Voting Systems

Now that we have a visual tool for representing weight distributions in the unit simplex, we can use the geometry to analyze weighted voting systems. Jones [9] and Haines and Jones [8] broke the simplex into regions based on the kinds of winning coalitions that could form. We will examine these ideas as they will help us in computing the Banzhaf power index for different weighted voting systems

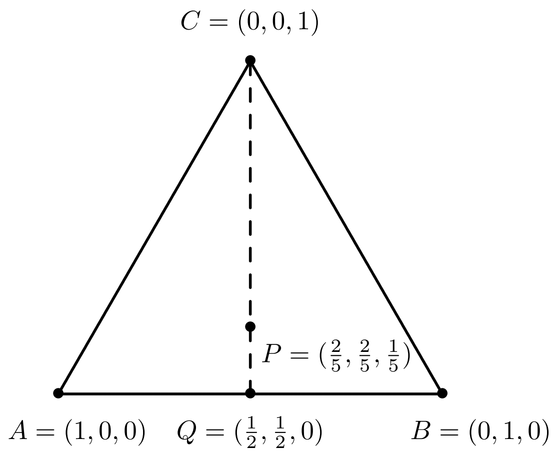


Figure 4: P lies $\frac{1}{5}$ of the way from Q to C.

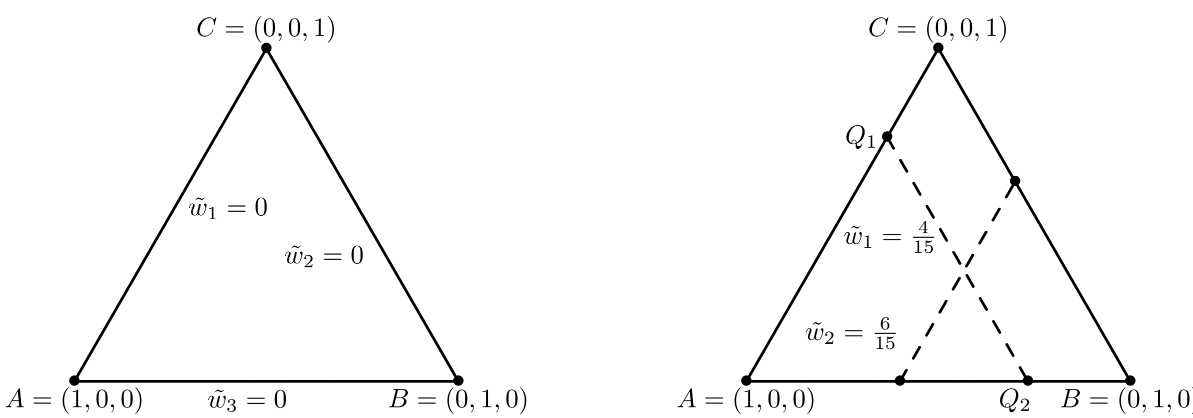


Figure 5: Example 2.6

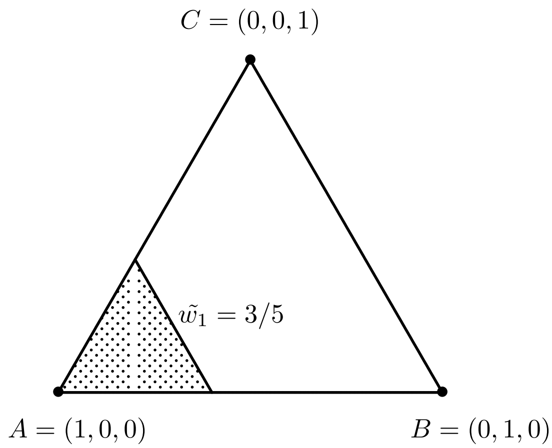


Figure 6: v_1 is the dictator for any weight distribution lying in the shaded region.

Example 2.8 *Consider a weighted voting system of three voters with a total weight of twenty and a quota of twelve. Under what weight distributions would v_1 be a dictator?*

The quota represents $\frac{12}{20} = \frac{3}{5}$ of the total weight; thus v_1 would be a dictator if she holds $\frac{3}{5}$ or more of the total weight. On the simplex, this represents the region on the triangle corresponding to the inequality $\tilde{w}_1 \geq \frac{3}{5}$ as in Figure 6. Thus any weighted voting system whose weight distribution lies in this region would include v_1 as dictator.

Example 2.9 *Consider again a weighted voting system whose quota represents $\frac{3}{5}$ of the total weight. Under what weight distributions would the winning coalitions be*

$$\{v_1, v_2\}, \{v_1, v_3\} \text{ and } \{v_1, v_2, v_3\}?$$

Since no voter represents a winning coalition on their own, $\tilde{w}_1 < \frac{3}{5}, \tilde{w}_2 < \frac{3}{5}$ and $\tilde{w}_3 < \frac{3}{5}$. Also, the fact that $\{v_2, v_3\}$ is not a winning coalition implies that $\tilde{w}_2 + \tilde{w}_3 < \frac{3}{5}$. Algebraically, this is equivalent to $\tilde{w}_1 \geq \frac{2}{5}$ (substitute $1 - \tilde{w}_1$ for $\tilde{w}_2 + \tilde{w}_3$ and rearrange the inequality).

Similarly, the inequalities $\tilde{w}_1 + \tilde{w}_2 \geq \frac{3}{5}$ and $\tilde{w}_1 + \tilde{w}_3 \geq \frac{3}{5}$ imply $\tilde{w}_3 < \frac{2}{5}$ and $\tilde{w}_2 < \frac{2}{5}$, respectively. Putting these facts together, we obtain $\frac{2}{5} \leq \tilde{w}_1 < \frac{3}{5}, \tilde{w}_2 < \frac{2}{5}$ and $\tilde{w}_3 < \frac{2}{5}$. The intersection of these inequalities is shown in Figure 7. Thus any weighted voting system whose weight distribution lies in this region has the required winning coalitions.

As the previous examples indicate, it is possible to decompose the triangle into different regions corresponding to different combinations of winning coalitions.

To explore this idea, let's continue to assume that the quota represents $\frac{3}{5}$ of the total weight. The lines corresponding to $\tilde{w}_1 = \frac{3}{5}, \tilde{w}_1 = \frac{2}{5}, \tilde{w}_2 = \frac{3}{5}, \tilde{w}_2 = \frac{2}{5}$, and $\tilde{w}_3 = \frac{3}{5}, \tilde{w}_3 = \frac{2}{5}$ are indicated in Figure 8.

Let's examine the region to the left of the $\tilde{w}_1 = \frac{3}{5}$ line (labeled R_1 in Figure 8). In this region, $\tilde{w}_1 > \frac{3}{5}, \tilde{w}_2 < \frac{2}{5}$, and $\tilde{w}_3 < \frac{2}{5}$. Hence $\{v_1\}$ is a winning coalition but neither $\{v_2\}$ nor $\{v_3\}$ are. Furthermore, since v_1 by itself is a winning coalition, so are $\{v_1, v_2\}$ and $\{v_1, v_3\}$. Finally, the inequality $\tilde{w}_1 > \frac{3}{5}$ is equivalent to $\tilde{w}_2 + \tilde{w}_3 \leq \frac{2}{5}$. Thus $\{v_2, v_3\}$ cannot be a winning coalition. To

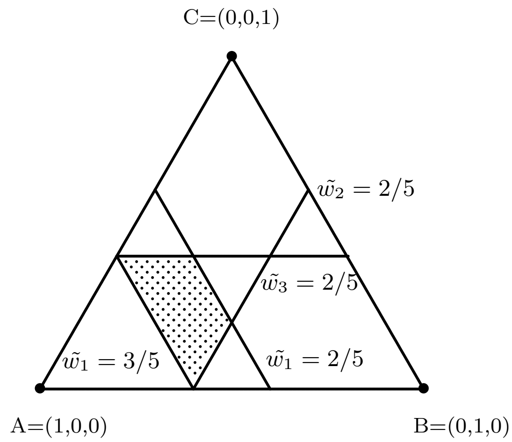


Figure 7: $\{v_1, v_2\}, \{v_1, v_3\}$ and $\{v_1, v_2, v_3\}$ are the only winning coalitions in the shaded region. Note that since $\tilde{w}_2 < \frac{2}{5}$, the shaded region must line on the side of the line $\tilde{w}_2 = \frac{2}{5}$ that is *away* from B.

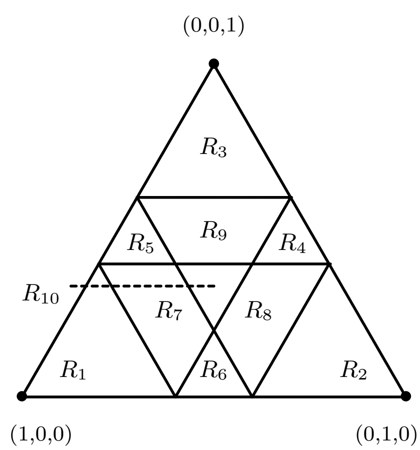


Figure 8: The graphs of the lines $\tilde{w}_1 = \frac{2}{5}, \tilde{w}_1 = \frac{3}{5}, \tilde{w}_2 = \frac{2}{5}, \tilde{w}_2 = \frac{3}{5}, \tilde{w}_2 = \frac{3}{5}, \tilde{w}_3 = \frac{2}{5}, \tilde{w}_3 = \frac{3}{5}$. These lines break the triangle into 10 regions, labeled as shown.

sum up, in R_1, the winning coalitions are: $\{v_1\}, \{v_1, v_2\}, \{v_1, v_3\}$, and $\{v_1, v_2, v_3\}$. Note that v_1 is a dictator in R_1.

By symmetry, regions R_2 and R_3 correspond to weighted voting systems where v_2 and v_3 are dictators, respectively.

Now let's look at the region R_4. Here, $\tilde{w}_1 < \frac{2}{5}$, and $\frac{2}{5} < \tilde{w}_2 < \frac{3}{5}$, and $\frac{2}{5} < \tilde{w}_3 < \frac{3}{5}$. Clearly, no individual voter can form a winning coalition. However the inequality $\tilde{w}_1 < \frac{2}{5}$ is equivalent to $\tilde{w}_2 + \tilde{w}_3 \geq \frac{3}{5}$; thus $\{v_2, v_3\}$ is a winning coalition. Similarly, $\{v_1, v_2\}$ and $\{v_1, v_3\}$ are not winning coalitions. Thus the only winning coalitions in R_4 are $\{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$. Note that v_1 is a dummy in R_4.

Again by symmetry, regions R_5 and R_6 correspond to weighted voting systems where v_2 and v_3 are dummies, respectively.

Continuing our analysis, in R_7, we have $\frac{2}{5} < \tilde{w}_1 < \frac{3}{5}, \tilde{w}_2 < \frac{2}{5}$, and $\tilde{w}_3 < \frac{2}{5}$. As before, $\tilde{w}_2 < \frac{2}{5}$ is equivalent to $\tilde{w}_1 + \tilde{w}_3 \geq \frac{3}{5}$ and $\tilde{w}_3 < \frac{2}{5}$ is equivalent to $\tilde{w}_1 + \tilde{w}_2 \geq \frac{3}{5}$. Thus $\{v_1, v_3\}$ and $\{v_1, v_2\}$ are winning coalitions, while $\{v_2, v_3\}$ is not. Thus the winning coalitions in R_7 are $\{v_1, v_2\}, \{v_1, v_3\}$, and $\{v_1, v_2, v_3\}$. Note that v_1 has veto power in R_7. (Recall that a voter has veto power if her inclusion in a coalition is necessary for the coalition to be winning.)

By symmetry, R_8 and R_9 correspond to weighted voting systems where v_2 and v_3 have veto power, respectively.

Finally, in $R_{10}, \tilde{w}_1 < \frac{2}{5}, \tilde{w}_2 < \frac{2}{5}$, and $\tilde{w}_3 < \frac{2}{5}$. These are equivalent to $\tilde{w}_2 + \tilde{w}_3 \geq \frac{3}{5}, \tilde{w}_1 + \tilde{w}_3 \geq \frac{3}{5}$, and $\tilde{w}_1 + \tilde{w}_2 \geq \frac{3}{5}$ respectively. Thus the winning coalitions in R_{10} are $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$, and $\{v_1, v_2, v_3\}$. This region might be characterized as "majority rules."

Table 1 summarizes these findings. The third column in the table lists only the minimal winning coalitions in each region. A *minimal winning coalition* is a coalition in which all of the voters are critical. For instance, in R_7, notice that in the winning coalition $\{v_1, v_2, v_3\}$, neither voter v_2 nor v_3 is critical; however all players are critical in the winning coalitions $\{v_1, v_2\}$ and $\{v_1, v_3\}$. Thus $\{v_1, v_2\}$ and $\{v_1, v_3\}$ are the only minimal winning coalitions in R_7. Listing only minimal winning coalitions is a more concise way of describing a weighted voting system since any winning coalition can be obtained by adding additional voters to the minimal winning coalitions. These results are also summarized in Table 2. Note that in column 2, the notation $N \setminus \{v_i\}$ refers to the set of all players except v_i. Hence a weighted voting system in R_4 has minimal winning coalition $\{v_2, v_3\}$.

Having the list of minimal winning coalitions in each region makes it easy to find the power indices of each player in these regions.

Exercise 2.10 *Suppose the quota is changed to correspond to $\frac{2}{3}$ of the total weight. Graph the corresponding lines and determine the winning coalitions in each region. What happens to R_{10}?*

Exercise 2.11 *Repeat Exercise 2.10 for a quota of $\frac{3}{4}$ of the total weight. Is the description "majority rules" still appropriate for R_{10}?*

Exercise 2.12 *Using the leftmost triangle in Figure 9, compute the BPI for a weighted voting system in each of the 10 regions. You may want to refer to exercise 1.20.*

The previous examples demonstrate the range of possible scenarios for weighted voting systems of three voters. There are three cases, determined by the ratio of the quota to the total weight of the voting system:

$$\frac{1}{2} < \frac{q}{w} < \frac{2}{3}, \quad \frac{q}{w} = \frac{2}{3}, \quad \text{and} \quad \frac{q}{w} > \frac{2}{3}$$

Region	Winning Coaltions	Minimal Winning Coalitions
R_1	$\{v_1\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}$	$\{v_1\}$
R_2	$\{v_2\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$	$\{v_2\}$
R_3	$\{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$	$\{v_3\}$
R_4	$\{v_2, v_3\}, \{v_1, v_2, v_3\}$	$\{v_2, v_3\}$
R_5	$\{v_1, v_3\}, \{v_1, v_2, v_3\}$	$\{v_1, v_3\}$
R_6	$\{v_1, v_2\}, \{v_1, v_2, v_3\}$	$\{v_1, v_2\}$
R_7	$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\}$	$\{v_1, v_2\}, \{v_1, v_3\}$
R_8	$\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$	$\{v_1, v_2\}, \{v_2, v_3\}$
R_9	$\{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$	$\{v_1, v_3\}, \{v_2, v_3\}$
R_{10}	$\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}$	$\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$

Table 1: List of winning coalitions by region for weighted voting systems with $q = 3/5$.

as shown in Figure 9.

Region	R_i for $i = 1$ to 3	R_{i+3} for $i = 1$ to 3	R_{i+6} for $i = 1$ to 3	R_{10}
MWCs	$\{v_i\}$	$N \backslash \{v_i\}$	$\{v_i, v_j\}, \{v_i, v_k\}$ where $i \neq j \neq k$	$\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}^*$ or $\{v_1, v_2, v_3\}^{**}$

Table 2: Regions and their corresponding minimal winning coalitions (MWCs). The coalition structure for R_{10} depends on whether $q \leq \frac{2}{3}^*$ or $q > \frac{2}{3}^{**}$.

3 The Simplectic Geometry of the Range for $n = 3$

Power indices such as the BPI and SSPI can be thought of as functions. The domain of the power index is the set of all normalized weighted voting systems. In the prior section, we analyzed the geometry of the domain. Since a normalized power index for three voters gives an ordered triplet whose coordinates sum to one, the range of the power index function is the set of ordered triples whose values sum to one. These values can be represented on the simplex as well. In Exercise 2.12

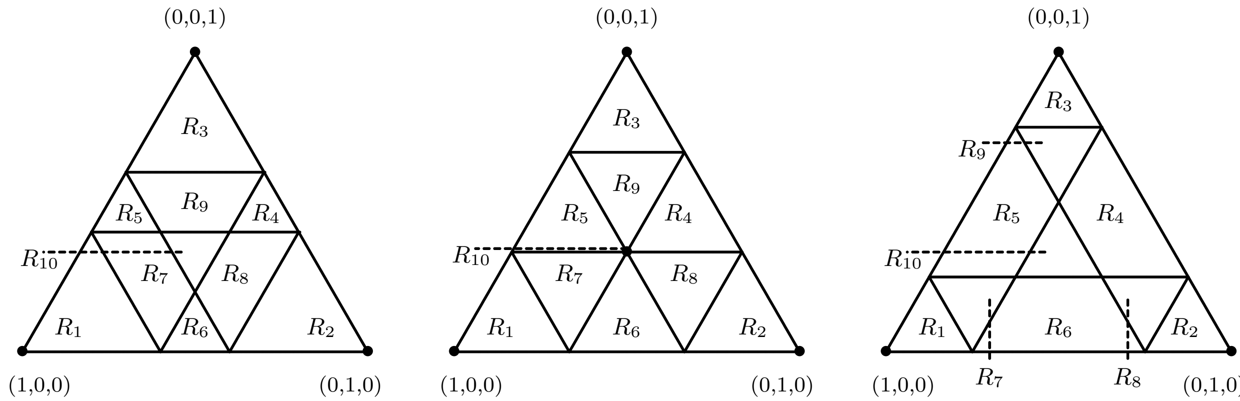


Figure 9: Shape of regions for $\frac{1}{2} < \frac{q}{w} < \frac{2}{3}$ (left), $\frac{q}{w} = \frac{2}{3}$ (middle), and $\frac{2}{3} < \frac{q}{w} < 1$ (right).

you used the geometry of the domain to analyze power indices. Now we will explore this concept of power indices using the geometry of the range.

Example 3.1 *Let's again consider the weighted voting system*

$$[3 : 2, 1, 1].$$

We have computed both the BPI and the SSPI of this weighted voting system. We calculated the Banzhaf score as $(3, 1, 1)$ *and the Shapley-Shubik score as* $(4, 1, 1)$. *We saw that normalizing these indices, gave us BPI* $= (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ *and SSPI* $= (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. *Figure 10 shows the geometric representation of each of these indices.*

Exercise 3.2 *What do you notice about the BPI and SSPI for this simple weighted voting game?*

Exercise 3.3 *Represent the BPI and SSPI for the weighted voting system*

$$[3 : 2, 2, 1]$$

on the simplex as in example 3.1.

With a power index, the essential information is the relative values of the power distribution. i.e., the rankings of who holds the most (or least) power. If we have three voters, v_1, v_2, and v_3 with power indices p_1, p_2, and p_3 then there are 6 possible different strict rankings of power:

$$\begin{aligned} p_1 &> p_2 &> p_3 \\ p_2 &> p_3 &> p_1 \\ p_3 &> p_1 &> p_2 \\ p_2 &> p_1 &> p_3 \\ p_1 &> p_3 &> p_2 \\ p_3 &> p_2 &> p_1. \end{aligned}$$

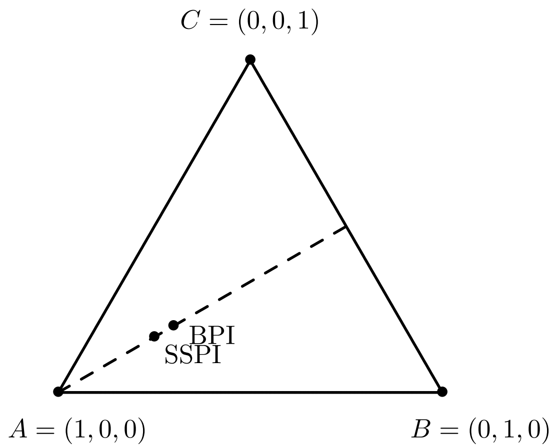


Figure 10: Representation of the BPI and SSPI for the weighted voting system $[3 : 2, 1, 1]$.

The *indifference lines*, where the power rankings are not strict

$$p_1 = p_2, \quad p_1 = p_3, \quad p_2 = p_3,$$

divide the triangle into 6 *ranking regions*. In any region, the power rankings among voters remains constant, no matter what power index is used to measure power.

Exercise 3.4 *Label the indifference lines*

$$p_1 = p_2, \quad p_1 = p_3, \quad p_2 = p_3$$

on the triangle in Figure 11.

Exercise 3.5 *In Q_1, the power ranking is $p_1 > p_2 > p_3$ because points in this region are closest to A and farthest from C. Determine the power rankings for the other 5 regions.*

Exercise 3.6 *Previously, you computed the BPI and SSPI for the weighted voting systems*

1. $[4 : 2, 2, 1]$
2. $[5 : 2, 5, 2]$
3. $[2 : 1, 1, 1]$

In which region do these indices fall?

Exercise 3.7 *What do you think happens to the power of an individual voter as the power index moves towards a vertex in the triangle?*

Exercise 3.8 *What does the center point (where the three indifference lines intersect) of the triangle represent in terms of the powers of the three voters?*

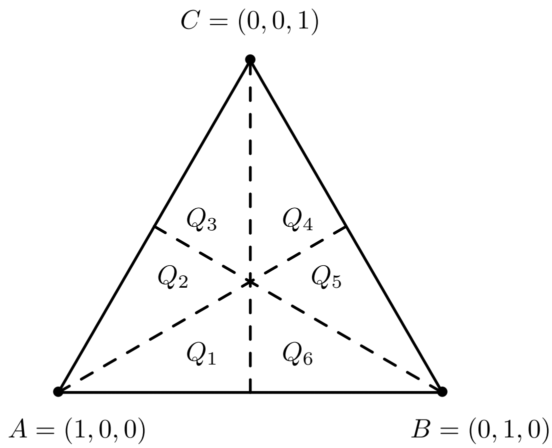


Figure 11: Power Ranking Regions Representation

4 General Power Indices and Scores

4.1 The general definition

The Banzhaf power index and the Shapley-Shubik power index are two different methods of calculating a voter's power. We saw in Exercise 1.11 that these do not always give the same result. There are many other power indices that quantify a voter's power. In order to describe these different power indices and to see geometrically what they describe, we first need to have a more general definition of a power index. Suppose that we have a weighted voting system with n voters, $v_1, v_2, \ldots, v_n$. Let S denote any coalition of voters, so S is just any *set* of voters. We define the set function

$$\nu(S) = \begin{cases} 1 & \text{if } S \text{ is a winning coalition} \\ 0 & \text{if } S \text{ is a losing coalition} \end{cases}$$

We denote the number of voters in S, or the cardinality of S, by $|S|$. The set $S - \{v_i\}$ is simply the set S with voter v_i removed.

Exercise 4.1 *In the weighted voting system* $[5 : 2, 5, 2]$*, let* $S = \{v_1, v_2\}$*.*

1. *What is* $\nu(S)$*?*
2. *What is* $|S|$*?*
3. *What is* $S - \{v_2\}$*?*
4. *What is* $\nu(S) - \nu(S - \{v_2\})$*?*

Note that the quantity $\nu(S) - \nu(S - \{v_i\})$ can be thought of as measuring the value that voter v_i contributes to the coalition S. For $1 \leq i \leq n$, let p_i denote the power of voter v_i. Define p_i of voter v_i as

$$p_i = \sum_{S \subseteq \{v_1, v_2, v_3, \ldots, v_n\}} \lambda_S \cdot (\nu(S) - \nu(S - \{v_i\})).$$

where the sum is taken over all subsets of voters $\{v_1, v_2, \ldots, v_n\}$ which contain v_i. The λ_S are coefficients which depend on the particular subset S. If the values of the λ_S's produce a set of p_i's that are normalized, then we will refer to them as power indices; if the values of λ_S's yield p_i's that are not normalized then we will refer to them as a power scores. An important class of indices are those where the λ_S depends only on how many people are in the coalition. This includes the BPI and SSPI. We will now examine these indices as well as a few others.

Let's start with the Banzhaf power index (BPI). Recall that to calculate the BPI of voter v_i, we began by computing the Banzhaf score. The Banzhaf score was found by simply counting up the number of winning coalitions in which v_i was critical. This is equivalent to letting $\lambda_S = 1$ in the general formula above.

Example 4.2 *To compute the Banzhaf score of v_1 in the weighted voting system*

$$[3 : 2, 1, 1],$$

we must compute

$$p_1 = \sum_{S \subseteq \{v_1, v_2, v_3\}} \lambda_S \cdot (\nu(S) - \nu(S - \{v_1\})) = \sum_{S \subseteq \{v_1, v_2, v_3\}} 1 \cdot (\nu(S) - \nu(S - \{v_1\}))$$

We begin by listing all possible coalitions S and identifying whether they are winning or losing coalitions. Then for any coalition S, we can easily determine whether $\nu(S)$ is 1 or 0. We also compute the second part of the above sum for the first voter : $\nu(S - \{v_1\})$.

S	S *Winning or Losing*	$\nu(S)$	$S - \{v_1\}$	$S - \{v_1\}$ *Winning or Losing*	$\nu(S - \{v_1\})$
$\{v_1\}$	*losing*	0	$\emptyset$	*losing*	0
$\{v_2\}$	*losing*	0	$\{v_2\}$	*losing*	0
$\{v_3\}$	*losing*	0	$\{v_3\}$	*losing*	0
$\{v_1, v_2\}$	*winning*	1	$\{v_2\}$	*losing*	0
$\{v_1, v_3\}$	*winning*	1	$\{v_3\}$	*losing*	0
$\{v_2, v_3\}$	*losing*	0	$\{v_2, v_3\}$	*losing*	0
$\{v_1, v_2, v_3\}$	*winning*	1	$\{v_2, v_3\}$	*losing*	0

Of the subsets listed above, only four contain v_1. Thus for p_i representing the Banzhaf score of v_i, we have

$$\begin{aligned} p_1 &= \sum_{S \subseteq \{v_1, v_2, v_3\}} 1 \cdot (\nu(S) - \nu(S - \{v_1\})) \\ &= 1 \cdot (0 - 0) + 1 \cdot (1 - 0) + 1 \cdot (1 - 0) + 1 \cdot (1 - 0) = 3 \end{aligned}$$

which agrees with our previous answer. Note that although we added only four terms, corresponding to the four subsets containing v_1, the answer would have remained the same had we included all seven subsets since for each of the additional terms, $\nu(S) - \nu(S - \{v_1\}) = 0$.

Notice that this formula can only be used to find the Banzhaf score. To find the BPI, the Banzhaf scores of all the players must be calculated, and then normalized as before. (Note that there are several versions of the Banzhaf index in current use. Some authors use the value $\lambda_S = \frac{1}{2^{n-1}}$ for all S, which makes the Banzhaf index a *semi-value*, see [18]. Others distinguish between the concept of a Banzhaf *measure* and a Banzhaf *index*, see [4]. We will return to this issue in Section 4.3.)

Next let's consider the Shapley-Shubik power index (SSPI) and determine the value of the λ_S. The wrinkle in computing the BPI was that it relied on the total number of critical voters which varies even with a fixed number of voters depending on the distribution of weights among those voters. This is not the case with the SSPI, since its score is normalized using the total number of permutations, which is fixed at $n!$ when there are n players. Thus we can determine λ_S that yield the SSPI instead of just the score.

Suppose we are trying to calculate the the Shapley-Shubik power index of voter v_i and want to determine the value of the λ_S.

Let S be a subset of size k with $\nu(S) - \nu(S - \{v_i\}) = 1$. Now the SSPI adds 1 for each permutation in which v_i is pivotal. Thus the quantity $\nu(S) - \nu(S - \{v_i\}) = 1$ should be multiplied by the number of arrangements of the voters in which v_i is the last (and pivotal) member of S. Consider a list of voters of the form

$$v_{j_1}, v_{j_2}, \ldots, v_{j_{k-1}}, v_i, v_{j_{k+1}}, v_{j_{k+2}}, \ldots, v_{j_n}$$

where the first $k-1$ voters are in S and the final $n-k$ voters are not in S. Since there are $(k-1)!$ permutations of the first $k-1$ voters and $(n-k)!$ permutations of the final $n-k$ voters, this set should be counted $(|S|-1)!(n-|S|)!$ times. Since the total number of permutations is equal to $n!$, we get the formula,

$$\lambda_S = \frac{(|S|-1)!(n-|S|)!}{n!}.$$

Example 4.3 *Let's compute the SSPI of v_1 in the weighted voting system*

$$[3 : 2, 1, 1].$$

Since we have three voters,

$$\lambda_S = \frac{(|S|-1)!(n-|S|)!}{n!} = \frac{(|S|-1)!(3-|S|)!}{3!}.$$

For each subset S, we can compute λ_S as follows:

S	$\lvert S\rvert$	$\lambda_S = \frac{(\lvert S\rvert-1)!(3-\lvert S\rvert)!}{3!}$
$\{v_1\}$	1	1/3
$\{v_2\}$	1	1/3
$\{v_3\}$	1	1/3
$\{v_1, v_2\}$	2	1/6
$\{v_1, v_3\}$	2	1/6
$\{v_2, v_3\}$	2	1/6
$\{v_1, v_2, v_3\}$	3	1/3

Thus, using our general definition, we get

$$\begin{aligned} p_1 &= \sum_{S \subseteq \{v_1, v_2, v_3\}} \lambda_S \cdot (\nu(S) - \nu(S - \{v_1\})) \\ &= \frac{1}{3} \cdot (0-0) + \frac{1}{6} \cdot (1-0) + \frac{1}{6} \cdot (1-0) + \frac{1}{3} \cdot (1-0) = \frac{2}{3}. \end{aligned}$$

Exercise 4.4 *Using the general definition above, compute the BPI and the SSPI for each of the three voters in the weighted voting system*

$$[2 : 1, 1, 1].$$

We now define two other scores that form power indices by normalizing. One is called the Dictatorial Power Index (DPI) and the other is the Marginal Power Index (MPI). The Dictatorial Power score is found using the general definition of a power index with

$$\lambda_S = \begin{cases} 1 & \text{if } |S| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

while the Marginal Power score is found using

$$\lambda_S = \begin{cases} 1 & \text{if } |S| = n \\ 0 & \text{otherwise.} \end{cases}$$

Since these indices do not always add up to 1 so if we wish to graph them, we must normalize as in Section 1.4.

Example 4.5 *Let's compute the Dictatorial Power score and Marginal Power score of voter v_1 in the weighted voting system:*

$$[3 : 2, 1, 1].$$

We need to compute the λ's for both indices.

S	$\lvert S\rvert$	λ_S *for DPI*	λ_S *for MPI*
$\{v_1\}$	1	1	0
$\{v_2\}$	1	1	0
$\{v_3\}$	1	1	0
$\{v_1, v_2\}$	2	0	0
$\{v_1, v_3\}$	2	0	0
$\{v_2, v_3\}$	2	0	0
$\{v_1, v_2, v_3\}$	3	0	1

Thus the Dictatorial Power score for the first voter is

$$\begin{aligned} p_1 &= \sum_{S \subseteq \{v_1, v_2, v_3\}} \lambda_S \cdot (\nu(S) - \nu(S - \{v_1\})) = \\ &= 1 \cdot (0-0) + 1 \cdot (0-0) + 1 \cdot (0-0) + 0 \cdot (1-0) + 0 \cdot (1-0) + 0 \cdot (0-0) + 0 \cdot (1-0) = 0. \end{aligned}$$

The Marginal Power score for the first voter is computed in the same manner:

$$\begin{aligned} p_1 &= \sum_{S \subseteq \{v_1, v_2, v_3\}} \lambda_S \cdot (\nu(S) - \nu(S - \{v_1\})) = \\ &= 0 \cdot (0-0) + 0 \cdot (0-0) + 0 \cdot (0-0) + 0 \cdot (1-0) + 0 \cdot (1-0) + 0 \cdot (0-0) + 1 \cdot (1-0) = 1. \end{aligned}$$

Exercise 4.6 *Explain why the Dictatorial Power Index is so named. Why might someone be tempted to give the nickname of "veto power index" to the Marginal Power Index?*

Exercise 4.7 *Compute the Dictatorial and Marginal Power Indices for the three voters in the weighted voting system*

$$[2 : 1, 1, 1].$$

Don't forget to first compute the scores and then normalize to obtain the indices.

Exercise 4.8 *What is the Dictatorial Power score for any of the five voters in the weighted voting system*

$$[3 : 1, 1, 1, 1, 1]?$$

Compare with the Banzhaf scores calculated in Exercise 1.10.

Exercise 4.9 *What is the Marginal Power score for all five voters in the weighted voting system*

$$[4 : 3, 1, 1, 1, 1]?$$

Compare with the Banzhaf scores calculated in Exercise 1.10.

There are many other power indices beyond the four mentioned above. In particular, there are numerous situations where it does not make sense to require the λ_S to depend only on the size of the set S. Consider a hiring committee, for example, when more value might be placed on a coalition that includes minority or other diverse members. Or, in the Supreme Court model outlined in Exercise 1.9 , it might be appropriate to weight coalitions relative to the probability of their occurring. (A winning coalition consisting of the conservative and liberal blocs, for instance, would be be highly unlikely.) Such considerations are beyond the scope of this module; interested readers might want to look at Weber [22].

4.2 Vector Representations

In this section we examine the ideas of Saari and Sieberg [18] who used ideas from linear algebra to compare power indices for weighted voting systems with three voters v_1, v_2 and v_3. We start by interpreting the formula for power indices in terms of vectors. Let us first assume that the λ_S depend only on the size of S. Consider how the calculation for the power score/index p_1 of voter v_1 is found

$$p_1 = \sum_{S \subseteq \{v_1, v_2, v_3\}} \lambda_{|S|} \cdot (\nu(S) - \nu(S - \{v_1\}))$$

where the sum is taken over all subsets of $\{v_1, v_2, v_3\}$ which contain v_1. Expanding this sum, we get

$$\begin{aligned} p_1 &= \sum_{k=1}^{3} \sum_{\substack{S \subseteq \{v_1, v_2, v_3\} \\ |S|=k}} \lambda_k \cdot (\nu(S) - \nu(S - \{v_1\})) \\ &= \lambda_1 \cdot \sum_{|S|=1} (\nu(S) - \nu(S - \{v_1\})) + \lambda_2 \cdot \sum_{|S|=2} (\nu(S) - \nu(S - \{v_1\})) + \lambda_3 \cdot \sum_{|S|=3} (\nu(S) - \nu(S - \{v_1\})) \\ &= \lambda_1[\nu(\{v_1\}) - \nu(\varnothing)] + \lambda_2[\nu(\{v_1, v_2\}) - \nu(\{v_2\}) + \nu(\{v_1, v_3\}) - \nu(\{v_3\})] + \lambda_3[\nu(\{v_1, v_2, v_3\}) - \nu(\{v_2, v_3\})]. \end{aligned}$$

Similar expansions exist for p_2 and p_3. These equations can be summarized as:

$$\overline{P} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} \nu(\{v_1\}) \\ \nu(\{v_2\}) \\ \nu(\{v_3\}) \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} \nu(\{v_1, v_2\}) - \nu(\{v_2\}) + \nu(\{v_1, v_3\}) - \nu(\{v_3\}) \\ \nu(\{v_1, v_2\}) - \nu(\{v_1\}) + \nu(\{v_2, v_3\}) - \nu(\{v_3\}) \\ \nu(\{v_1, v_3\}) - \nu(\{v_1\}) + \nu(\{v_2, v_3\}) - \nu(\{v_2\}) \end{bmatrix}$$

$$+\lambda_3 \cdot \begin{bmatrix} \nu(\{v_1, v_2, v_3\}) - \nu(\{v_2, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_2\}) \end{bmatrix}.$$

Thus we can see that the set of power scores or indices (depending on whether they result in normalized values or not) can be written as a linear combination of three column vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ where

$$\overline{P}_1 = \begin{bmatrix} \nu(\{v_1\}) \\ \nu(\{v_2\}) \\ \nu(\{v_3\}) \end{bmatrix}, \overline{P}_2 = \begin{bmatrix} \nu(\{v_1, v_2\}) - \nu(\{v_2\}) + \nu(\{v_1, v_3\}) - \nu(\{v_3\}) \\ \nu(\{v_1, v_2\}) - \nu(\{v_1\}) + \nu(\{v_2, v_3\}) - \nu(\{v_3\}) \\ \nu(\{v_1, v_3\}) - \nu(\{v_1\}) + \nu(\{v_2, v_3\}) - \nu(\{v_2\}) \end{bmatrix},$$

$$\overline{P}_3 = \begin{bmatrix} \nu(\{v_1, v_2, v_3\}) - \nu(\{v_2, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_2\}) \end{bmatrix}.$$

The components of $\overline{P}_1$ are the contributions of each voter to coalitions of size one. Notice that a component of $\overline{P}_1$ will be non-zero only when the weight of a voter exceeds the quota. Similarly, the components of $\overline{P}_2$ are the contributions of each voter to coalitions of size two and the components of $\overline{P}_3$ are the contributions of each voter to coalitions of size three. So given any weighted voting system with three voters, we can find the three vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ by looking at winning and losing coalitions.

Example 4.10 *Find the three vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ for the weighted voting system*

$$[3 : 2, 1, 1].$$

As in example 4.2, we find the winning and losing coalitions. The winning coalitions are

$$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_2, v_3\},$$

and the losing coalitions are

$$\{v_1\}, \{v_2\}, \{v_3\}, \{v_2, v_3\}.$$

Thus we can compute

$$\overline{P}_1 = \begin{bmatrix} \nu(\{v_1\}) \\ \nu(\{v_2\}) \\ \nu(\{v_3\}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\overline{P}_2 = \begin{bmatrix} \nu(\{v_1, v_2\}) - \nu(\{v_2\}) + \nu(\{v_1, v_3\}) - \nu(\{v_3\}) \\ \nu(\{v_1, v_2\}) - \nu(\{v_1\}) + \nu(\{v_2, v_3\}) - \nu(\{v_3\}) \\ \nu(\{v_1, v_3\}) - \nu(\{v_1\}) + \nu(\{v_2, v_3\}) - \nu(\{v_2\}) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

and

$$\overline{P}_3 = \begin{bmatrix} \nu(\{v_1, v_2, v_3\}) - \nu(\{v_2, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_2\}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Exercise 4.11 *Find $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ for the weighted voting system*

$$[4 : 2, 2, 1].$$

Example 4.12 *Calculate the Banzhaf scores of the weighted voting system $[3 : 2, 1, 1]$ by writing them as a linear combination of the three vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$. Since $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we see that the Banzhaf scores are given by*

$$\overline{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Example 4.13 *Compute the SSPI of the weighted voting system*

$$[3 : 2, 1, 1].$$

We know that

$$\overline{P} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \lambda_1 \overline{P}_1 + \lambda_2 \overline{P}_2 + \lambda_3 \overline{P}_3.$$

But from our previous calculations, we know that for the SSPI,

$$\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{1}{6}, \textit{ and } \lambda_3 = \frac{1}{3}.$$

Thus,

$$\overline{P} = \frac{1}{3}\overline{P}_1 + \frac{1}{6}\overline{P}_2 + \frac{1}{3}\overline{P}_3$$

$$= \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \\ 1/6 \end{bmatrix}$$

Exercise 4.14 *Write both the Banzhaf scores and the Shapley-Shubik indices of the weighted voting system*

$$[4 : 2, 2, 1]$$

as a linear combination of $\overline{P}_1, \overline{P}_2$, *and* $\overline{P}_3$.

How can we use the vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ to *compare* power indices geometrically? We graph the vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ on the triangle (of course)! This isn't as straight-forward as it might sound. Not only are vectors not triples, their coordinates may not add up to 1. The first issue is easy to deal with; we simply think of a vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as an ordered triple (x, y, z). How do we get the sum of its coordinates to be one? Normalize! Simply add up the components, and then divide each component by this sum. We can do this as long as the sum is not zero.

Example 4.15 *For example, if we wish to represent the vector*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

on the triangle, we note that the sum of it's components is six so the normalized vector is

$$\begin{bmatrix} 1/6 \\ 2/6 \\ 3/6 \end{bmatrix}.$$

We then represent this vector on the triangle as the ordered triple $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$.

What happens if all three of the coordinates are zero? That is, how do we represent the vector (0, 0, 0) on the triangle? In this case we identify the vector with the point of intersection of the three indifference lines: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

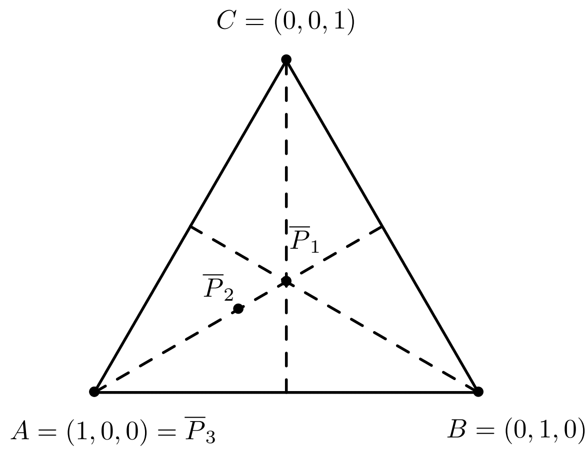


Figure 12: Representing $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ of the weighted voting system $[3:2,1,1]$ on the triangle.

Example 4.16 *Represent the vectors $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ of the weighted voting system $[3:2,1,1]$ on the triangle.*

Since the vector $\overline{P}_1 = \overline{0}$, it is represented at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Since

$$\overline{P}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

it is represented by the point $(\frac{2}{4}, \frac{1}{4}, \frac{1}{4})$.

The vector $\overline{P}_3$ is already normalized and is thus represented on the triangle as $(1,0,0)$. We plot these points in Figure 12.

Exercise 4.17 *What do you notice about this figure?*

Exercise 4.18 *Represent $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ of the weighed voting system*

$$[4:2,2,1]$$

on the triangle as in Figure 12.

In both Example 4.16 and Exercise 4.18, you should have noticed that the vectors $\overline{P}_1, \overline{P}_2$ and $\overline{P}_3$ lay along a line. In fact, this is true for any weighted voting system with three players. This can be seen by considering the five different non-equivalent weighted voting systems you found in Exercise 1.20. Example 4.16 and Exercise 4.18 correspond to two of these five. It is easy to check that in the other three cases, all the $\overline{P}_i$ will also lie along a line. (In two cases, the voters all have equivalent roles in the winning coalitions hence all the $\overline{P}_i$ lie at $(1/3, 1/3, 1/3)$; in the remaining case, one player is a dictator hence all the $\overline{P}_i$ lie at one of the vertices of the triangle.)

Example 4.19 *Now that we know how to represent $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ on the triangle, how can we use this to compare power indices? Consider the weighted voting system $[3:2,1,1]$. In Example 1.21, we found the BPI and SSPI to be $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ and $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$, respectively. Including the BPI and SSPI in Figure 12 produces Figure 13.*

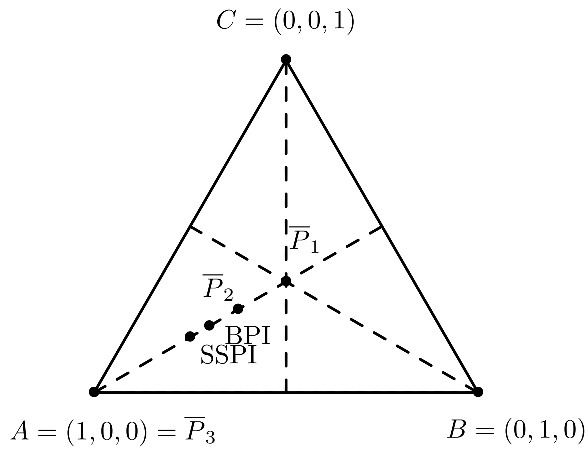


Figure 13: Representing the BPI, SSPI, $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ of the weighted voting system $[3 : 2, 1, 1]$ on the triangle.

Exercise 4.20 *What do you notice about Figure 13?*

Exercise 4.21 *Now represent the BPI, SSPI, $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ of the weighed voting system $[4 : 2, 2, 1]$ on the triangle. What do you notice about this figure?*

Example 4.22 *In Example 4.5, we computed the Dictatorial and Marginal Power scores of the first voter in the weighted voting system*

$$[3 : 2, 1, 1].$$

It's not difficult to compute these scores for the other two voters in this weighted voting system. We see that the Dictatorial scores of all three voters are 0 and the Marginal Power scores of v_1, v_2, and v_3 are 1,0,0, respectively. In order to represent these indices on the triangle, we must represent DPI $= (0, 0, 0)$ as $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and the already normalized MPI as $(1, 0, 0)$. We see these on the triangle in Figure 14.

4.3 Convex Hulls

In Examples 4.19 and 4.22, the SSPI, BPI, MPI and DPI all lay along the same line as the $\overline{P}_i$ vectors. In fact, this will be true of any weighted voting system. In the previous section, we determined that the $\overline{P}_i$ vectors will always lie along a line by considering each of the five non-equivalent voting systems. To understand why this is also true for all the different power indices requires the idea of a convex hull. In general, the convex hull of vectors $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n$ is the set of points P that can be written as

$$\overline{P} = \sum_k \gamma_k \overline{P}_k \text{ where } \gamma_k \geq 0 \text{ and } \sum_k \gamma_k = 1.$$

See Figure 15. (Note that if the vectors $\overline{P}_i$ all lie along a line, so will the convex hull.)

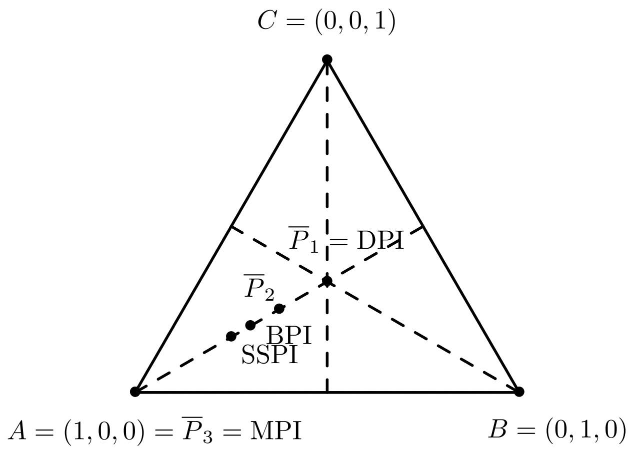


Figure 14: Representing the BPI, SSPI, DPI, MPI, $\overline{P}_1, \overline{P}_2$, and $\overline{P}_3$ of the weighted voting system $[3 : 2, 1, 1]$ on the triangle.

Figure 15: Geometrically, the convex hull of a set of points can be represented by the region bounded by the lines joining the set of points.

Thus the convex hull consists of all positive linear combinations of the vectors $\bar{P}_i$ such that the sum of the coefficients equals one. This is very similar to our definition of power index. We have shown that for three voters, the set of power indices satisfies

$$\overline{P} = \sum_{i=1}^{3} \lambda_k \overline{P}_k$$

as long as the λ_k depend only on the size of $|S| = k$. However the λ_k may not always sum to one. For instance, for the SSPI,

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \frac{5}{6}.$$

Let's see what happens in the case where a voter (say v_1) is a dictator, and suppose $p_1 = 1$. Then

$$p_1 \quad = \quad \lambda_1[\nu(\{v_1\}) - \nu(\varnothing)] + \lambda_2[\nu(\{v_1, v_2\}) - \nu(\{v_2\})] + \lambda_2[\nu(\{v_1, v_3\}) - \nu(\{v_3\})]$$

$$\begin{aligned} &+ \quad \lambda_3[\nu(\{v_1, v_2, v_3\}) - \nu(\{v_2, v_3\})] \\ &= \quad \lambda_1 + \lambda_2[1+1] + \lambda_3. \end{aligned}$$

Thus

$$1 = \lambda_1 + 2\lambda_2 + \lambda_3.$$

This is true for the SSPI, since

$$\lambda_1 + 2\lambda_2 + \lambda_3 = \frac{1}{3} + \frac{2}{6} + \frac{1}{3} = 1.$$

It is also true for the DPI and the MPI since they satisfy $\lambda_1 + 2\lambda_2 + \lambda_3 = 1 + 2 \cdot 0 + 0 = 1$ and $\lambda_1 + 2\lambda_2 + \lambda_3 = 0 + 2 \cdot 0 + 1 = 1$ respectively. (We'll look at the BPI in a moment.) Thus for power indices such as these, we can use a change of variables to make the λ_k sum to 1. Let

$$\gamma_1 = \lambda_1, \gamma_2 = 2\lambda_2 \text{ and } \gamma_3 = \lambda_3.$$

Then

$$\sum_k \gamma_k = 1.$$

Furthermore, if we let

$$\hat{\overline{P}}_1 = \overline{P}_1, \hat{\overline{P}}_2 = 1/2\overline{P}_2 \text{ and } \hat{\overline{P}}_3 = \overline{P}_3,$$

then

$$\hat{\overline{P}} = \sum_{i=1}^{3} \gamma_k \hat{\overline{P}}_k.$$

This shows that the SSPI, the DPI and the MPI, (and any power index for which $p_1 = 1$), lies in the convex hull of the $\hat{\overline{P}}_k$.

The situation for the BPI is a little more subtle. In this case

$$p_1 = \lambda_1 + 2\lambda_2 + \lambda_3 = 1 + 2 + 1 = 4,$$

so the change of variables will not work for the BPI. However, we have seen in the examples that the BPI does lie along the same line as the other power indices. Why is this the case? Notice that if we altered the BPI slightly by letting the $\lambda_k = \frac{1}{4}$ for all k, then we would have

$$p_1 = \lambda_1 + 2\lambda_2 + \lambda_3 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1.$$

This variation of the BPI is often called the Banzhaf *measure* and is defined for each voter in a system of n voters to be the voter's Banzhaf score divided by $2^{(n-1)}$. Like the other power indices, it will lie in the convex hull of the $\hat{\overline{P}}_k$. It is clearly different from the Banzhaf score, but its geometry is the same. (Recall that there are three related variations of Banzhaf power: the Banzhaf score, the BPI, and the Banzhaf measure.) Notice that the λ_k values for the Banzhaf measure and the Banzhaf score are still in the same proportion (in each case, they are equal). Thus they measure the same *relative* power between the players. More importantly, when they are normalized, they both equal the BPI, and hence lie on the same point in the simplex. Thus, since the Banzhaf measure lies in the convex hull of the $\hat{\overline{P}}_k$, so does the Banzhaf score.

Exercise 4.23 *Represent the vectors* $\hat{\overline{P}}_1, \hat{\overline{P}}_2,$ *and* $\hat{\overline{P}}_3$ *of the weighted voting system*

$$[4 : 2, 2, 1]$$

on the triangle. Use the $\hat{\overline{P}}_k$ *vectors to compute the BPI, SSPI, DPI and MPI and plot them on the triangle.*

In the previous exercise, you may have noticed that the γ_k's for the Shapley-Shubik index were all equal to $\frac{1}{3}$. This means that the Shapley-Shubik index weights all contributions of size k equally: $\frac{1}{3}$ of the power is derived from each of the three coalition sizes. Geometrically, this causes the Shapley-Shubik index to lie at the barycenter (the "center of gravity") of the vectors $\hat{\overline{P}}_k$. More importantly, the exercise demonstrates the fact that the power indices lie along a line as a result of the fact that all the $\hat{\overline{P}}_k$ vectors lie along a line. One consequence of this is that if there are three voters, all power indices result in the same ranking of the voters' power. In fact, since the power indices either all lie at the barycenter, or a vertex of a triangle, or lie along one of the indifference lines, in any weighted voting system with three players, at least two players must have equal power.

In more general situations when the set function v can be equal to more than just 0 or 1, we will see that this is not always the case. This is the subject of the next section.

5 Generalized Power Indices for Non-simple Weighted Voting Systems

In the weighted voting systems considered so far, there were only two possible outcomes: a "yes" vote or a "no" vote; either a measure would pass or it wouldn't. In order to reflect this, when we look at a power index as a function, there are only two outcomes when a coalition S is evaluated using the set function v: $v(S) = 1$ if S is a winning coalition, and $v(S) = 0$ if S is a losing coalition. However, sometimes we want to measure a person's power or value in a situation where the voting is not just yes or no. In fact, the situation may not involve voting at all. Power indices (including the Shapley-Shubik and the Banzhaf) can be use in these situations as well. Examples of these sorts of situations include measuring the amount of value added by different units in designing a company's new product, determining the relative worth of a basketball player to their league in order to rank all players, or even measuring how much a student contributes to a group project.

The method for adapting power indices to these new settings is to use a generalized definition of power indices as in section 4 but to consider the set function as the characteristic function of a *cooperative games*. A cooperative game consists of a set of players (or voters) $N = \{v_1, v_2, \ldots, v_n\}$ and a characteristic function ν that assigns a real number $\nu(S)$ to each subset $S \subset N$ such that $\nu(\emptyset)$. Notice in this definition, the value of $\nu(S)$ can be any real number. Frequently, the characteristic function is normalized so that $\nu(N) = 1$ as in the example below.

Example 5.1 *Three students, Amy, Benito and Calvin, are assigned to work together on a calculus project. Amy is an expert at using the computer program Maple, Benito is an excellent writer and Calvin asks good questions and serves as a peacemaker (since Amy and Benito tend to fight when they are together without the peacemaker). We'll assign reasonable values to the possible coalitions of Amy (A), Benito (B), and Calvin (C), based on their contributions to their joint work:*

$$
\begin{array}{lcll}
v(\emptyset) & = & 0 & \\
v(\{A\}) & = & 0.3 & \textit{since Amy has useful skills on her own.} \\
v(\{B\}) & = & 0.2 & \textit{since Benito is a good writer, but needs something to write about!} \\
v(\{C\}) & = & 0.1 & \textit{since Calvin asks good questions, but has trouble answering them by himself!} \\
v(\{A,B\}) & = & 0.2 & \textit{since Amy and Benito argue a lot when they try to work together and waste time.} \\
v(\{A,C\}) & = & 0.5 & \textit{since Amy and Calvin work fairly well together.} \\
v(\{B,C\}) & = & 0.4 & \textit{since Benito and Calvin also work well together, but they need computational skills.} \\
v(\{A,B,C\}) & = & 1.0 & \textit{since all three students work well together and have complimentary skills.}
\end{array}
$$

The situation will usually determine the values ν assigns to each subset. Using this new valuation and our generalized definition off power indices, we can compute the BPI, as the next example shows.

Example 5.2 *Using the same values as in Example 5.1 above, we can determine the Banzhaf score of Benito. Recall that $\lambda_S = 1$ for all subsets S, so Benito's Banzhaf score is*

$$
\begin{array}{ll}
 & \sum_{S\in\{A,B,C\}}(v(S) - v(S - \{B\})) \\
= & (v(\{A,B,C\}) - v(\{A,C\})) + (v(\{A,B\}) - v(\{A\})) + (v(\{B,C\}) - v(\{C\})) + (v(\{B\}) - v(\emptyset)) \\
= & 1 - 0.5 + 0.2 - 0.3 + 0.4 - 0.1 + 0.2 - 0 = 0.9.
\end{array}
$$

Using the vector representation, we compute the Banzhaf scores for the system and then normalize the result to obtain the BPI:

$$
\overline{P}_1 = \begin{bmatrix} v(\{A\}) - v(\emptyset) \\ v(\{B\}) - v(\emptyset) \\ v(\{C\}) - v(\emptyset) \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \end{bmatrix},
$$

$$
\overline{P}_2 = \begin{bmatrix} v(\{A,B\}) - v(\{B\}) + v(\{A,C\}) - v(\{C\}) \\ v(\{A,B\}) - v(\{A\}) + v(\{B,C\}) - v(\{C\}) \\ v(\{A,C\}) - v(\{A\}) + v(\{B,C\}) - v(\{B\}) \end{bmatrix} = \begin{bmatrix} 0.2 - 0.2 + 0.5 - 0.1 \\ 0.2 - 0.3 + 0.4 - 0.1 \\ 0.5 - 0.3 + 0.4 - 0.2 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix},
$$

and

$$
\overline{P}_3 = \begin{bmatrix} v(\{A,B,C\}) - v(\{B,C\}) \\ v(\{A,B,C\}) - v(\{A,C\}) \\ v(\{A,B,C\}) - v(\{A,B\}) \end{bmatrix} = \begin{bmatrix} 1 - 0.4 \\ 1 - 0.5 \\ 1 - 0.2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.8 \end{bmatrix}.
$$

Then

$$
\overline{P} = \overline{P}_1 + \overline{P}_2 + \overline{P}_3 = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.5 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.9 \\ 1.3 \end{bmatrix}.
$$

The BPI is

$$
BPI = \left[\ \tfrac{1.3}{3.5}, \tfrac{0.9}{3.5}, \tfrac{1.3}{3.5} \ \right] = \left[\ \tfrac{13}{35}, \tfrac{9}{35}, \tfrac{13}{35} \ \right].
$$

Notice that while Amy and Calvin contribute equally to the group, Benito's contribution is comparatively lower.

The SSPI can be extended in the same way. For the same example as above, we would use $\lambda_S = \frac{(|S|-1)!(n-|S|)!}{n!}$ to find each person's power as measured by the SSPI. We can use the vector representation that we computed above, since the only quantities that are different are the $\lambda_{|S|}$.

Example 5.3 *Using the SSPI to measure the contributions of each student to the various groups, our $\lambda_{|S|}$ will be as in Example 4.3.*

$$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = \frac{1}{6}, \quad \text{and} \quad \lambda_3 = \frac{1}{3}.$$

Thus the

$$SSPI = \frac{1}{3}\overline{P}_1 + \frac{1}{6}\overline{P}_2 + \frac{1}{3}\overline{P}_3 = \frac{1}{3}\begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0.6 \\ 0.5 \\ 0.8 \end{bmatrix} = \begin{bmatrix} \frac{11}{30} \\ \frac{8}{30} \\ \frac{11}{30} \end{bmatrix}.$$

Once again Amy and Calvin add equally to the group (as measured by the Shapley-Shubik power index).

Exercise 5.4 *Compute the BPI and SSPI for A, B and C using the following values for the set function v:*

$$\begin{aligned} v(\emptyset) &= 0 \\ v(\{A\}) &= 0.1 \\ v(\{B\}) &= 0.2 \\ v(\{C\}) &= 0.1 \\ v(\{A,B\}) &= 0.3 \\ v(\{A,C\}) &= 0.5 \\ v(\{B,C\}) &= 0.6 \\ v(\{A,B,C\}) &= 1.0. \end{aligned}$$

In the previous examples, although the values were different under each power index, the rankings of the person who added the most value were the same for each power index. However even with just three people, different power indices can give different rankings.

Example 5.5 *Brothers Thomas (aged two) and Patrick (aged four) are trying to get Grandma to buy them a new treehouse for Christmas. They could try to get Mom to help sway Grandma as well, but Grandma will only get the treehouse if one or more of the kids requests it. (She doesn't want to get the boys a gift that only Mom thinks is a good idea.) The value for the set function describing how much influence different coalitions would have when they approach Grandma follow (P for Patrick, T for Thomas and M for Mom):*

$$\begin{array}{ll} v(\emptyset) = 0 & v(\{T,P\}) = 0.35 \\ v(\{M\}) = 0 & v(\{T,M\}) = 0.7 \\ v(\{T\}) = 0.15 & v(\{P,M\}) = 0.4 \\ v(\{P\}) = 0.2 & v(\{T,P,M\}) = 1. \end{array}$$

As before each person's contribution to the coalitions of size i are given by the vector $\overline{P}_i$:

$$\overline{P}_1 = \begin{bmatrix} v(\{P\}) - v(\emptyset) \\ v(\{T\}) - v(\emptyset) \\ v(\{M\}) - v(\emptyset) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.15 \\ 0 \end{bmatrix},$$

$$\overline{P}_2 = \begin{bmatrix} v(\{P,T\}) - v(\{T\}) + v(\{P,M\}) - v(\{M\}) \\ v(\{P,T\}) - v(\{P\}) + v(\{T,M\}) - v(\{M\}) \\ v(\{M,T\}) - v(\{T\}) + v(\{P,M\}) - v(\{P\}) \end{bmatrix} = \begin{bmatrix} 0.35 - 0.15 + .4 - 0 \\ 0.35 - 0.2 + 0.7 - 0 \\ 0.7 - 0.15 + 0.4 - 0.2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.85 \\ 0.75 \end{bmatrix},$$

and

$$\overline{P}_3 = \begin{bmatrix} v(\{P,T,M\}) - v(\{T,M\}) \\ v(\{P,T,M\}) - v(\{M,P\}) \\ v(\{P,M,T\}) - v(\{T,P\}) \end{bmatrix} = \begin{bmatrix} 1 - 0.7 \\ 1 - 0.4 \\ 1 - 0.35 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.65 \end{bmatrix}.$$

The total amount of value Patrick (in row one of each of these vectors) adds to the entire collection of coalitions of size one is 0.2. Patrick adds a total of 0.6 value to the collection of all coalitions of size two. Similarly he adds a value of 0.3 to the sole coalition of size three. However, we would usually be more interested in a single coalition of, say, size two, and so more useful is the notion of one's average contribution to a coalition. These vectors containing the average amount added to a coalition are exactly the same as the $\hat{\overline{P}}_i$ *in the previous section. Since there is only one coalition of size three and one coalition of size one, but two of size two, the only vector affected in the three player case is* $\overline{P}_2$*. We leave* $\overline{P}_1$ *and* $\overline{P}_3$ *alone, but divide* $\overline{P}_2$ *by two. In general we average according to the formula:*

$$\hat{P}_i = \frac{\overline{P}_i}{\begin{pmatrix} n-1 \\ i-1 \end{pmatrix}}.$$

Thus our averaged vectors in this case are:

$$\hat{\overline{P}}_1 = \overline{P}_1 = \begin{bmatrix} 0.2 \\ 0.15 \\ 0 \end{bmatrix},$$

$$\hat{\overline{P}}_2 = \frac{\overline{P}_2}{2} = \begin{bmatrix} 0.3 \\ 0.425 \\ 0.375 \end{bmatrix},$$

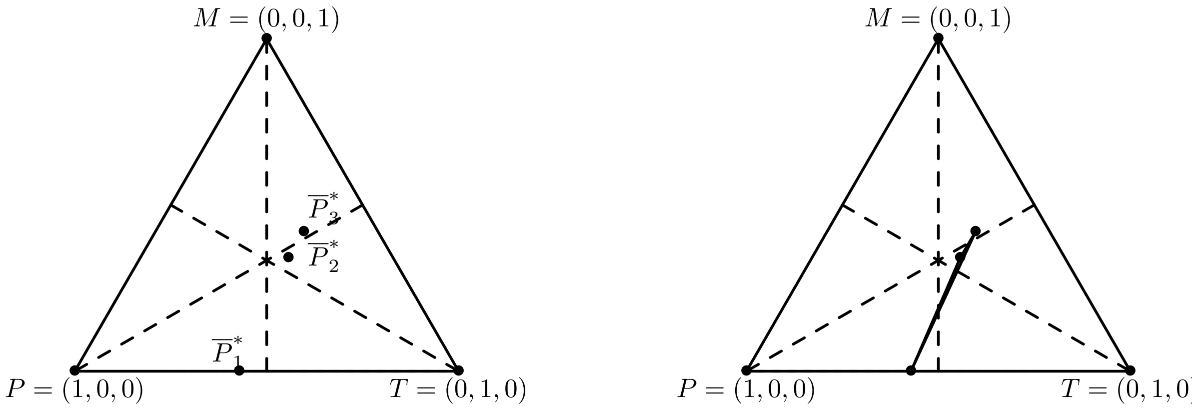
and

$$\hat{\overline{P}}_3 = \overline{P}_3 = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.65 \end{bmatrix}.$$

We now normalize each of these in order to graph them on the simplex and draw the convex hull. The normalized versions are

$$\overline{P}_1^* = \begin{bmatrix} \frac{0.2}{0.35} \\ \frac{0.15}{0.35} \\ \frac{0}{0.35} \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \\ 0 \end{bmatrix},$$

$$\overline{P}_2^* = \begin{bmatrix} \frac{0.3}{1.1} \\ \frac{0.425}{1.1} \\ \frac{0.375}{1.1} \end{bmatrix} = \begin{bmatrix} \frac{12}{44} \\ \frac{17}{44} \\ \frac{15}{44} \end{bmatrix},$$



The vectors $\hat{\overline{P}}_i$ — The convex hull of the vectors $\hat{\overline{P}}_i$.

Figure 16: Example 5.5

and

$$\overline{P}_3^* = \begin{bmatrix} \frac{0.3}{1.55} \\ \frac{0.6}{1.55} \\ \frac{0.65}{1.55} \end{bmatrix} = \begin{bmatrix} \frac{6}{31} \\ \frac{12}{31} \\ \frac{13}{31} \end{bmatrix}.$$

Notice that if a power index only counted how much a player contributed to size one coalitions (perhaps Grandma will only listen to individuals), then $p_P > p_T > p_M$, but if a different one counted only contributions to size two coalitions (Grandma is most influenced by pairs), then for that index, $p_T > p_M > p_P$, and finally if a different power index concerns itself with three person coalitions (Grandma only listens if all three approach her at once) then the ranking is $p_M > p_T > p_P$. So the index that is being used to measure added-value very much affects who adds the most value or who holds the most power. Additionally, if we graph the vectors on the simplex as in Figure 16, and compute the convex hull, we find that any of ***seven different rankings is possible*** *(three of them ties) depending on the index used.*

Exercise 5.6 *Compute the BPI and the SSPI for Example 5.5 using the vectors $\hat{\overline{P}}_1, \hat{\overline{P}}_2$, and $\hat{\overline{P}}_3$. Plot the vectors and the power indices and check that the power indices lie within the convex hull of $\hat{\overline{P}}_1, \hat{\overline{P}}_2$, and $\hat{\overline{P}}_3$. What do you observe?*

6 Additional Topics/Projects

6.1 The Geometry of Paradoxes

Power indices have been used to analyze a wide variety of political and economic institutions, from the International Monetary Fund [5, 13], to the European Council of Ministers [3,12], to the composition of County Boards [23]. Nevertheless, they frequently give rise to counter-intuitive results. Some of the most famous of these have been labeled paradoxes. In this section we will look at how the geometry of power indices can be used to understand two of these paradoxes.

The Paradox of Redistribution was first discussed by Fischer and Schotter in 1978 [7]. It can arise in certain weighted voting systems when the weights are distributed, causing one voter's weight to increase while his power decreases (or vice versa).

Example 6.1 *Consider the weighted voting system*

$$[7 : 4, 4, 4].$$

The winning coalitions are $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$ *and* $\{v_1, v_2, v_3\}$; *thus the Banzhaf score of each voter is equal to two. Suppose, now, that the weight is redistributed to form a new weighted voting system*

$$[7 : 1, 6, 5].$$

The winning coalitions are now $\{v_1, v_2\}, \{v_2, v_3\}$ *and* $\{v_1, v_2, v_3\}$, *and the Banzhaf score of players* v_1 *and* v_3 *has been reduced to one. Normalizing, we see that the set of Banzhaf Power indices has been transformed from* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ *to* $\left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right)$. *In particular, the power of* v_3 *has decreased while the percentage of its total weight has increased.*

Exercise 6.2 *Check that the Shapley-Shubik power index also exhibits the Paradox of Redistribution in this example.*

Exercise 6.3 *By using the right-most drawing in Figure 9, create an example of a weighted voting system that manifests the Paradox of Redistribution by moving from* R_{10} *to* R_5.

The Paradox of Quarreling Members, introduced by Kilgour in 1974, refers to a situation when two voters refuse to belong to the same coalition, so withdraw from a winning coalition, yet find their power increased [10]. This result is considered paradoxical because intuitively, we would expect the voters to suffer a decrease in power corresponding to their decreased ability to form winning coalitions.

Example 6.4 *Consider the weighted voting system*

$$[3 : 1, 1, 2].$$

The winning coalitions are $\{v_1, v_3\}$, $\{v_2, v_3\}$ *and* $\{v_1, v_2, v_3\}$, *and the Banzhaf scores are* $(1, 1, 3)$. *If voters* v_1 *and* v_2 *quarrel then the coalition* $\{v_1, v_2, v_3\}$ *is no longer possible and the Banzhaf scores become* $(1, 1, 2)$. *Normalizing, we see that both* v_1 *and* v_2 *have benefited from the quarrel by increasing their relative share of power from* $\frac{1}{5}$ *to* $\frac{1}{4}$.

A look at the right-hand drawing in Figure 9 shows what has happened. The weighted voting system lies at a vertex between R_9 *and* R_{10}. *The effect of the quarrel is to remove the line corresponding to* $\tilde{w}_1 + \tilde{w}_2 = \frac{3}{4}$, *resulting in changes to the shapes of the regions. Regions* R_7 *and* R_8 *get subsumed by regions* R_5 *and* R_4 *respectively. And regions* R_6 *and* R_{10} *form a new region that has no winning coalitions. Thus the weighted voting system has only the minimal winning coalitions of* R_9.

Exercise 6.5 *Does the Shapley-Shubik index exhibit the Paradox of Quarreling Members in this example?*

There are several other paradoxes named in the literature. Interested students might want to research the Quota Paradox, the Paradox of a New Member or the Paradox of Large Size and analyze some examples using the geometric ideas introduced in this module. For a comprehensive treatment of the paradoxes, see Felsenthal and Machover [6].

6.2 The Case $n = 4$

The previous sections have focused almost entirely on weighted voting systems with three voters. While most of the results are true for more than three voters, the geometry becomes more difficult to visualize, for obvious reasons. One possible project for those interested in investigating these areas more fully, might be to explore the geometry of four voters. For four voters, the two-dimensional simplex is replaced with the three-dimensional simplex $x+y+z+w=1$ which can be visualized as a tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. (Points in the interior of the tetrahedron satisfy $x+y+z<1$; thus w represents the distance from the boundary.) It is possible, by dividing the tetrahedron into several "slices," to analyze the interior of the tetrahedron by region as we did for the simplex, listing the winning coalitions, and power indices in each region. Visualizing the convex hull for four voters is more difficult, but the power indices can still be calculated as a linear combination of basis vectors. One problem that can be fun to explore is to investigate the number of different power rankings possible as the λ_S range over all possible values.

7 Selected Solutions

Solution to Exercise 1.1 Winning coalitions consist of the five permanent members and four or more of the non-permanent members. If any of the permanent members is not in a coalition, that coalition does not have sufficient weight to be a winning coalition.

Solution to Exercise 1.2 A coalition consisting of any one of the permanent members is a blocking coalition. One such example is: {China}. A coalition of at least 7 non-permanent members is also a blocking coalition.

Solution Exercise 1.3 The minimum number of votes is 7.

Solution to Exercise 1.11

1. the Banzhaf scores: $(2,2,0)$ and the Shapley-Shubik scores: $(3,3,0)$.
2. the Banzhaf scores: $(0,4,0)$ and the Shapley-Shubik scores: $(0,6,0)$.
3. the Banzhaf scores: $(2,2,2)$ and the Shapley-Shubik scores: $(2,2,2)$.
4. the Banzhaf scores: $(1,3,1)$ and the Shapley-Shubik scores: $(1,4,1)$.

Solution to Exercise 1.12 An example in which the weights are kept the same but an increase in the quota results in no change to the Banzhaf scores is the system: $[5:2,2,2]$ which has Banzhaf scores $(1,1,1)$. If the quota is increased to 6, but the weights are kept the same, the Banzhaf scores don't change. The system $[5:2,5,2]$ has Banzhaf scores $(0,4,0)$ as we saw in the second part of Exercise 1.11. If we change the quota to 6 and keep the weights the same, the Banzhaf scores change to: (1, 3, 1).

Solution to Exercise 1.13 The Shapley-Shubik score of a dummy voter in an n-voter weighted voting system is 0. The Shapley-Shubik score of a dictator in an n-voter weighted voting system is $n!$.

Solution to Exercise 1.14 Here are two possibilities: $[4 : 4, 1, 1]$ and $[6 : 6, 3, 2]$.

Solution to Exercise 1.19 Equivalent weighted voting systems have the same Banzhaf power indices. Because the winning and losing coalitions of two equivalent voting systems are the same, the proportion of times each voter is a critical voter is the same for the two systems. This is the BPI for each voter. Similarly, the SSPI for each voter is the same for equivalent voting systems.

Solution to Exercise 1.20 We present the examples of weighted voting systems with the voters' weights given in decreasing order. Examples of the "non-equivalent" three-voter weighted voting systems are:

1. $[2 : 1, 1, 1]$ with winning coalitions $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$, and $\{v_1, v_2, v_3\}$.
2. $[4 : 2, 2, 1]$ with winning coalitions $\{v_1, v_2\}$ and $\{v_1, v_2, v_3\}$.
3. $[5 : 5, 2, 2]$ with winning coalitions $\{v_1\}, \{v_1, v_2\}, \{v_1, v_3\}$, and $\{v_1, v_2, v_3\}$.
4. $[7 : 6, 5, 1]$ with winning coalitions $\{v_1, v_2\}, \{v_1, v_3\}$ and $\{v_1, v_2, v_3\}$. $\{v_1, v_2, v_3\}$.
5. $[3 : 1, 1, 1]$ with winning coalition $\{v_1, v_2, v_3\}$.

Solution to Exercise 1.23 Examples (2) and (6) from the solution to Exercise 1.20 are two non-equivalent weighted voting systems with the same BPI: $(1/2, 1/2, 0)$.

Solution to Exercise 1.24 If we compute the BPI for the different equivalent weighted voting systems found in Exercise 1.20 we see that possible BPI values for three-voter weighted voting systems are:

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{2}, 0\right), (1, 0, 0), \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right), \text{ and } (0, 0, 0).$$

Notice that in each of these systems, at least two voters have the same BPI.

Solution to Exercise 2.2

1. The normalized weight distribution is $(\frac{6}{20}, \frac{8}{20}, \frac{6}{20})$. Since $\tilde{w}_1 = \tilde{w}_3$, P lies on the perpendicular bisector between A and C. Since $\tilde{w}_2$ is greater than either $\tilde{w}_1$ or $\tilde{w}_3$, P lies closer to vertex B than side AC.
2. The normalized weight distribution is $(1, 0, 0)$. Since $\tilde{w}_1 = 1$ which is the largest value possible on the simplex (and $\tilde{w}_2, \tilde{w}_3 = 0$ are the smallest values possible), P coincides with point A.
3. The normalized weight distribution is $(\frac{1}{2}, \frac{1}{2}, 0)$. Since $\tilde{w}_1 = \tilde{w}_2$, P lies on the perpendicular bisector between A and B. Since $\tilde{w}_3 = 0$, point P is as far from vertex C as possible. Thus P lies on side AB, halfway between the two vertices.
4. The normalized weight distribution is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Since $\tilde{w}_1 = \tilde{w}_2 = \tilde{w}_3$, P lies at the point where all three perpendicular bisectors meet. This point is known as the *barycenter* of the simplex.

Solution to Exercise 2.3 Point P lies on one of the sides of the triangle if one its coordinates is equal to zero.

Solution to Exercise 2.4 Point P lies on one of the sides of the triangle if one its coordinates is equal to zero.

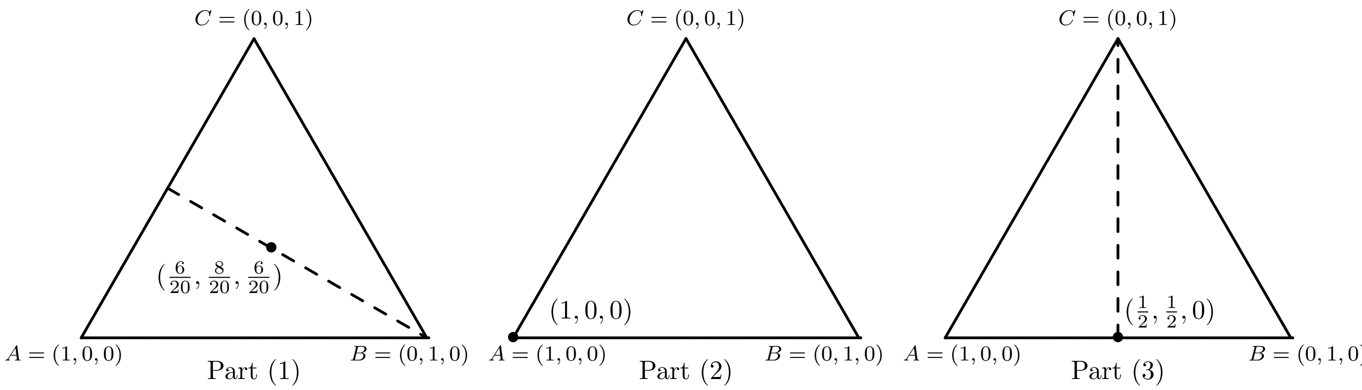


Figure 17: Solutions to Exercise 2.2

Solution to Exercise 2.5 Point P lies on one of the vertices of the triangle if two of its coordinates are equal to zero.

Solution to Exercise 2.7 1. Point P is at the barycenter, where the lines $\tilde{w}_1 = \tilde{w}_2$, $\tilde{w}_1 = \tilde{w}_3$ and $\tilde{w}_2 = \tilde{w}_3$ meet.

2. The line $\tilde{w}_1 = \frac{3}{11}$ has endpoints $(\frac{3}{11}, 0, \frac{8}{11})$ and $(\frac{3}{11}, \frac{8}{11}, 0)$. The line $\tilde{w}_2 = \frac{4}{11}$ has endpoints $(0, \frac{4}{11}, \frac{7}{11})$ and $(\frac{7}{11}, \frac{4}{11}, 0)$. Point P lies at the intersection of these lines.

3. The line $\tilde{w}_1 = \frac{1}{2}$ has endpoints $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$. The line $\tilde{w}_3 = \frac{3}{10}$ has endpoints $(0, \frac{7}{10}, \frac{3}{10})$ and $(\frac{7}{10}, 0, \frac{3}{10})$. Point P lies at the intersection of these lines.

Solution to Exercise 2.10 If a winning coalition must have at least $\frac{2}{3}$ of the total weight, a blocking coalition must have more than $1 - \frac{2}{3} = \frac{1}{3}$ of the total weight. The picture looks like the center triangle in Figure 9. Region R_{10} shrinks to the single point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Solution to Exercise 2.11 The picture looks like the rightmost triangle in Figure 9. In region R_{10}, the only winning coalition is $\{v_1, v_2, v_3\}$. The region might be called the "unanimity" region now since instead of just a majority of voters needed to form a winning coalition, you need all the voters in order to form a winning coalition.

Solution to Exercise 2.12 In region R_1, the winning coalitions are $\{v_1\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_1, v_2, v_3\}$. Voter v_1 is critical in all four winning coalitions and voters v_2 and v_3 are never critical, so the BPI is $(1, 0, 0)$. The regions R_2 and R_3 are the same as region R_1 except that the voters' roles reversed. Hence the BPI's in regions R_2 and R_3 are $(0, 1, 0)$ and $(0, 0, 1)$ respectively. In region R_4, the winning coalitions are $\{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$. Voter v_1 is never critical and voters v_2 and v_3 are critical twice, so the BPI is $(0, \frac{1}{2}, \frac{1}{2})$. By symmetry, BPI's in regions R_5 and R_6 are $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$ respectively. In region R_7, the winning coalitions are $\{v_1, v_2\}, \{v_1, v_3\}$ and $\{v_1, v_2, v_3\}$. Player v_1 is critical three times, and players v_2 and v_3 are critical twice, so the BPI is $(\frac{3}{7}, \frac{2}{7}, \frac{2}{7})$. By symmetry, BPI's in regions R_8 and R_9 are $(\frac{2}{7}, \frac{3}{7}, \frac{2}{7})$ and $(\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$ respectively. And in region R_{10}, the winning coalitions

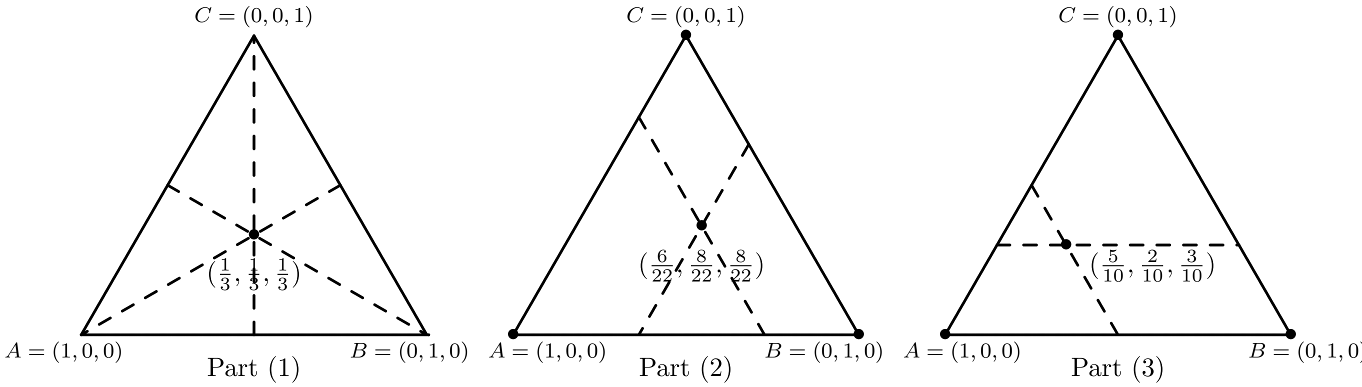


Figure 18: Solutions to Exercise 2.7

are $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$. Each player is critical one time, so the BPI is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Solution to Exercise 3.2 BPI and SSPI for the weighted voting system $[3 : 2, 1, 1]$ lie on the line $p_2 = p_3$.

Solution to Exercise 3.5

Region	Power Ranking
Q_2	$p_1 > p_3 > p_2$
Q_3	$p_3 > p_1 > p_2$
Q_4	$p_3 > p_2 > p_1$
Q_5	$p_2 > p_3 > p_1$
Q_6	$p_2 > p_1 > p_3$

Solution to Exercise 3.6 All of the indexes fall on one of the indifference lines $p_i = p_j, i \neq j, i, j = 1, 2, 3$.

1. The BPI and SSPI both fall on the line $p_1 = p_3$, and on the line joining vertices A and B.
2. The BPI and SSPI both lie at vertex B.

Solution to Exercise 3.7 As the power index moves towards a vertex, the power of the voter corresponding to that vertex increases while the powers of the other voters decrease.

Solution to Exercise 3.8 The center point is the point where the powers of the three voters is equal: $p_1 = p_2 = p_3$.

Solution to Exercise 4.1

1. Since $v_1 + v_2 = 7 \geq 5$, S is a winning coalition. Hence $\nu(S) = 1$.
2. $|S| = 2$.
3. $S - \{v_2\} = \{v_1, v_2\} - \{v_2\} = \{v_1\}$.

4. Since $S - \{v_2\} = \{v_1\}$ which is not a winning coalition, $\nu(S - \{v_2\}) = 0$. Hence $\nu(S) - \nu(S - \{v_2\}) = 1$.

Solution to Exercise 4.4 Since each voter has equal weight, each voter will have equal power, regardless of the power index. Since there are three voters, both the BPI and SSPI of a specific voter will be 1/3. To see this for the specific voter v_2 using the general definition, note that in order for

$$\nu(S) - \nu(S - \{v_2\})$$

to be non-zero, S must be a winning coalition in which v_2 is critical (so that $\nu(S) = 1$ but $\nu(S - \{v_2\}) = 0$). The only coalitions that satisfy this are $S = \{v_1, v_2\}$ and $S = \{v_2, v_3\}$. Thus Banzhaf score for v_2 is

$$1\dot{(}1 - 0) + 1\dot{(}1 - 0) = 2.$$

Similarly, the Banzhaf scores for v_1 and v_3 are also 2. We normalize the score by dividing by 6, hence the BPI of v_2 is $p_2 = \frac{2}{6} = \frac{1}{3}$. To find the SSPI, since $|S| = 2$ for both coalitions, $\lambda_{|S|} = \frac{(2-1)!(3-2)!}{3!} = \frac{1}{6}$. Hence

$$p_2 = \frac{1}{6} \cdot (1 - 0) + \frac{1}{6} \cdot (1 - 0) = \frac{1}{3}.$$

Solution to Exercise 4.6 The dictatorial power index gets its name because it only assigns weight to coalitions with one voter. That is, when calculating the power of player v_i, the only non-zero term in the sum is $\nu(\{i\}) - \nu(\emptyset)$. In general, this quantity will be equal to zero, meaning that player v_i has zero dictatorial power. The only time it will not be equal to zero is if $\nu(\{i\}) = 1$, in which case player v_i is a dictator. The marginal power index gets its name because it only assigns weight to the coalition consisting of all voters. Thus, v_i has non-zero power if and only if $\nu(\{v_1, \ldots, v_n\}) - \nu(\{v_1, \ldots, v_n\} \backslash \{v_i\}) = 1$. In general this quantity will be equal to zero, meaning that v_i has zero marginal power. The only time it will not be equal to zero is if v_i is critical in the coalition consisting of all voters. This would occur only if the voter has veto power, since a coalition of all voters except that voter would be a losing coalition.

Solution to Exercise 4.7 As explained in the previous exercise, the only non-zero term in the sum for the *dictatorial* power of v_2 is $\nu(\{v_2\}) - \nu(\emptyset)$. Hence $p_2 = 1 \cdot (0 - 0) = 0$. The only non-zero term in the sum for the *marginal* power of v_2 is $\nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_3\})$. Hence $p_2 = 1 \cdot (1 - 1) = 0$.

Solution to Exercise 4.11

$$\overline{P_1} = \begin{bmatrix} \nu(\{v_1\}) \\ \nu(\{v_2\}) \\ \nu(\{v_3\}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\overline{P_2} = \begin{bmatrix} \nu(\{v_1, v_2\}) - v(\{v_2\}) + \nu(\{v_1, v_3\}) - v(\{v_3\}) \\ \nu(\{v_1, v_2\}) - v(\{v_1\}) + \nu(\{v_2, v_3\}) - v(\{v_3\}) \\ \nu(\{v_1, v_3\}) - v(\{v_1\}) + \nu(\{v_2, v_3\}) - v(\{v_2\}) \end{bmatrix} = \begin{bmatrix} 1 - 0 + 0 - 0 \\ 1 - 0 + 0 - 0 \\ 0 - 0 + 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\overline{P_3} = \begin{bmatrix} \nu(\{v_1, v_2, v_3\}) - \nu(\{v_2, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_3\}) \\ \nu(\{v_1, v_2, v_3\}) - \nu(\{v_1, v_2\}) \end{bmatrix} = \begin{bmatrix} 1 - 0 \\ 1 - 0 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

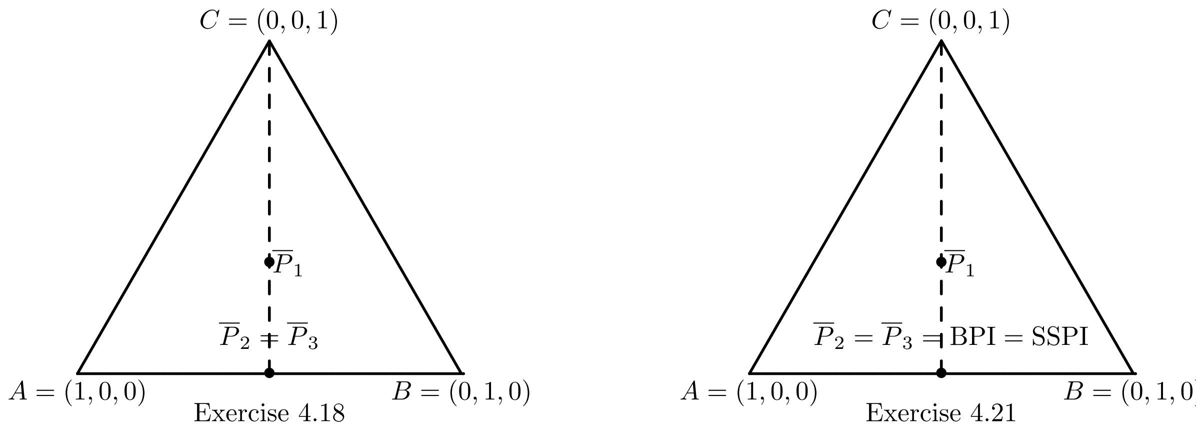


Figure 19: Solutions to Exercise 4.18 and 4.21

Solution to Exercise 4.14 For the Banzhaf scores,

$$\overline{P} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}.$$

For the SSPI,

$$\overline{P} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

Solution to Exercise 4.17 The vectors all lie along a line.

Solution to Exercise 4.18 See Figure 19.

Solution to Exercise 4.20 All the points lie along a line.

Solution to Exercise 4.21 See Figure 19 above. The BPI and SSPI correspond to $\overline{P}_1$ and $\overline{P}_2$.

Solution to Exercise 4.23

$$\overline{\hat{P}_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \overline{\hat{P}_2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \quad \overline{\hat{P}_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Hence for the BPI and SSPI

$$\overline{P} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{4}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{5}{4} \\ 0 \end{bmatrix}, \quad \text{and} \quad \overline{P} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

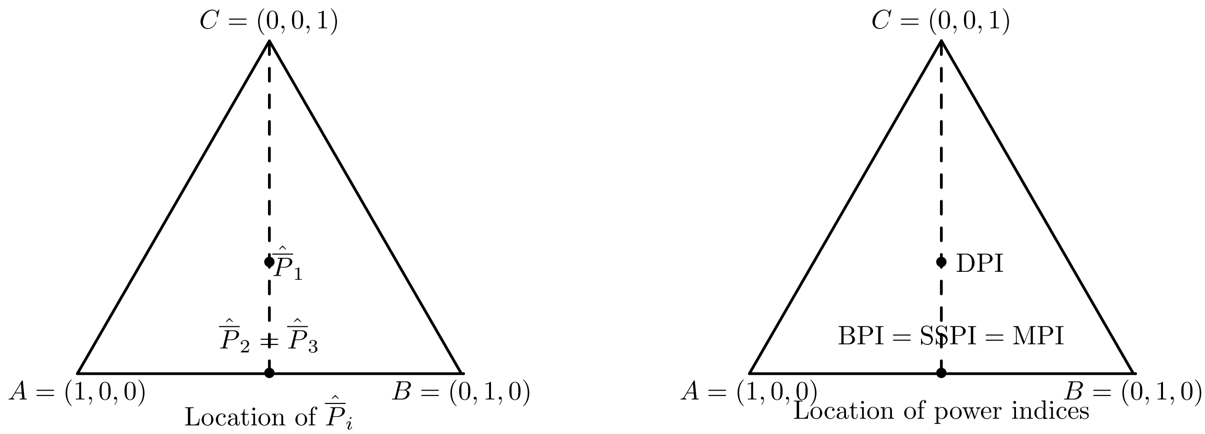


Figure 20: Solution to Exercise 4.23

For the DPI and MPI

$$\overline{P} = 1\cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0\cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + 0\cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \overline{P} = 0\cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0\cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + 1\cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

When all these power indices are normalized, we have

Solution to Exercise 5.4

$$\overline{P}_1 = [0.1, 0.2, 0.1],\ \overline{P}_2 = [0.5, 0.7, 0.8],\ \overline{P}_3 = [0.4, 0.5, 0.7],$$

$$\overline{P} = [1.0, 1.4, 1.6], \quad \text{BPI} = \text{SSPI} = [0.25, 0.35, 0.40].$$

Solution to Exercise 5.6 The Banzhaf scores are

$$\begin{aligned} \hat{\mathbf{P}} &= \hat{\gamma}_1 \hat{\overline{\mathbf{P}}}_1 + \hat{\gamma}_2 \hat{\overline{\mathbf{P}}}_2 + \hat{\gamma}_3 \hat{\overline{\mathbf{P}}}_3 \\ &= \frac{1}{4} \begin{bmatrix} 0.2 \\ 0.15 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0.3 \\ 0.425 \\ 0.375 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 0.3 \\ 0.6 \\ 0.65 \end{bmatrix} \\ &= \begin{bmatrix} 0.275 \\ 0.4 \\ 0.35 \end{bmatrix}. \end{aligned}$$

If we normalize this, we get $\begin{bmatrix} 0.268 \\ 0.390 \\ 0.341 \end{bmatrix}$.

The SSPI is

$$\hat{\mathbf{P}} = \hat{\gamma}_1 \hat{\overline{\mathbf{P}}}_1 + \hat{\gamma}_2 \hat{\overline{\mathbf{P}}}_2 + \hat{\gamma}_3 \hat{\overline{\mathbf{P}}}_3$$

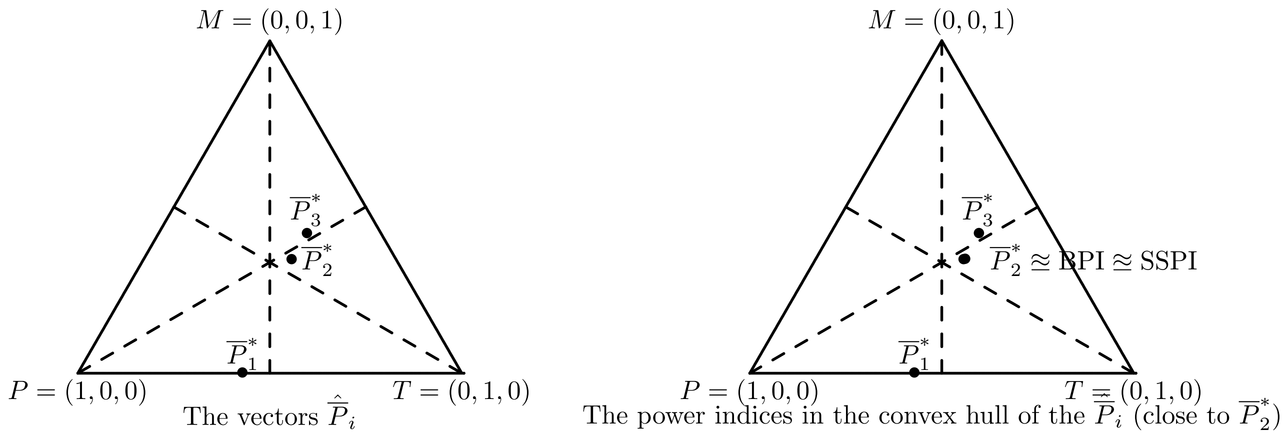


Figure 21: Solution to Exercise 5.6

$$= \frac{1}{3}\begin{bmatrix} 0.2 \\ 0.15 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0.3 \\ 0.425 \\ 0.375 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 0.3 \\ 0.6 \\ 0.65 \end{bmatrix}$$

$$= \begin{bmatrix} 0.267 \\ 0.392 \\ 0.342 \end{bmatrix}.$$

This is very close to the BPI.

Solution to Exercise 6.2 In the original weighted voting system $[7 : 4, 4, 4]$, all players have equal weight and hence the SSPI is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In the weighted voting system $[7 : 1, 6, 5]$, the SSPI is $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$. Again, the power of v_3 has been reduced even though its percentage of the total weight has increased.

Solution to Exercise 6.3 The right-most drawing in Figure 9 corresponds to weighted voting systems in which the quota is more than two-thirds of the total weight. In R_{10}, the only winning coalition is $\{v_1, v_2, v_3\}$, so the BPI is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In R_5, the winning coalitions are $\{v_1, v_3\}$ and $\{v_1, v_2, v_3\}$. In these coalitions only v_1 and v_3 are critical, hence the BPI is $(\frac{2}{4}, 0, \frac{2}{4}) = (\frac{1}{2}, 0, \frac{1}{2})$. In the previous exercise, the Paradox of Redistribution occurred when a voter's power *decreased* even though their fraction of the weight *increased*. This cannot happen here, since the only voter whose power decreases by moving to R_5 is v_2, yet by moving to R_5, v_2 will simultaneously lose weight as well. We can look for a different instance of the Paradox of Redistribution by finding a voter, (such as v_1), whose power *increases* while their power *decreases*.

To make it easy, start with weighted voting system where all the voters have equal weight and the quota is more than two-thirds of the weight, such as $[11 : 5, 5, 5]$. Since a winning coalition must have at least eleven-fifteenths of the weight, a blocking coalition must have at least four-fifteenths of the weight. Thus R_5 corresponds to the region $\tilde{w}_2 \leq \frac{4}{15}$, $\tilde{w}_1 \geq \frac{4}{15}$ and

$\tilde{w}_3 \geq \frac{4}{15}$. We are looking for a second voting system that satisfies these constraints and in which $\tilde{w}_1$ decreases. One possibility is $\tilde{w}_1 = \frac{4.5}{15}$, $\tilde{w}_2 = \frac{3.5}{15}$ and $\tilde{w}_3 = \frac{7}{15}$. This corresponds to the weighted voting system $[11 : 4.5, 3.5, 7]$. (There will be many other answers.)

Solution to Exercise 6.5 The SSPI for the weighted voting system $[3; 1, 1, 2]$ is $(\frac{1}{6}, \frac{1}{6}, \frac{4}{6})$. If v_1 and v_2 quarrel, then list of sequential orderings of possible coalitions is reduced to v_1v_3, v_3v_1, v_2v_3, and v_3v_2. In each case, the second voter in the sequence is pivotal, so the SSPI is $(\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$. Thus the power of v_1 and v_2 has increased despite their 'quarrel'.

8 Supplementary Reading and References

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