Graph-theoretical Models of the Spread and Control of Disease and of Fighting Fires

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Image credits:
Ebola: Army Medicine
Forest fire: USFS Region 5
No changes made in any image

DIMACS
Center for Discrete Mathematics & Theoretical Computer Science
Founded as a National Science Foundation Science and Technology Center

Happy 80th Birthday Boris!

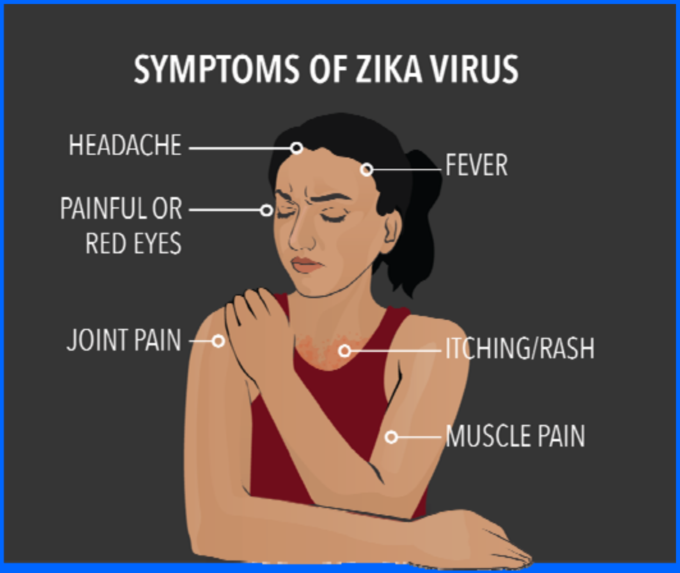
Spread and Control of Disease

•The spread of COVID-19 is just the latest and most devastating example of a newly emerging disease that threatens not only lives but our economy and our social systems.

Image credit: Wikimedia commons
https://www.youtube.com/watch?v=SBboFVjLQak , 1:10
Chinanews.com/China News Service
Unchanged

Spread and Control of Disease

Ebola



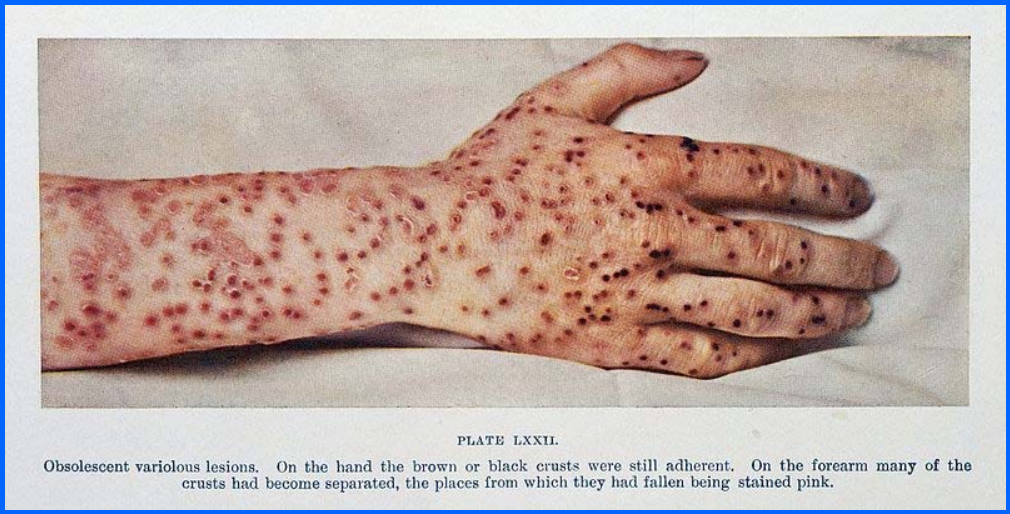
Zika

- Ebola, Zika are other recent examples

Image credits: Wikimedia commons
Ebola treatment unit: CDC Global; Zika: Beth.herlin no changes made

Mathematical Models of Disease Spread

Mathematical models of infectious diseases go back to Daniel Bernoulli's mathematical analysis of smallpox in 1760.



Smallpox

Image credit:
https://wellcomeimages.org/indexplus/obf_images/b2/a8/9ca500938fc44f77d4c4e49a4d90.jpg

Mathematical models have become important tools in analyzing the spread and control of infectious diseases, especially when combined with powerful, modern computer methods for analyzing and/or simulating the models. They have played a nontrivial role in the fight against COVID-19.

Bubonic Plague

AIDS

Credit for both: CDC

Great concern about the deliberate introduction of diseases by bioterrorists has led to new challenges for mathematical modelers.

anthrax

Credit: CDC

Spread and Control of Disease

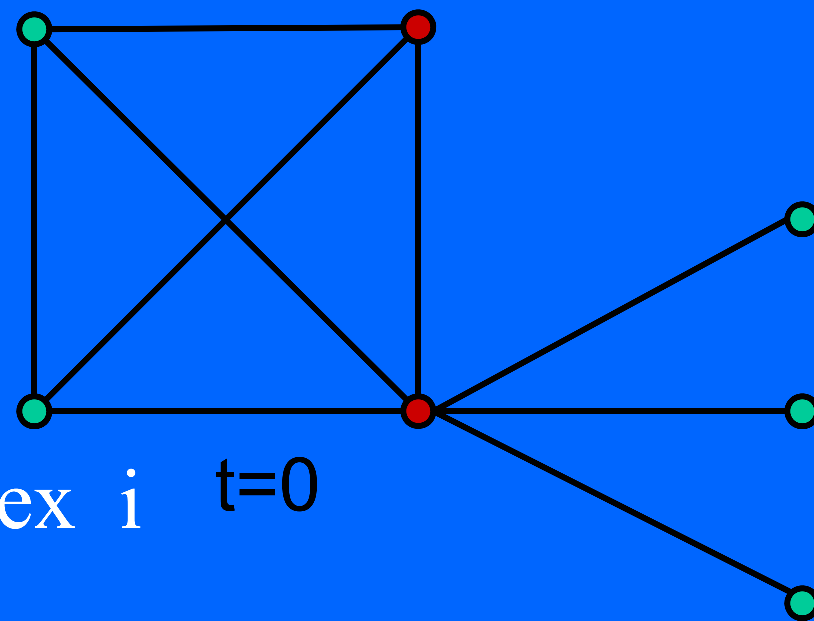
•Modern transportation systems allow for rapid spread of diseases.
•Diseases are spread through social networks.
•"*Contact tracing*" is an important part of any strategy to combat outbreaks of infectious diseases, whether naturally occurring or resulting from bioterrorist attacks.
•I will illustrate the ideas with some fairly simple ”toy” models that lead to fascinating graph-theoretical problems.
•***They reflect key themes of this conference and Boris' work: data analysis, optimization, and applications***
•The emphasis is on the graph theory.
•However, even toy models for spread of disease lead to interesting insights.

The Model: Moving From State to State

Social Network = Graph

Vertices = People

Edges = contact



Let $s_i(t)$ give the state of vertex i at time t.

Simplified Model: Two states:

= susceptible, = infected (SI Model)

Times are discrete: $t = 0, 1, 2, \ldots$

The Model: Moving From State to State

More complex models: SI, SEI, SEIR, etc.

measles

S = susceptible, E = exposed, I = infected, R = recovered (or removed)

SARS

Credit: measles: Wikimedia.org
SARS: Medical News Today

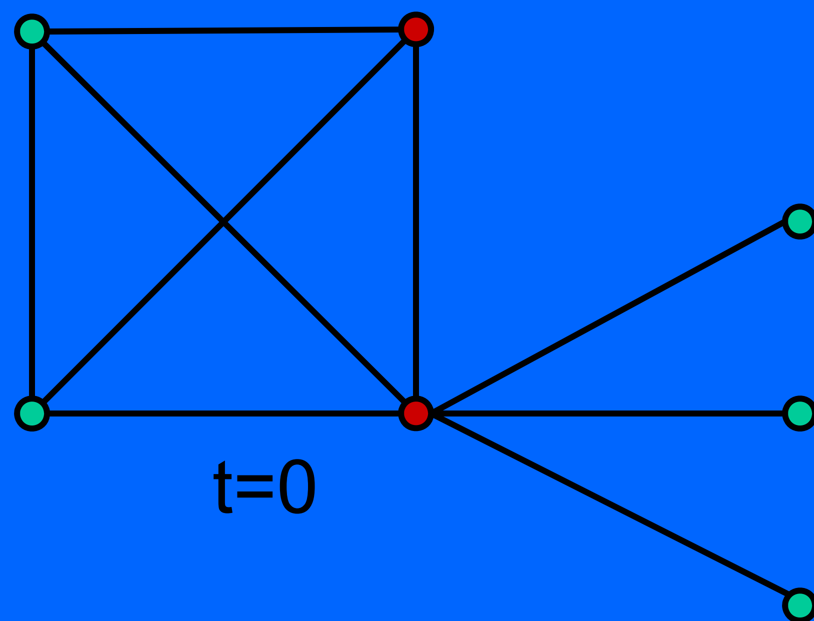
Threshold Processes

Irreversible k-Threshold Process: You change your state from ● to ● at time t+1 if at least k of your neighbors have state ● at time t. You never leave state ● .

Disease interpretation? Infected if sufficiently many of your neighbors are infected.

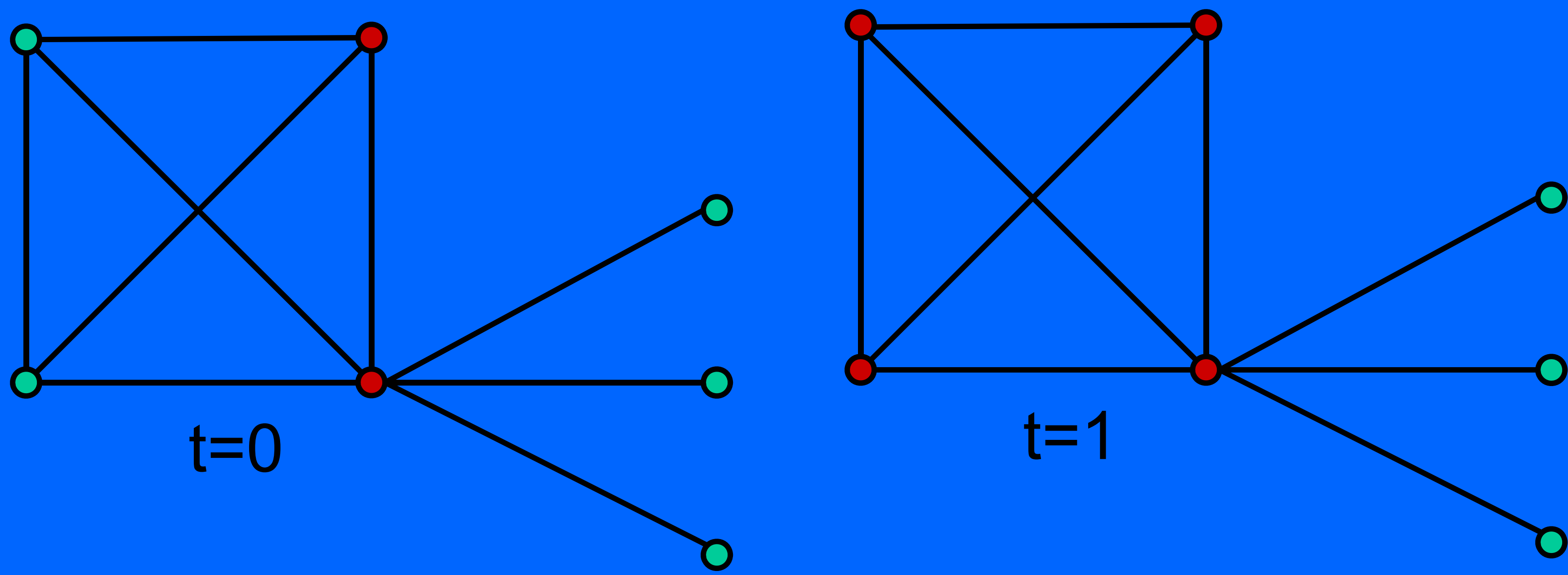
Special Case k = 1: Infected if any of your neighbors is infected.

Irreversible 2-Threshold Process

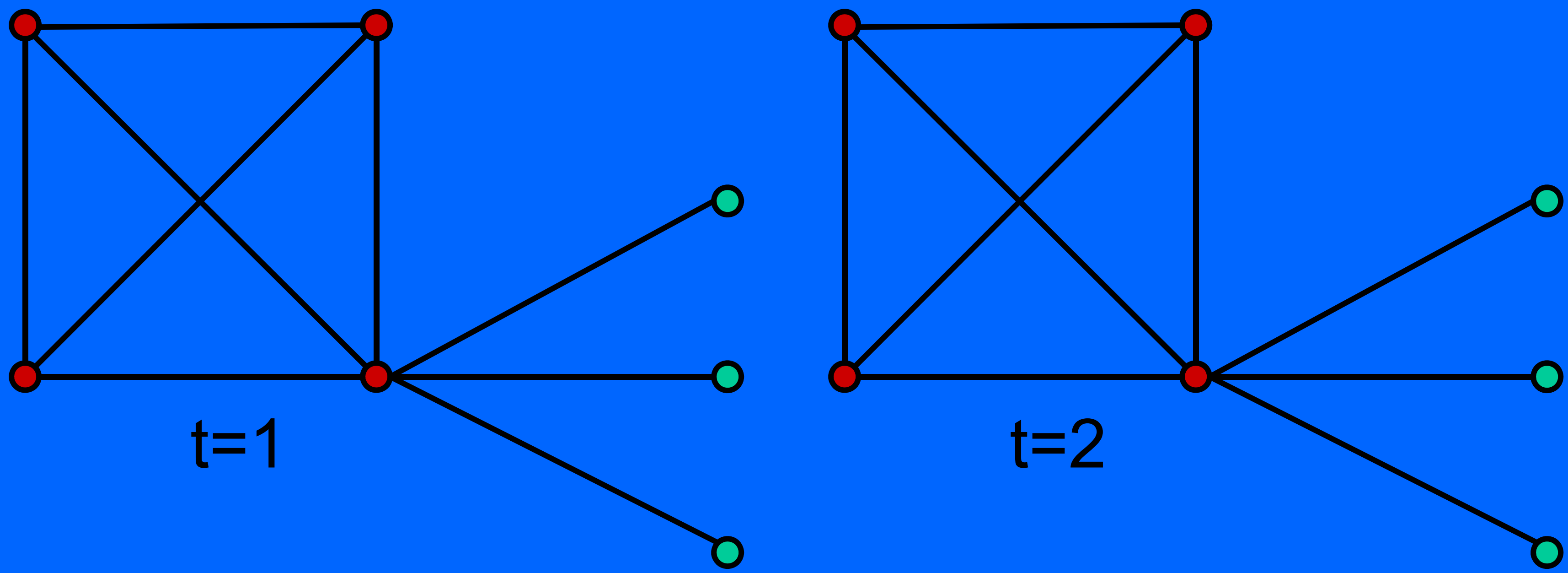


t=0

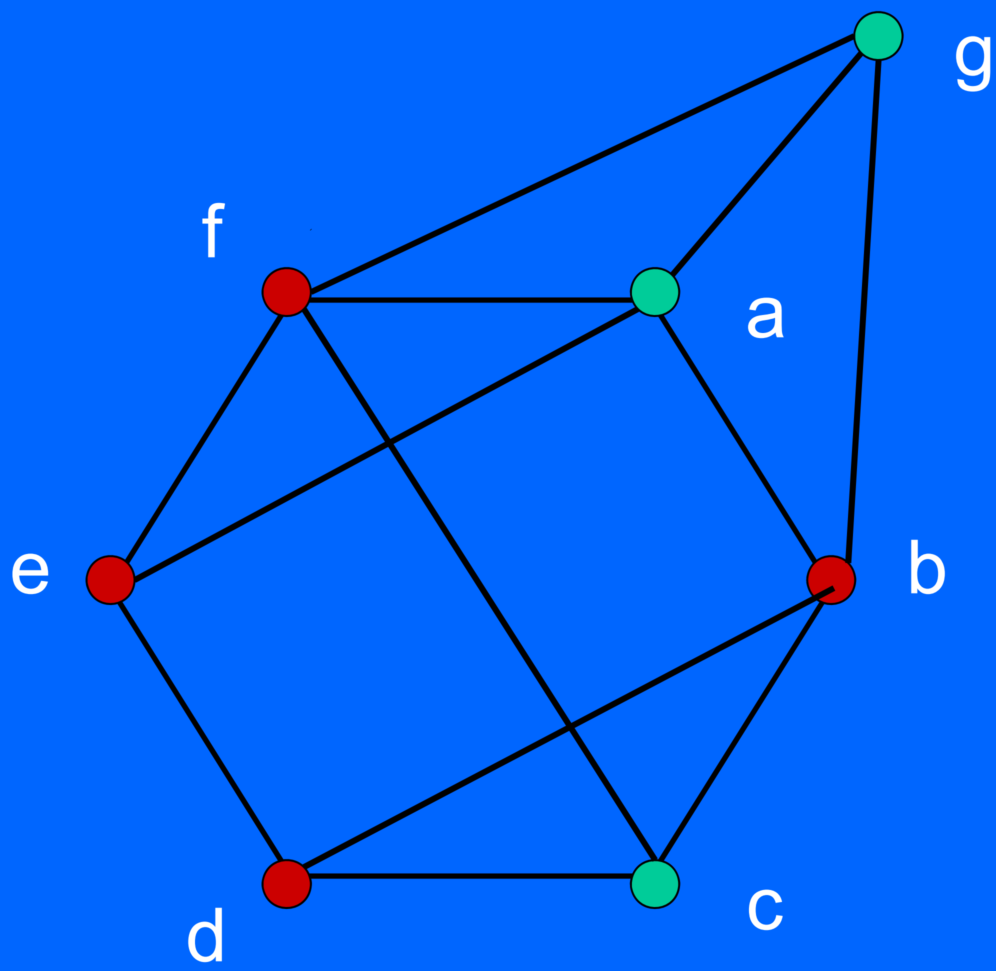
Irreversible 2-Threshold Process



Irreversible 2-Threshold Process

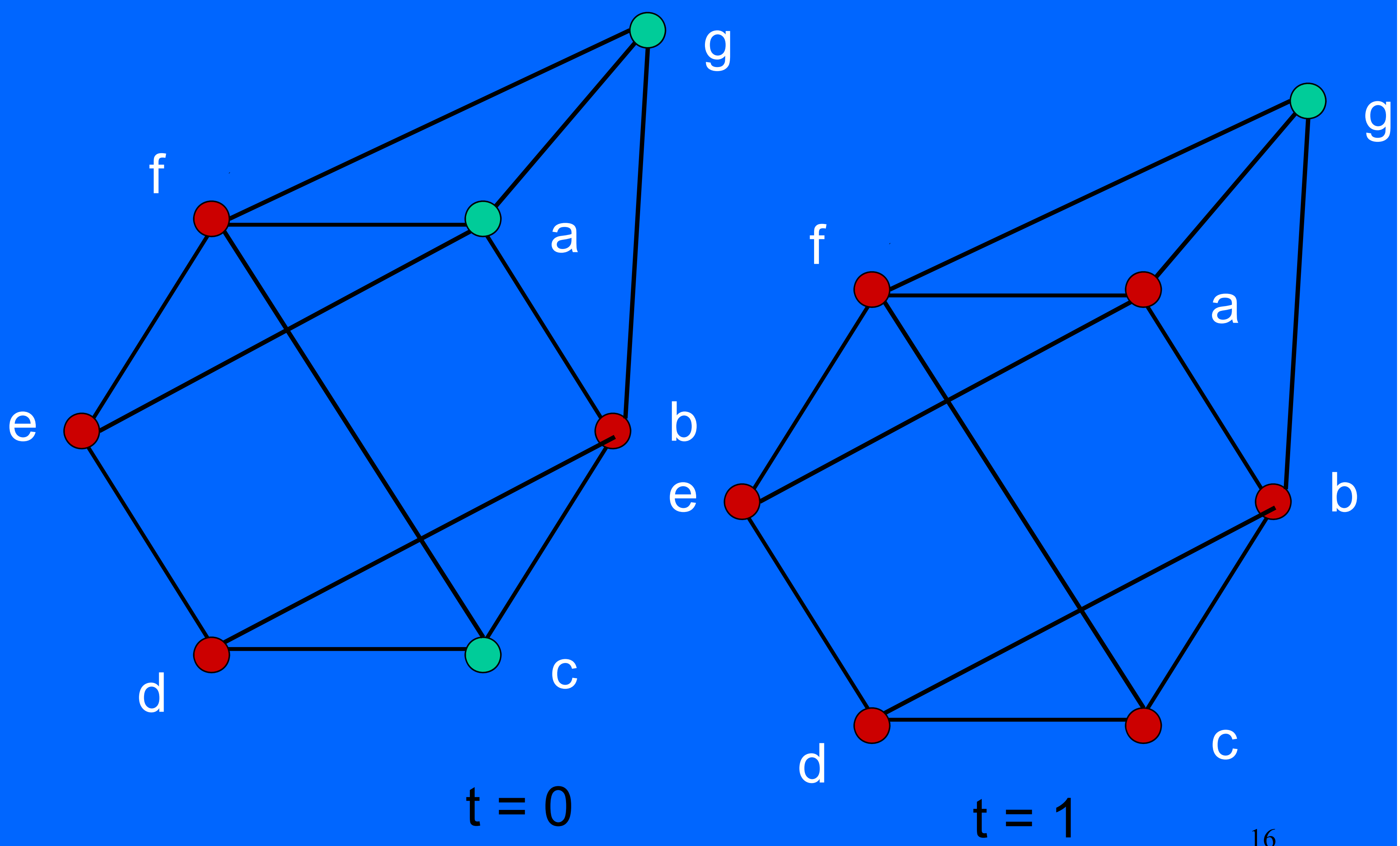


Irreversible 3-Threshold Process

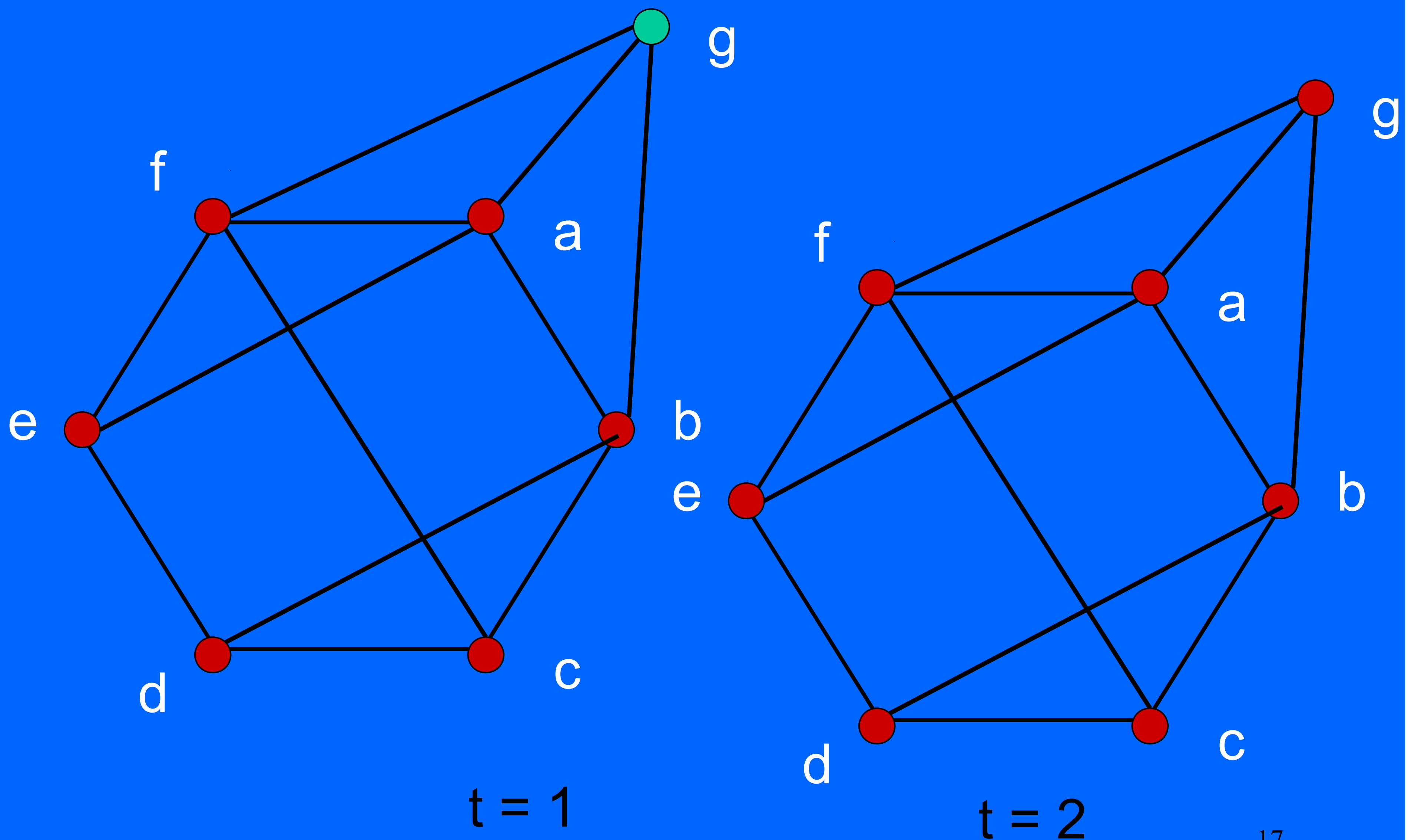


t = 0

Irreversible 3-Threshold Process



Irreversible 3-Threshold Process



Complications to Add to Model

- k = 1, but you only get infected with a certain probability.
- k could be different for different vertices.
- You are automatically cured after you are in the infected state for d time periods.
- A public health authority has the ability to "vaccinate" a certain number of vertices, making them immune from infection.

COVID-19 vaccination queue

Credit: Wikimedia commons, Ganesh Dhamodkar, no changes

The Saturation Problem

Attacker's Problem: Given a graph, what subsets S of the vertices should we plant a disease with so that ultimately the maximum number of people will get it?

Economic interpretation: What set of people do we place a new product with to guarantee "saturation" of the product in the population?

Defender's Problem: Given a graph, what subsets S of the vertices should we vaccinate to guarantee that as few people as possible will be infected?

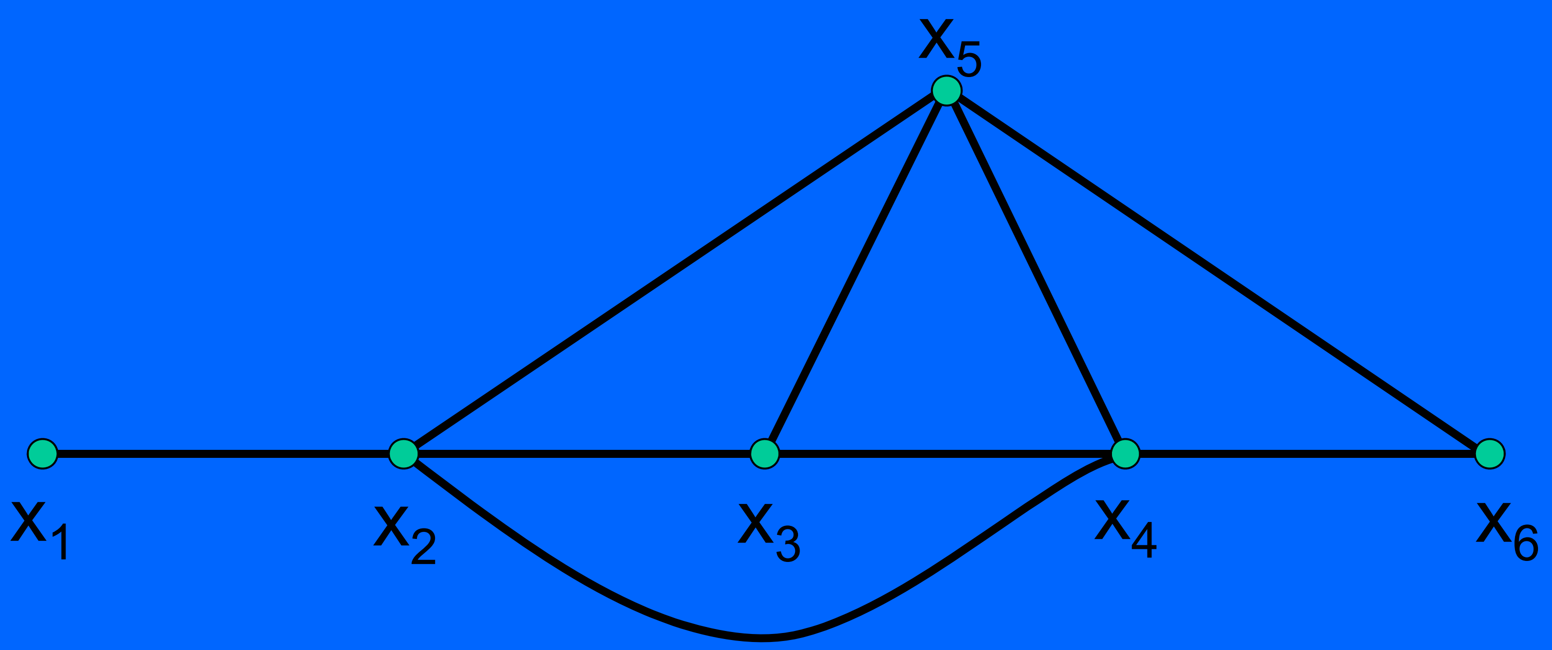
k-Conversion Sets

Attacker's Problem: Can we guarantee that ultimately everyone is infected?

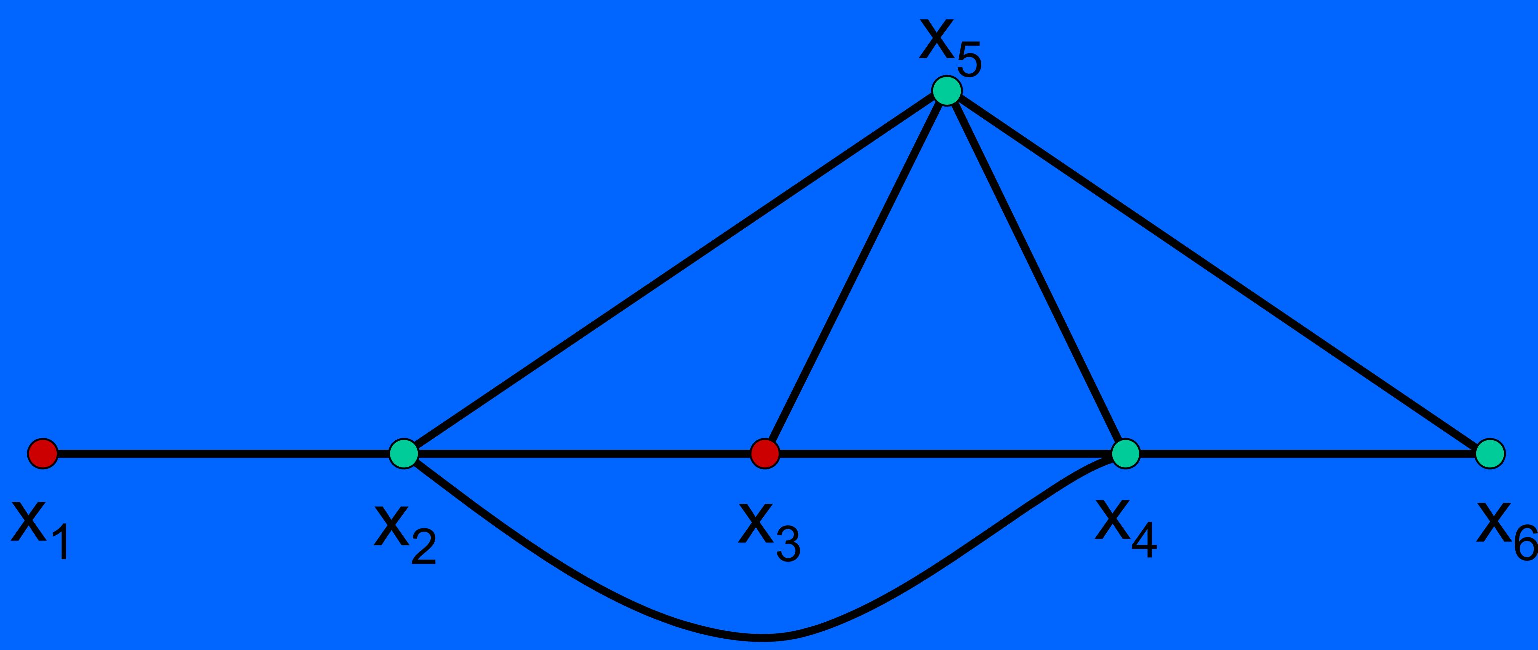
Irreversible k-Conversion Set: Subset S of the vertices that can force an irreversible k-threshold process to the situation where every state $s_i(t)$ = ●

Comment: If we can change back from ● to ○ at least after awhile, we can also consider the Defender's Problem: Can we guarantee that ultimately no one is infected, i.e., all $s_i(t)$ = ○?

What is an irreversible 2-conversion set for the following graph?

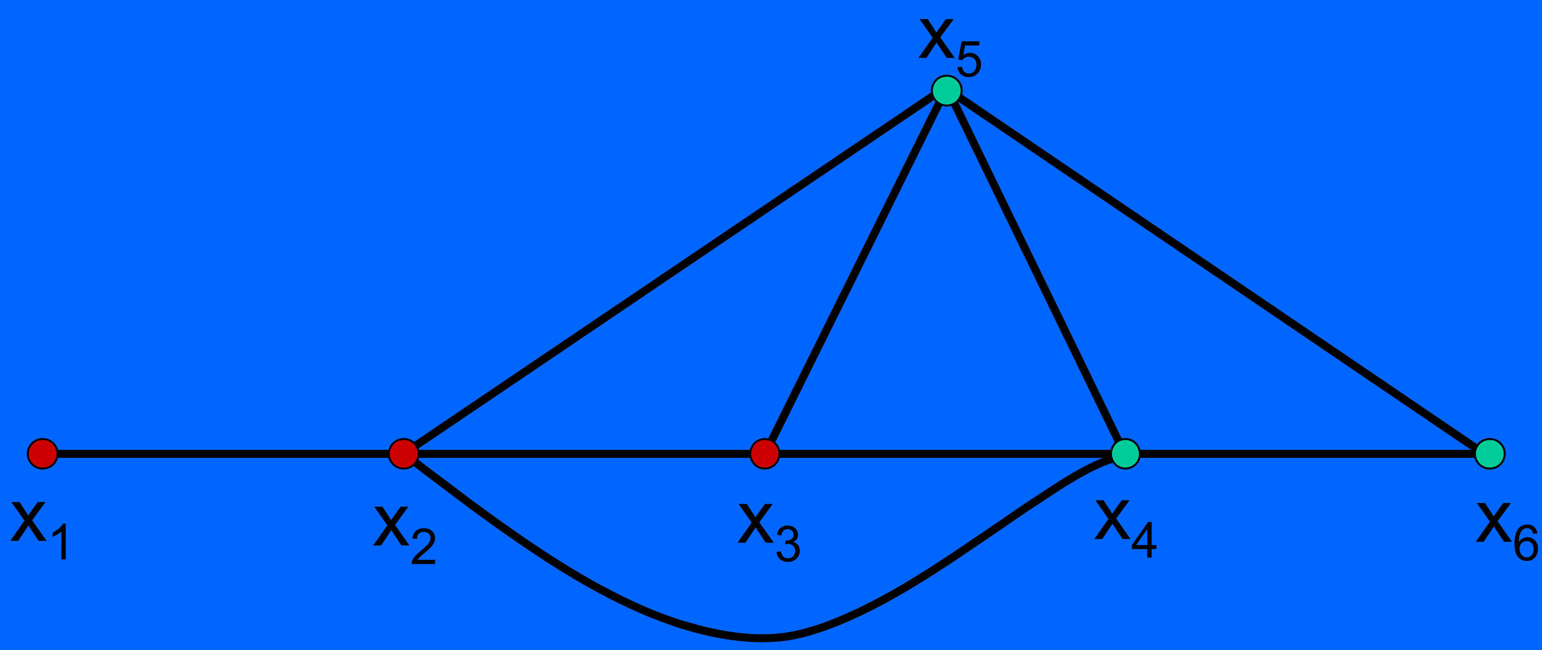


x_1, x_3 is an irreversible 2-conversion set.



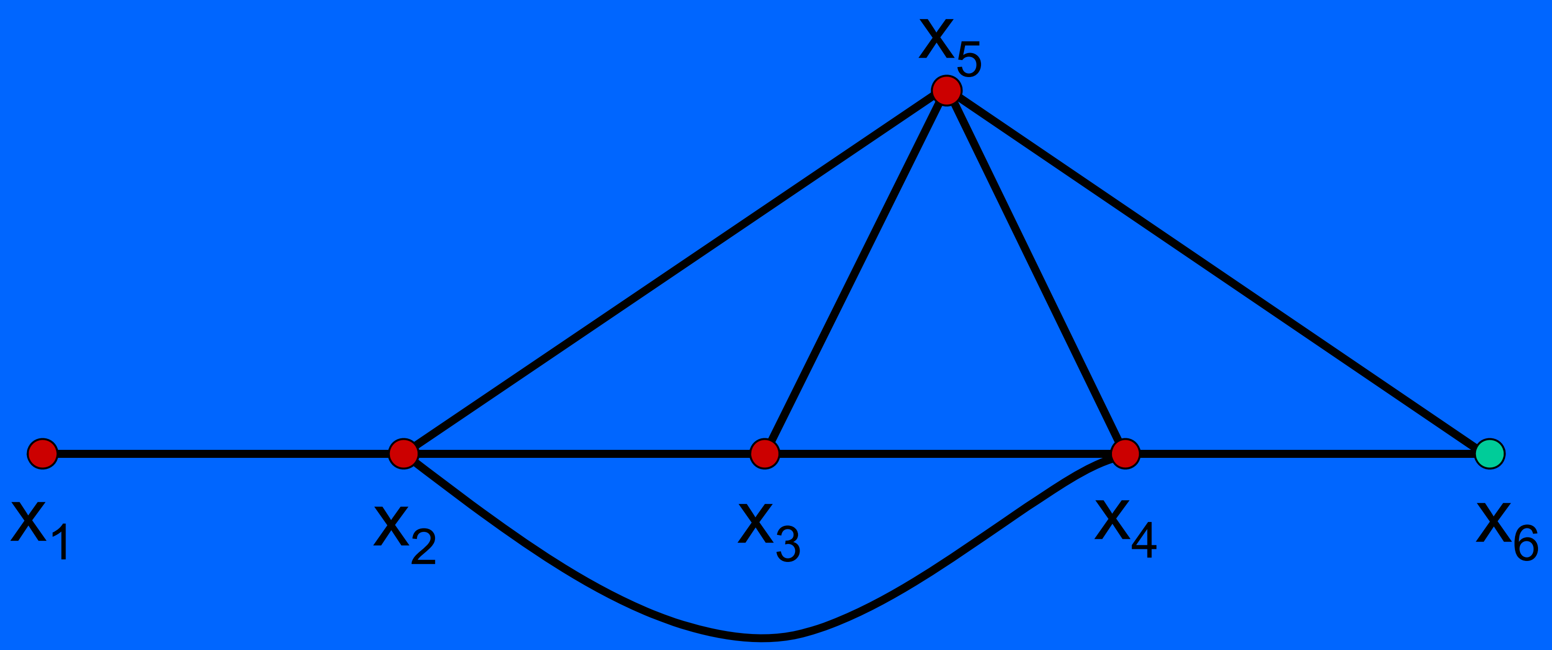
t = 0

x_1, x_3 is an irreversible 2-conversion set.



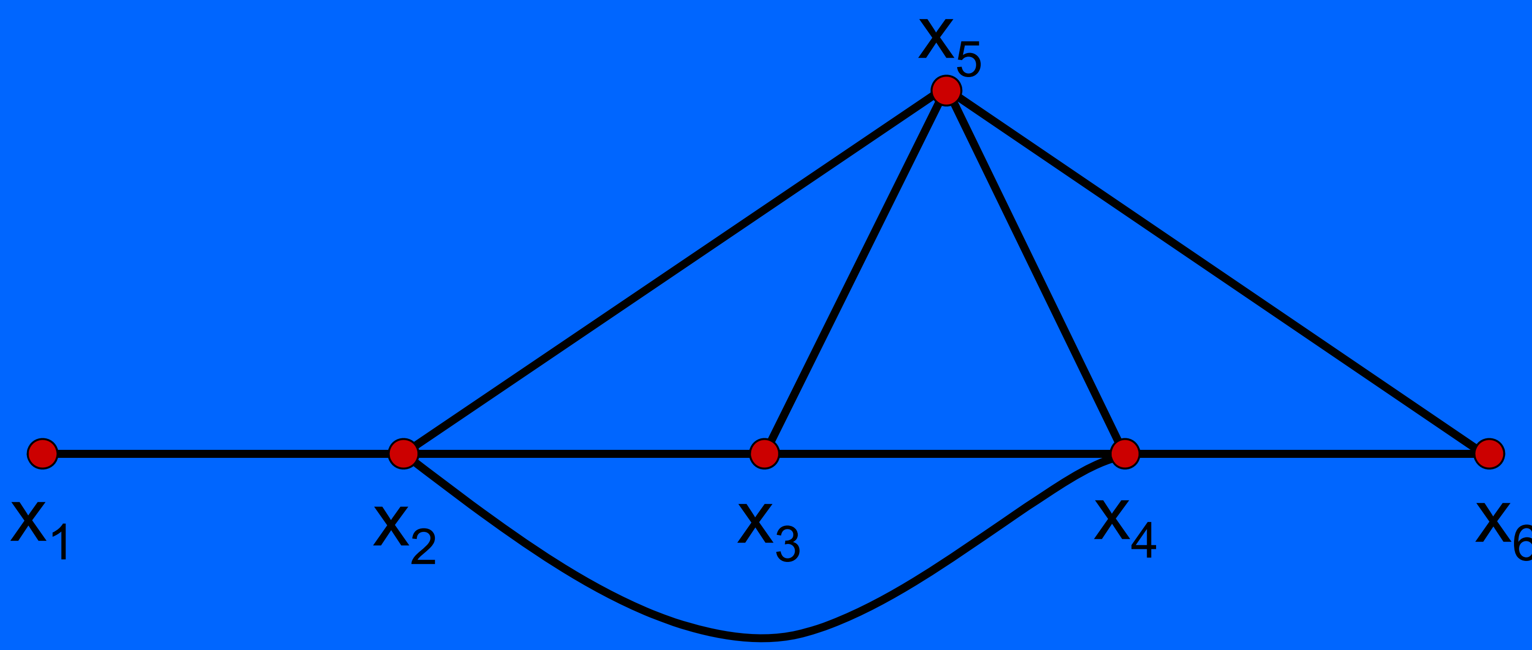
t = 1

x_1, x_3 is an irreversible 2-conversion set.



t = 2

x_1, x_3 is an irreversible 2-conversion set.



t = 3

How Hard is it to Find out if There is an Irreversible k-Conversion Set of Size at Most p?

Problem **IRREVERSIBLE k-CONVERSION SET**: Given a positive integer p and a graph G, does G have an irreversible k-conversion set of size at most p?

How hard is this problem?

Difficulty of Finding Irreversible Conversion Sets

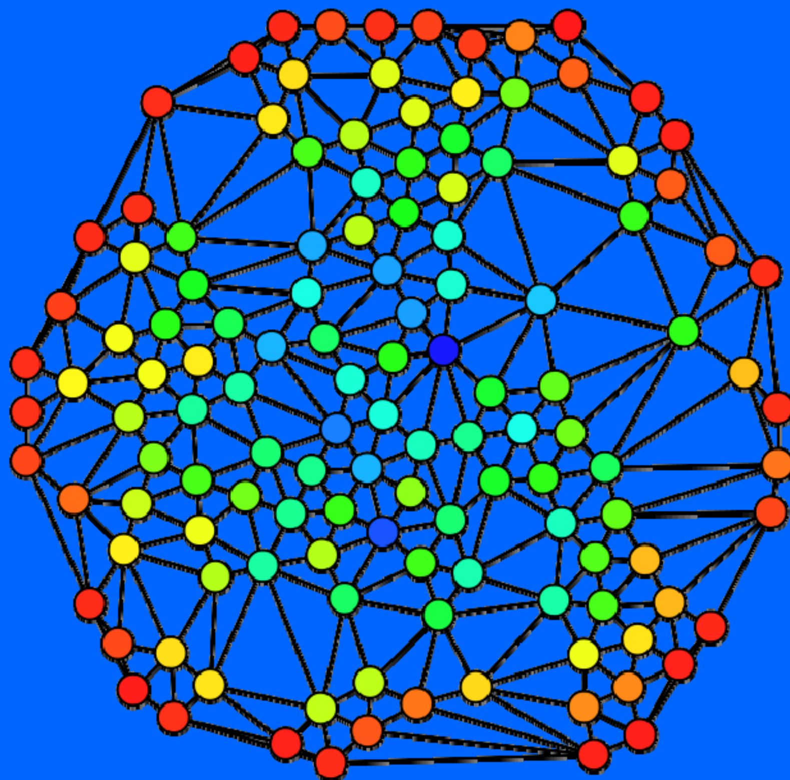
Problem **IRREVERSIBLE k-CONVERSION SET**: Given a positive integer p and a graph G, does G have an irreversible k-conversion set of size at most p?

Theorem (Dreyer): IRREVERSIBLE k-CONVERSION SET is NP-complete for fixed $k > 2$.

Theorem (Kyncl, Lidicky, Vyskocil): IRREVERSIBLE k-CONVERSION SET is NP-complete for $k = 2$ even for graphs of maximum degree 4.

Irreversible k-Conversion Sets in Special Graphs

Studied for many special graphs.



Credit: Wikimedia commons, Claudio Rocchini, no changes

Irreversible k-Conversion Sets in Special Graphs

Studied for many special graphs.

G is ***r-regular*** if every vertex has degree r.
Set of vertices is ***independent*** if there are no edges.

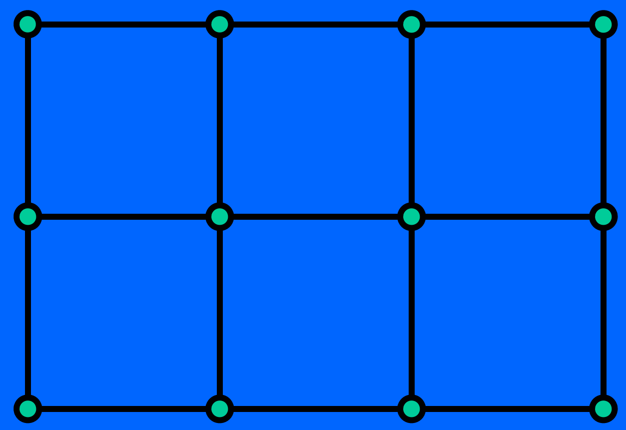
Theorem (Dreyer): Let G = (V,E) be a connected r-regular graph and D be a set of vertices. Then D is an irreversible r-conversion set iff V-D is an independent set.

Note: same r

Irreversible k-Conversion Sets in Special Graphs

Studied for many special graphs.

Let G(m,n) be the ***rectangular grid graph*** with m rows and n columns.

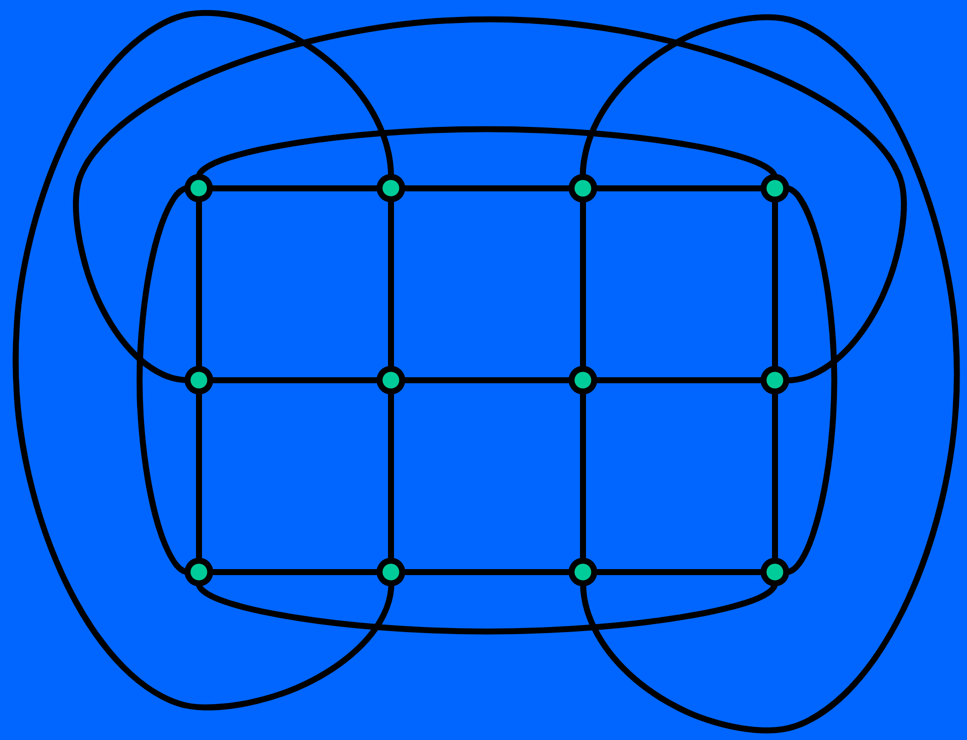


G(3,4)

Toroidal Grids

The ***toroidal grid*** T(m,n) is obtained from the rectangular grid G(m,n) by adding edges from the first vertex in each row to the last and from the first vertex in each column to the last.

Toroidal grids are easier to deal with than rectangular grids because they form regular graphs: Every vertex has degree 4. Thus, we can make use of the results about regular graphs.



T(3,4)

Irreversible4-Conversion Sets in Toroidal Grids

Theorem (Dreyer): In a toroidal grid T(m,n), the size of the smallest irreversible 4-conversion set is

$$\begin{cases} \max\{n(\text{ceiling}[m/2]), m(\text{ceiling}[n/2])\} & \text{m or n odd} \\ mn/2 & \text{m, n even} \end{cases}$$

Part of the Proof: Recall that D is an irreversible 4-conversion set in a 4-regular graph iff V-D is independent.

V-D independent means that every edge {u,v} in G has u or v in D. In particular, the ith row must contain at least ceiling[n/2] vertices in D and the ith column at least ceiling[m/2] vertices in D (alternating starting with the end vertex of the row or column).

We must cover all rows and all columns, and so need at least max{n(ceiling[m/2]), m(ceiling[n/2])} vertices in an irreversible 4-conversion set.

Irreversible k-Conversion Sets for Rectangular Grids

Let $C_k(G)$ be the size of the smallest irreversible k-conversion set in graph G.

Theorem (Dreyer):

$$C_4[G(m,n)] = 2m + 2n - 4 + \text{floor}[(m-2)(n-2)/2]$$

Theorem (Flocchini, Lodi, Luccio, Pagli, and Santoro):

$$C_2[G(m,n)] = \text{ceiling}([m+n]/2)$$

Irreversible 3-Conversion Sets for Rectangular Grids

For 3-conversion sets, the best we have are bounds:

<u>Theorem (Flocchini, Lodi, Luccio, Pagli, and Santoro):</u>

$$[(m-1)(n-1)+1]/3 \leq C_3[G(m,n)] \leq [(m-1)(n-1)+1]/3 + [3m+2n-3]/4 + 5$$

Finding the exact value is an open problem.

Vaccination Strategies

Stephen Hartke and others worked on a different problem:

Defender: can vaccinate v people ***per time period***.

Attacker: can only infect people at the beginning.
Irreversible k-threshold model.

What vaccination strategy minimizes number of people infected?

Sometimes called the ***firefighter problem***:

alternate fire spread and firefighter placement.

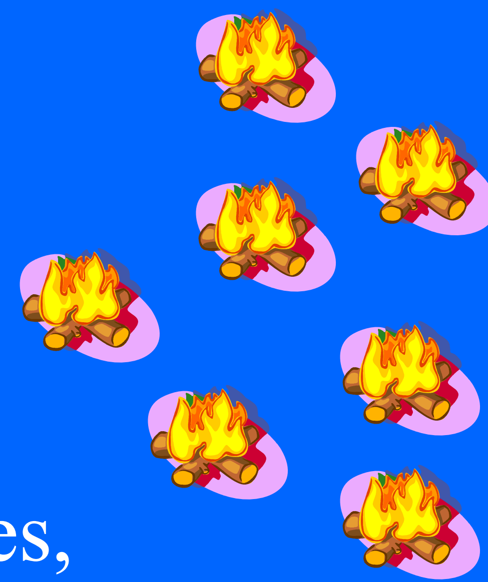
Usual assumption: k = 1. (We will assume this.)

Variation: The vaccinator and infector alternate turns, having v vaccinations per period and i doses of pathogen per period. What is a good strategy for the vaccinator?

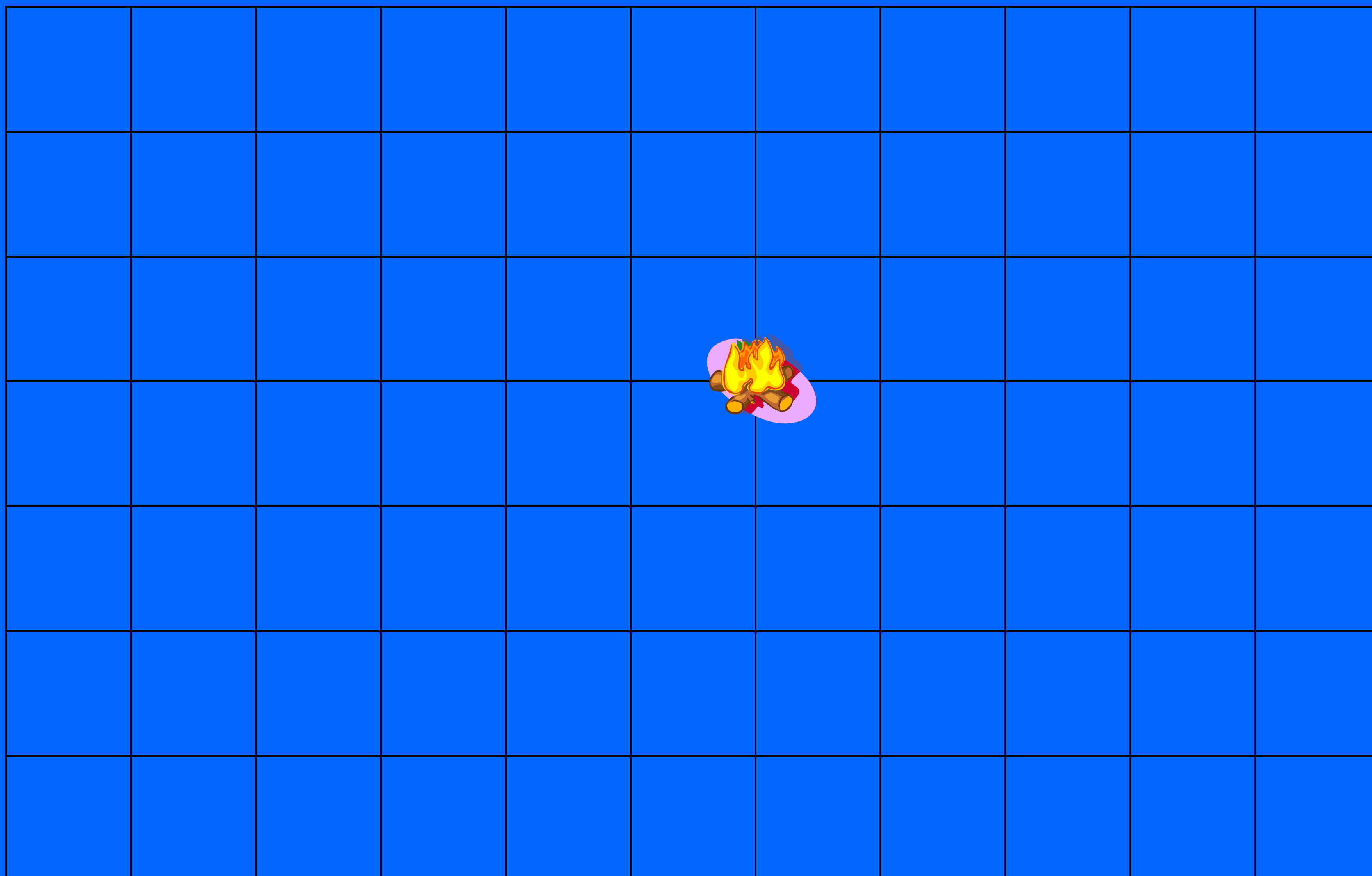
Problem goes back to Bert Hartnell 1995.

A Survey of Some Results on the Firefighter Problem

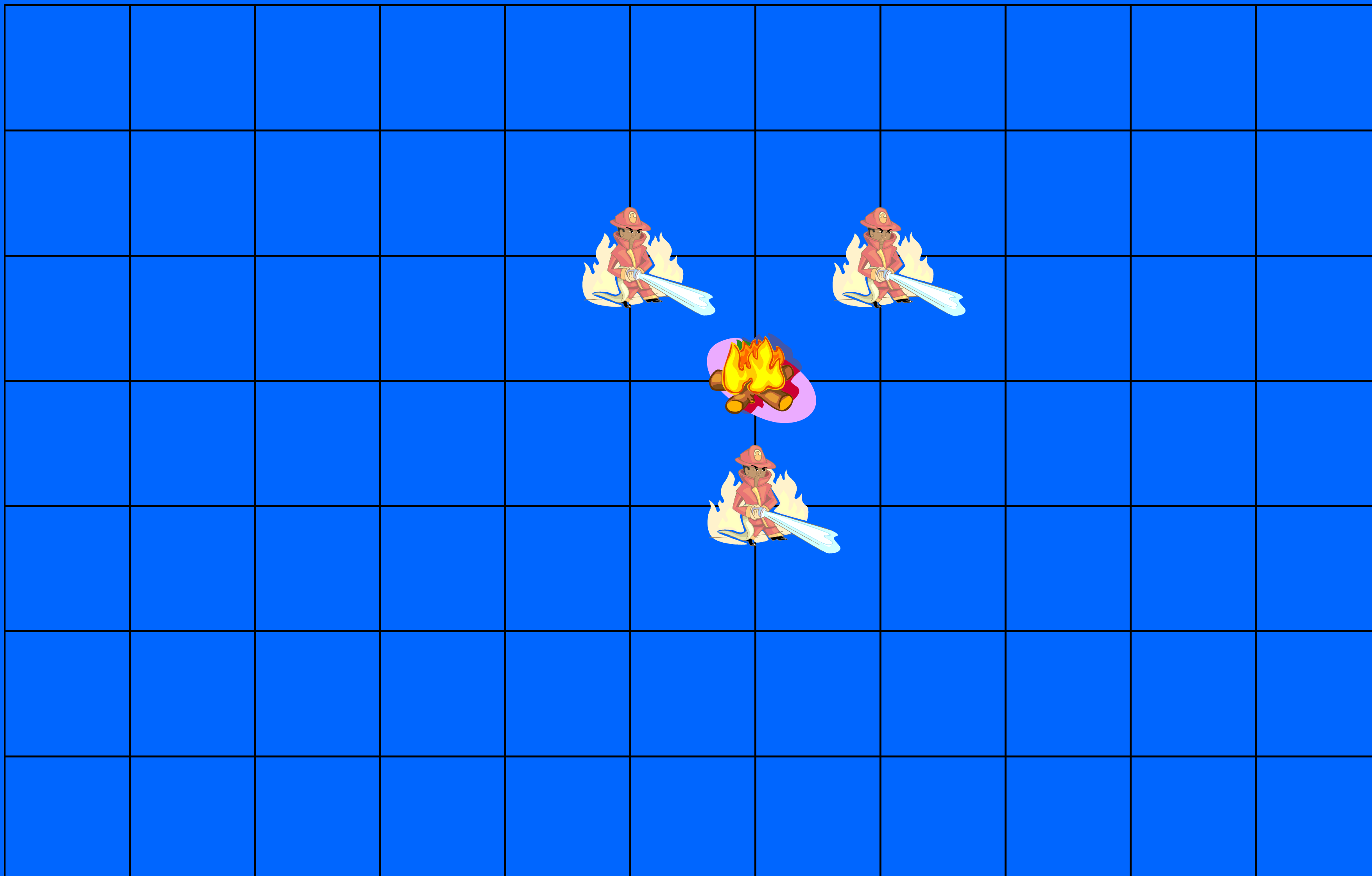
Thanks to
Kah Loon Ng
DIMACS
For the animated slides,
slightly modified by me



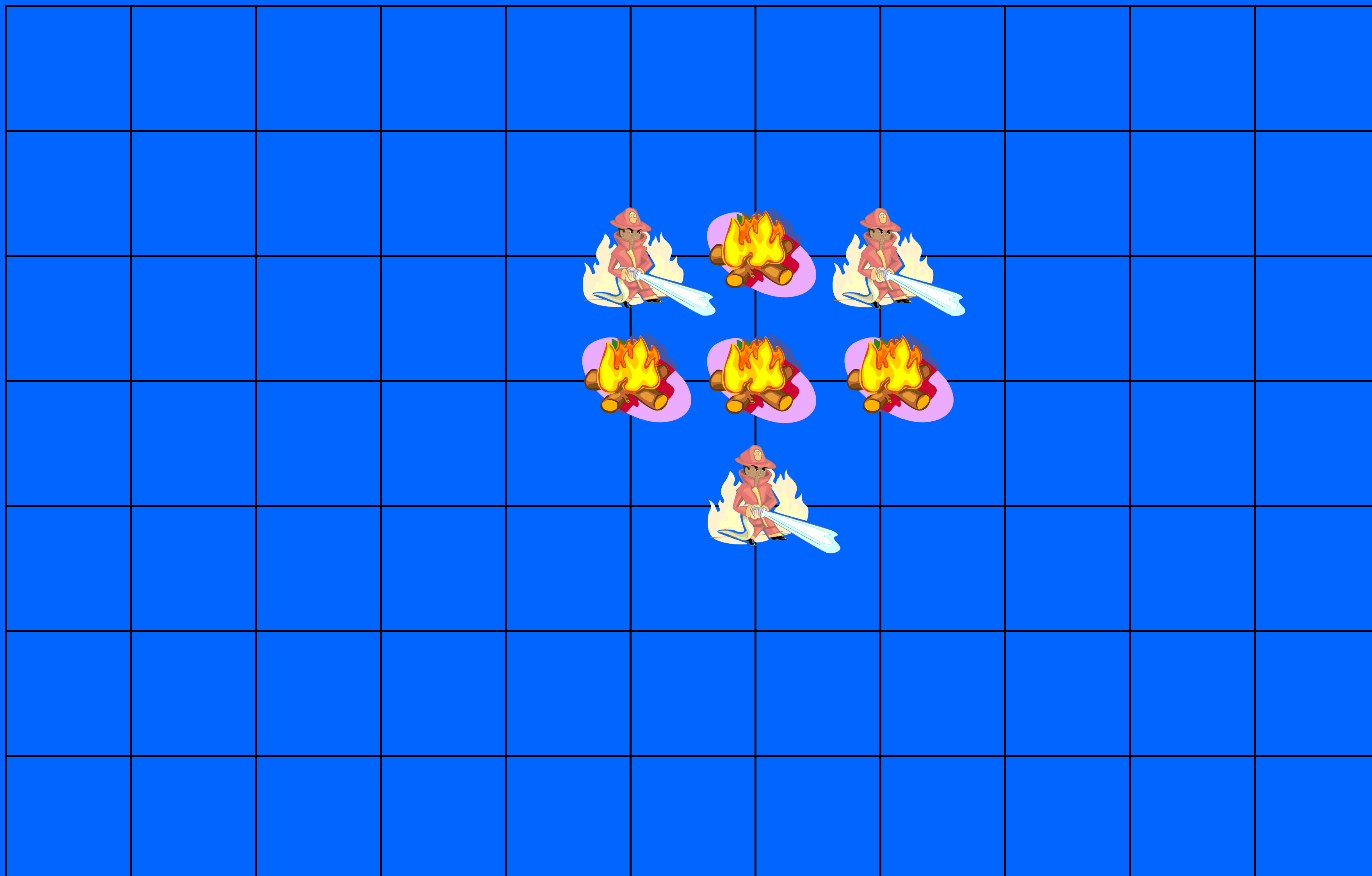
A Simple Model (k = 1) (v = 3)



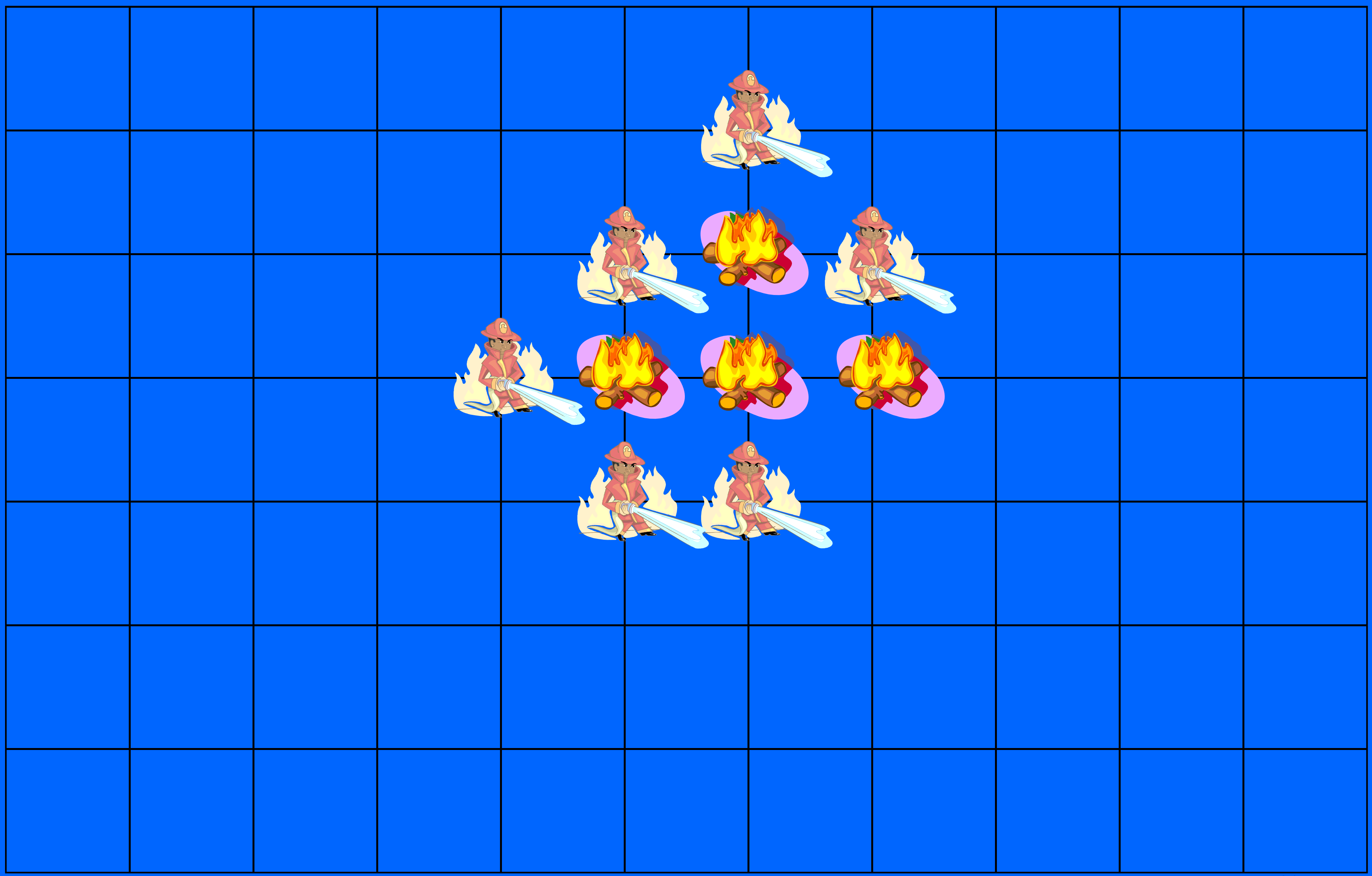
A Simple Model



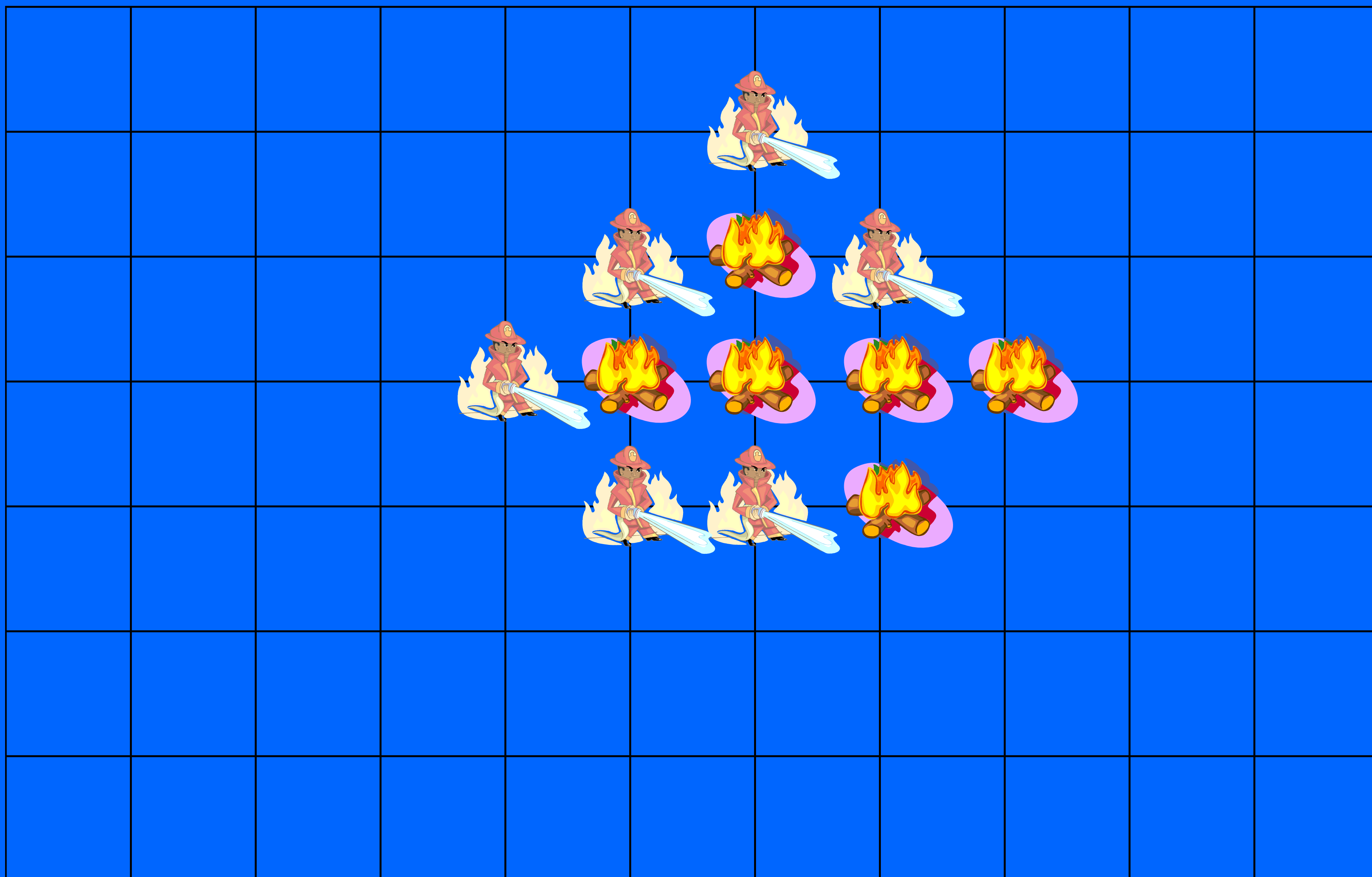
A Simple Model



A Simple Model

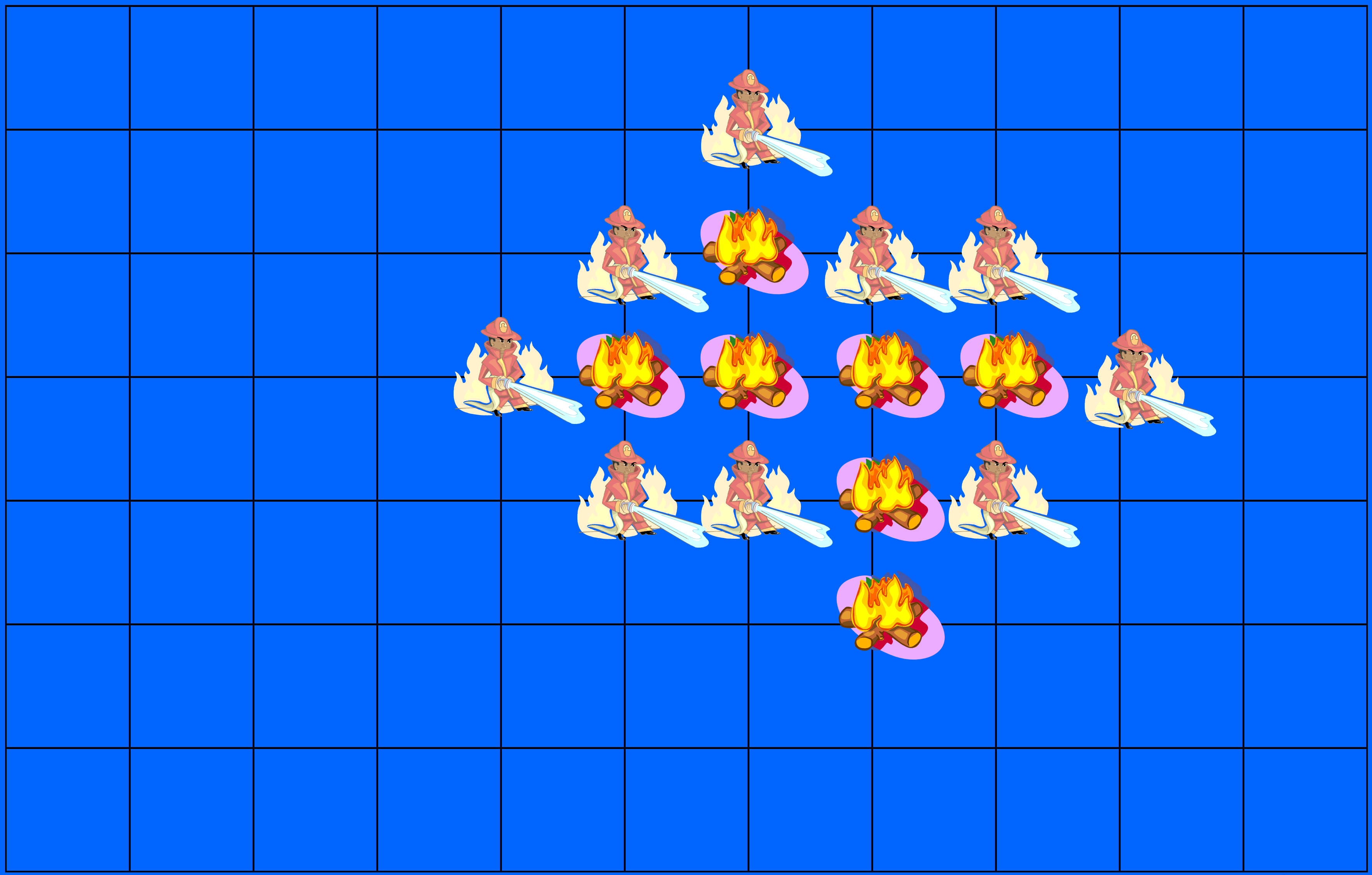


A Simple Model

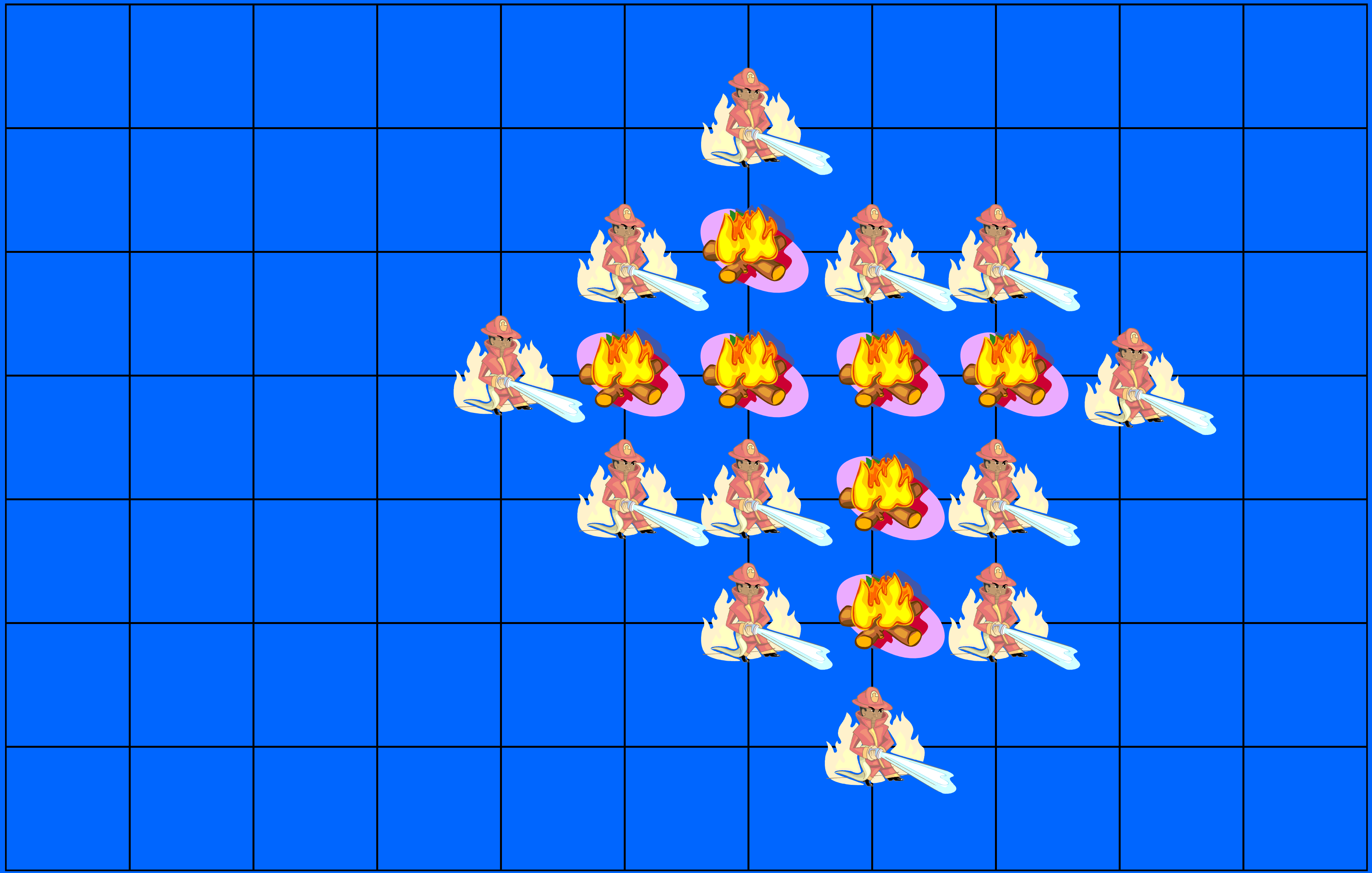


A Simple Model

A Simple Model



A Simple Model



Some questions that can be asked (but not necessarily answered!)

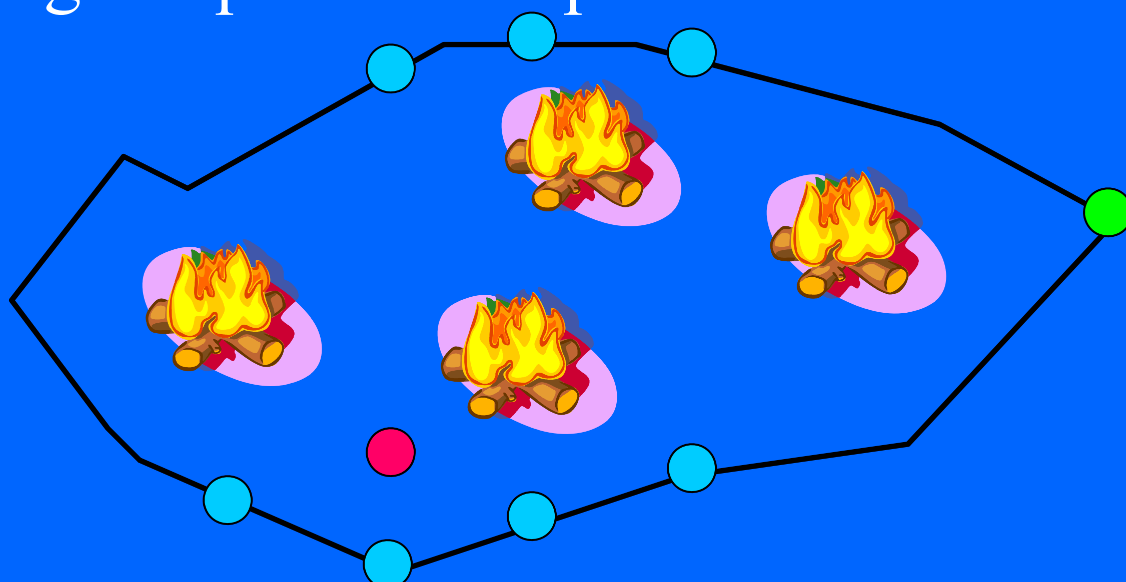
- Can the fire be contained?
- How many time steps are required before fire is contained?
- How many firefighters per time step are necessary?
- What fraction of all vertices will be saved (burnt)?
- Does where the fire breaks out matter?
- Fire starting at more than 1 vertex?
- Consider different graphs. Construction of (connected) graphs to minimize damage.
- Complexity/Algorithmic issues

Containing Fires in Infinite Grids L_d

Fire starts at only one vertex:

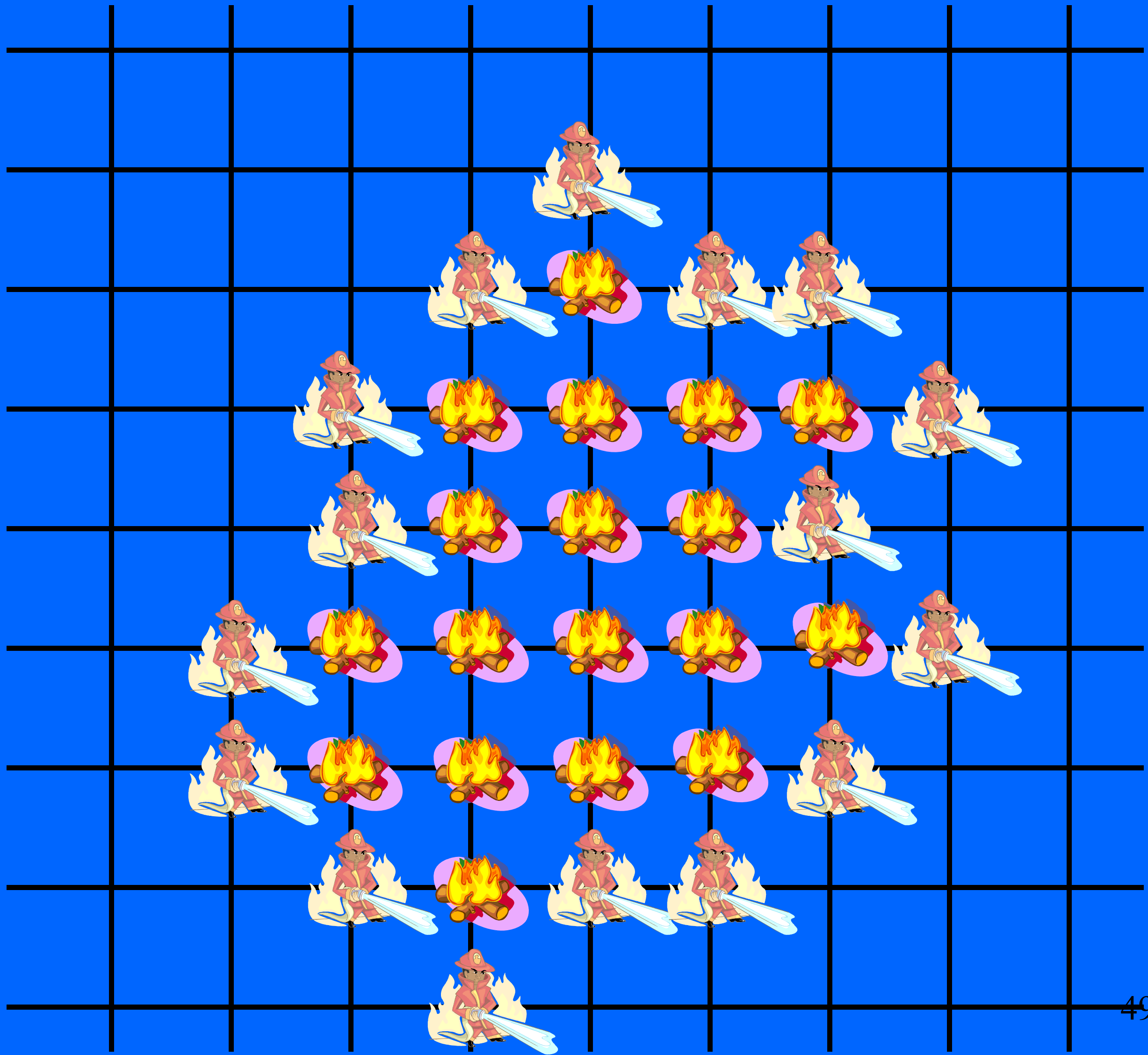
d =1: Trivial.

d = 2: Impossible to contain the fire with 1 firefighter per time step



Containing Fires in Infinite Grids L_d

$d = 2$: Two firefighters per time step needed to contain the fire.



8 time steps

18 burnt vertices

Containing Fires in Infinite Grids L_d

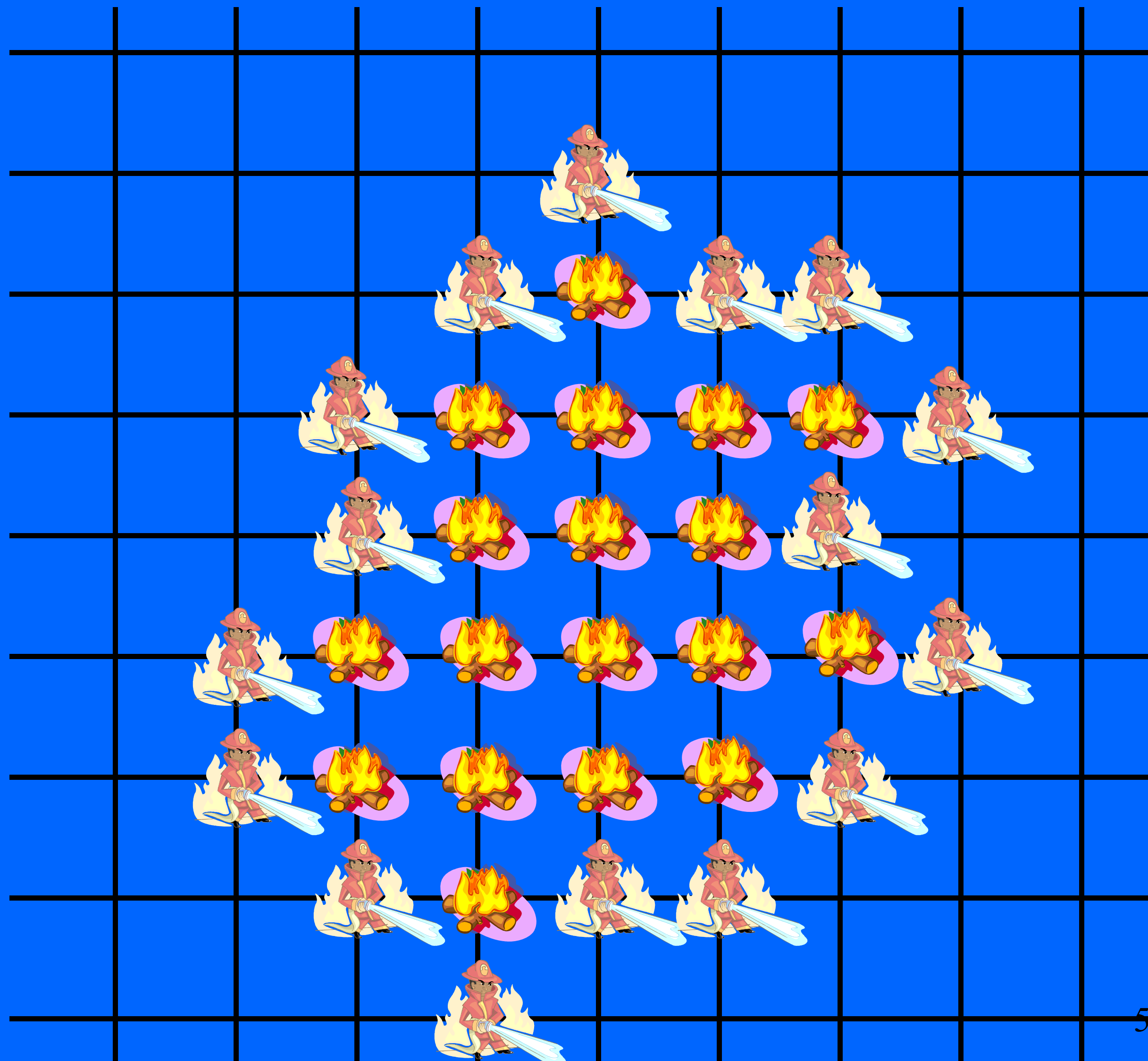
$d = 2$: Two firefighters per time step needed to contain the fire.

8 time steps

18 burnt vertices

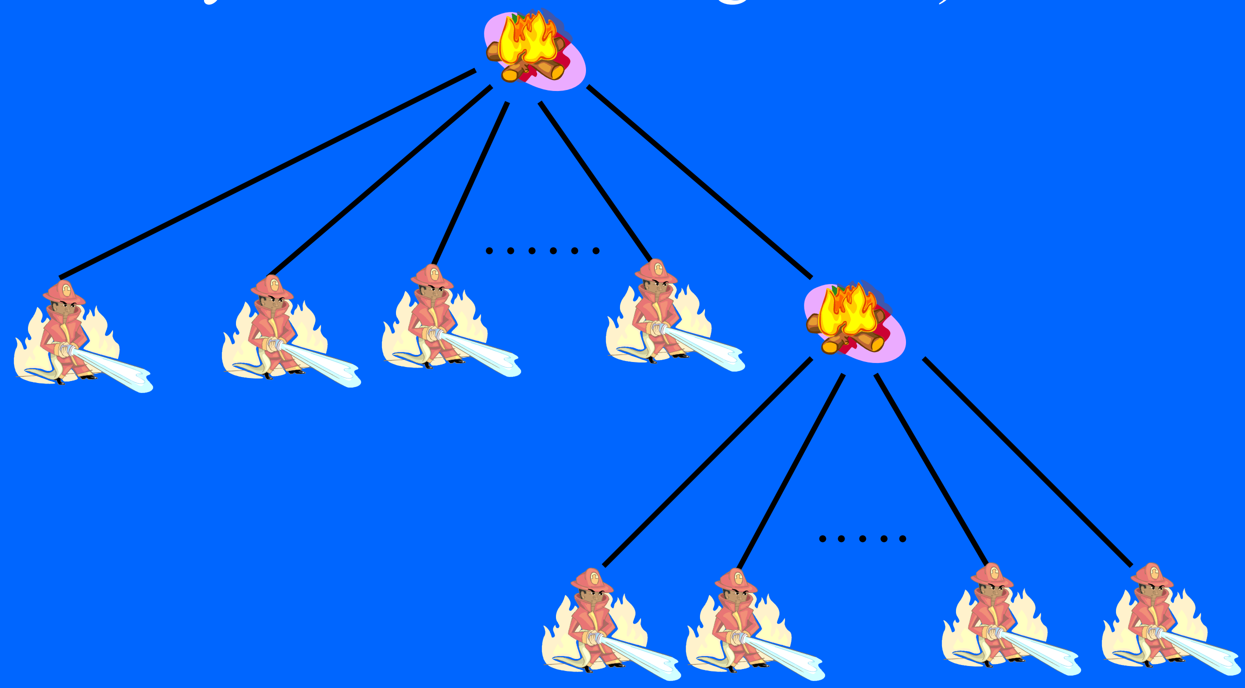
Develin & Hartke: Cannot do better than 18

Wang & Moeller: Cannot contain fire in < 8 steps



Containing Fires in Infinite Grids L_d

$d \geq 3$: Wang and Moeller: If G is an r-regular graph, $r - 1$ firefighters per time step is always sufficient to contain any fire outbreak (at a single vertex) in G.
(***r-regular***: every vertex has r neighbors.)



Containing Fires in Infinite Grids L_d

$d \geq 3$: In L_d, every vertex has degree $2d$.

Thus: $2d$-1 firefighters per time step are sufficient to contain any outbreak starting at a single vertex.

Theorem (Develin and Hartke): If $d \geq 3$, $2d - 2$ firefighters per time step are not enough to contain an outbreak in L_d.

Thus, $2d - 1$ firefighters per time step is the minimum number required to contain an outbreak in L_d and containment can be attained in 2 time steps.

Containing Fires in Infinite Grids L_d

Fire can start at more than one vertex.

$d = 2$: Fogarty: Two firefighters per time step are sufficient to contain any outbreak at a finite number of vertices.

$d \geq 3$: Hartke: For any $d \geq 3$ and any positive integer f, f firefighters per time step is not sufficient to contain all finite outbreaks in L_d. In other words, for $d \geq 3$ and any positive integer f, there is an outbreak such that f firefighters per time step cannot contain the outbreak.

Containing Fires in Infinite Grids L_d

The case of a different number of firefighters per time step.

Let f(t) = number firefighters available at time t.

Assume f(t) is periodic with period p_f.

Possible motivations for periodicity:

•Firefighters arrive in batches.

•Firefighters need to stay at a vertex for several time periods before redeployment.

Containing Fires in Infinite Grids L_d

The case of a different number of firefighters per time step.

$N_f = f(1) + f(2) + \ldots + f(p_f)$

$R_f = N_f/p_f$

(average number firefighters available per time period)

<u>Theorem (Ng and Raff):</u> If $d = 2$ and f is periodic with period p_f □ 1 and $R_f > 1.5$, then an outbreak at any number of vertices can be contained at a finite number of vertices.

Containing Fires in Infinite Grids L_d

The case of a different number of firefighters per time step.

Conjecture (Develin and Hartke): Suppose that $f(t)/t^{d-2}$ goes to 0 as t gets large. Then there is some fire on L_d that cannot be contained by deploying f(t) firefighters at time t.

Containing Fires in Infinite Grids

Other work has been done on infinite triangular grids and infinite hexagonal grids

Saving Vertices in Finite Grids *G*

Assumptions:

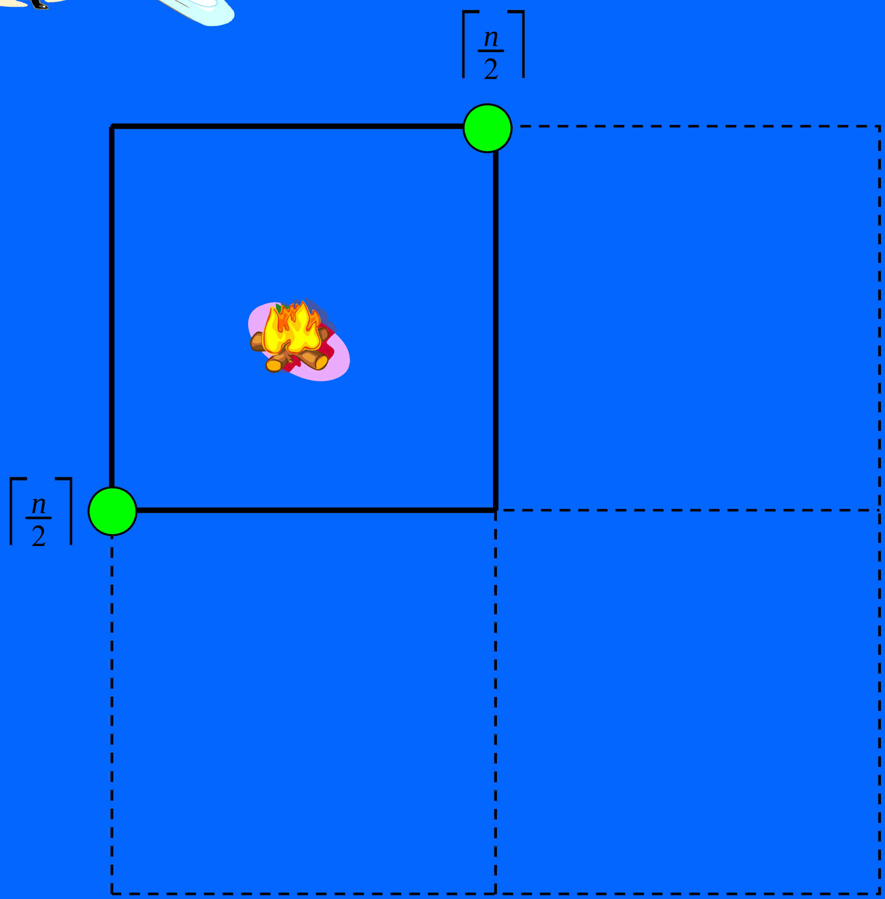
1. 1 firefighter is deployed per time step
2. Fire starts at one vertex

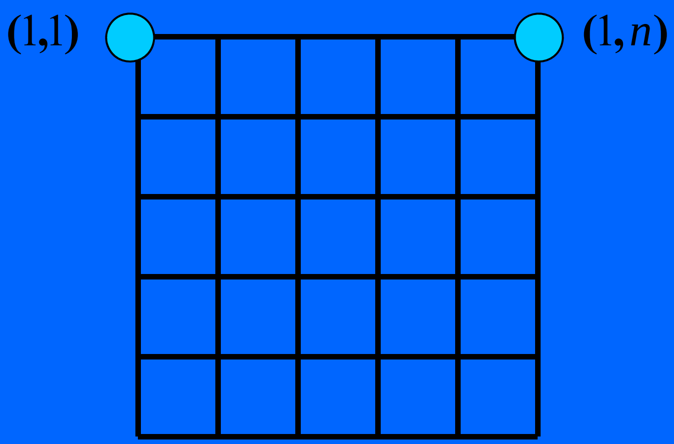
Let

***MVS*(*G*, *v*)** = maximum number of vertices that can be saved in *G* if fire starts at *v*.

Saving Vertices in Finite Grids G

$$G = P_n \times P_n \qquad V(G) = \{(a,b) \mid 1 \le a,b \le n\}$$

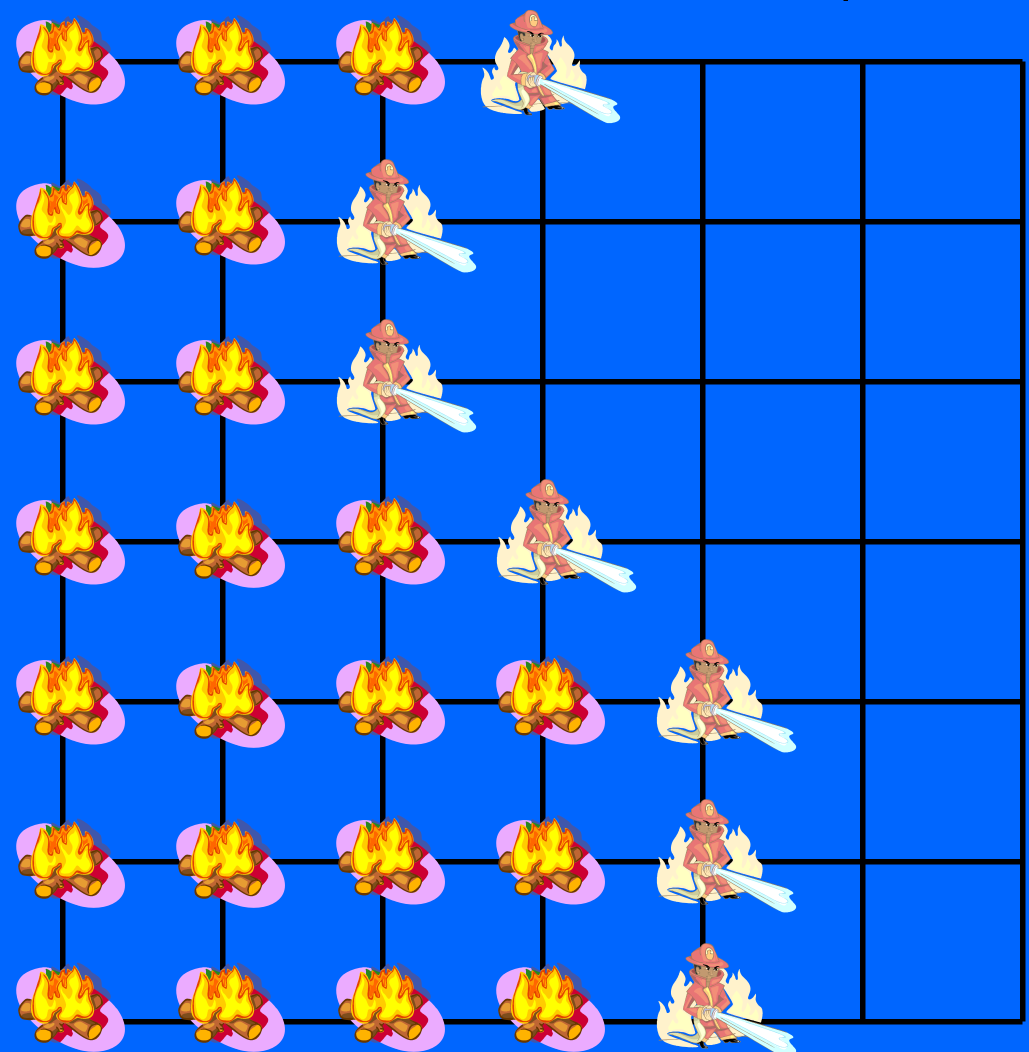




$$MVS(P_n \times P_n,(a,b)) \ge n(n-b)-(a-1)(n-a) \qquad 1 \le b \le a \le \left\lceil \frac{n}{2} \right\rceil$$

Saving Vertices in Finite Grids G

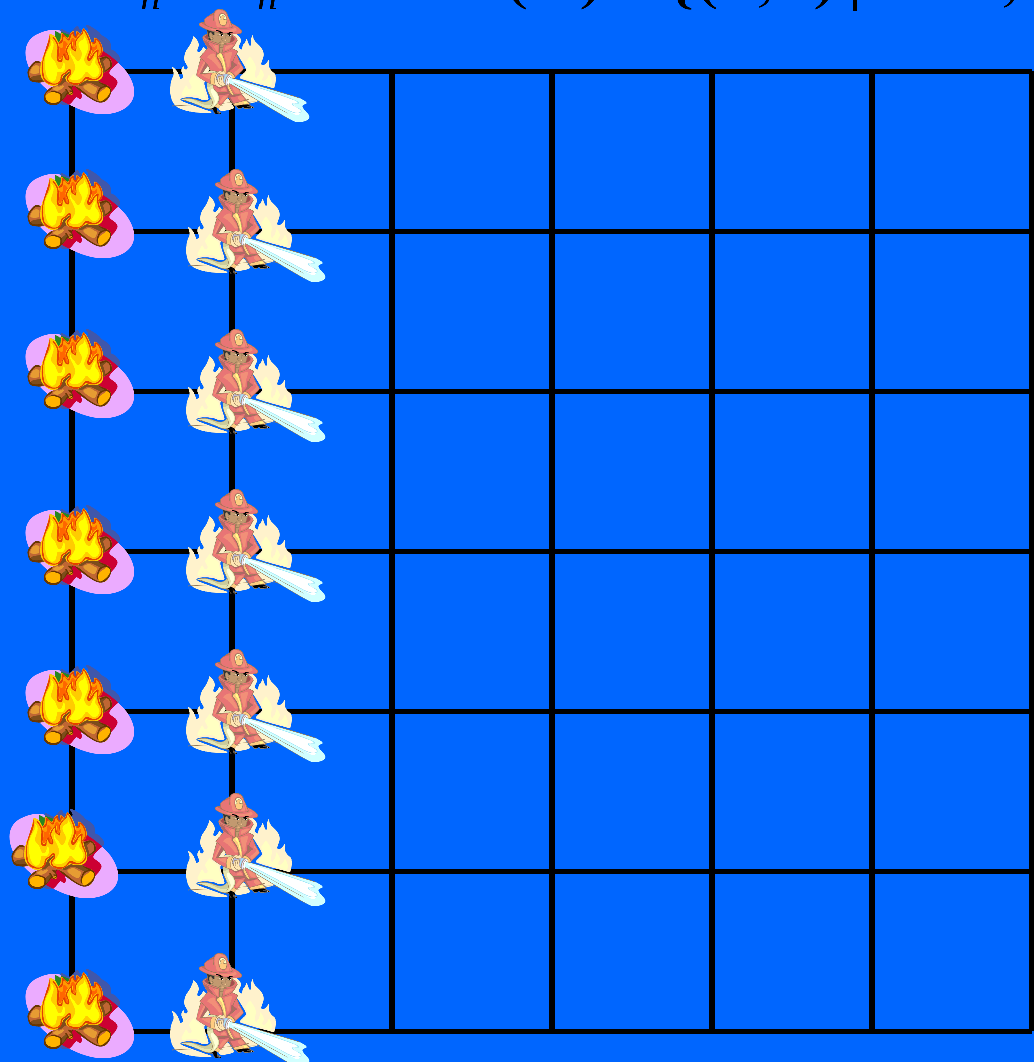
$G = P_n \times P_n$ $\quad$ $V(G) = \{(a,b) \mid 1 \le a,b \le n\}$



$MVS(P_n \times P_n, (a,b)) \ge n(n-b) - (a-1)(n-a)$ $\quad$ $1 \le b \le a \le \left\lceil \frac{n}{2} \right\rceil$

Saving Vertices in Finite Grids G

$$G = P_n \times P_n \qquad V(G) = \{(a,b) \mid 1 \le a, b \le n\}$$



$$MVS(P_n \times P_n, (1,1)) = n(n-1) = n^2 - n$$

More Realistic Models

Many oversimplifications in both of our models. For instance:

- What if you stay infected (burning) only a certain number of days?
- What if you are not necessarily infective for the first few days you are sick?

Chicken pox

Credit: Wikimedia commons, Øyvind Holmstad
no changes

- What if your threshold k for changes from to (uninfected to infected) changes depending upon how long you have been uninfected?

More Realistic Models

Consider an irreversible process in which you stay in the infected state (state ●) for d time periods after entering it and then go back to the uninfected state (state ●).

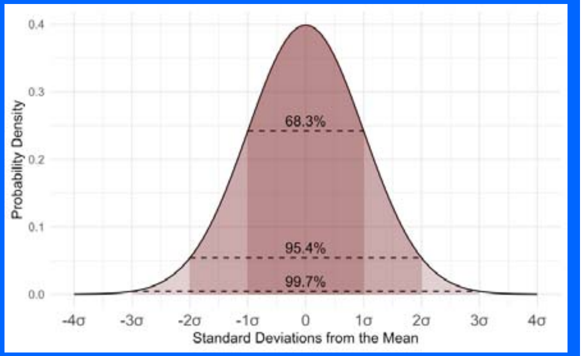
Consider an irreversible k-threshold process in which we vaccinate a person in state ● once k-1 neighbors are infected (in state ●).

Etc. – experiment with a variety of assumptions

Credit: PowerPoint stock image

More Realistic Models

Our models are *deterministic*. How do *probabilities* enter?



Credit: Wikimedia commons, D Wells no changes

•What if you only get infected with a certain probability if you meet an infected person?

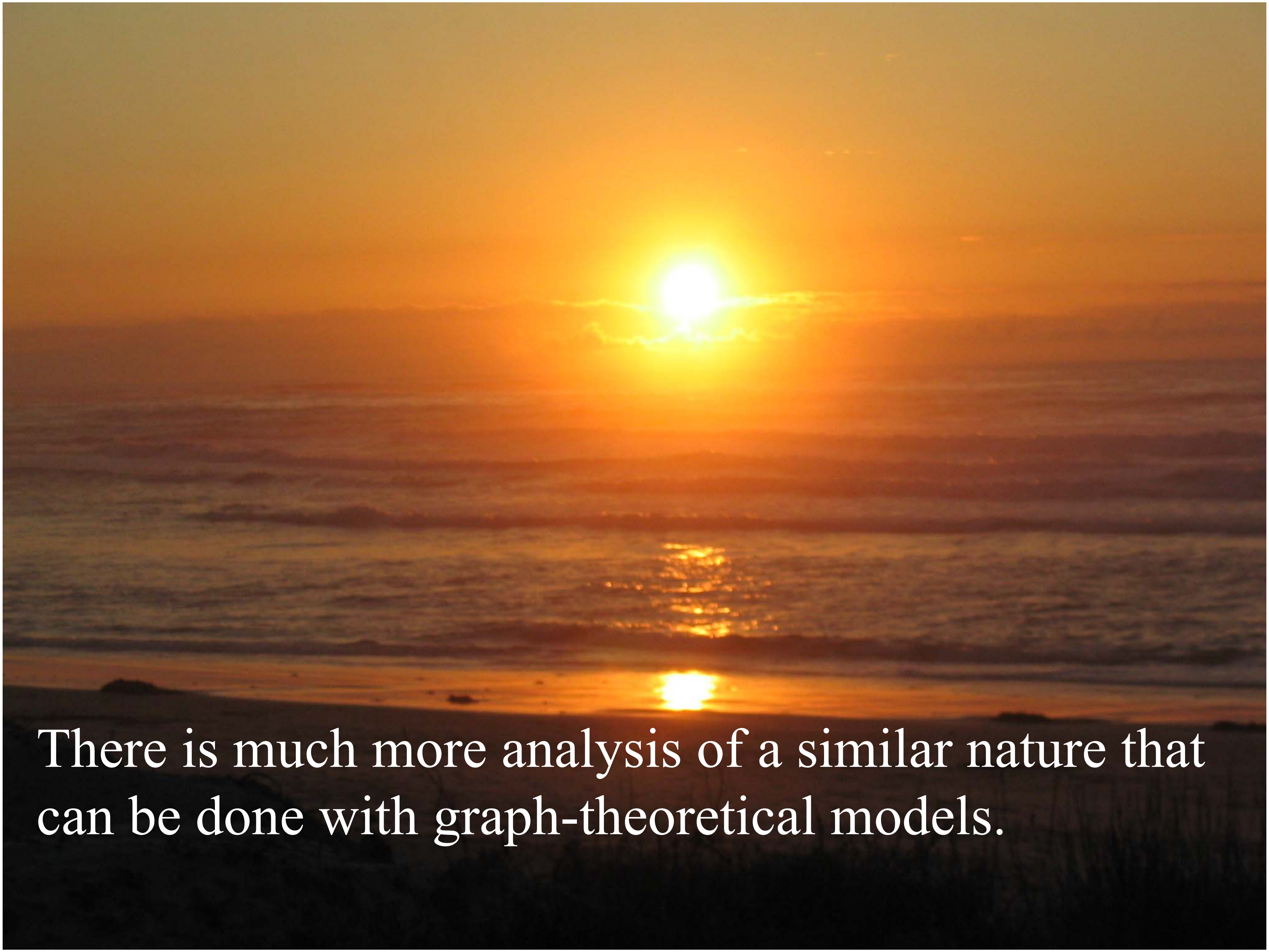
•What if vaccines only work with a certain probability?

•What if the amount of time you remain infective exhibits a probability distribution?

What about Quarantine?

Can you use graph-theoretical models to analyze the effect of different quarantine strategies?

Credit: Wikimedia commons, Tisnec no changes


There is much more analysis of a similar nature that can be done with graph-theoretical models.

Congratulations Boris!!!

Good Health!!!

Many more successes!!!