Energy Balance Models

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Module Summary: This module introduces the student to the process of mathematical modeling. It shows how the process starts in the "real world" with a physical system and some observations or an experiment. When the laws of physics that are thought to govern the behavior of the system are translated in mathematical terms, the result is what is called a *mathematical model*. The mathematical model is subsequently analyzed for its properties and used to generate predictions about the behavior of the system in a changing environment. These predictions are tested against observations, and, if there is agreement between predictions and observations, the model is accepted; otherwise, the model is refined, for example by bringing in more details of the physics, and the process is repeated. Thus, mathematical modeling is an *iterative process.*

To illustrate this iterative process, this module builds a series of *zero-dimensional energy balance models* for the Earth's climate system. In a zero-dimensional energy balance model, the Earth's climate system is described in terms of a single variable, namely the temperature of the Earth's surface averaged over the entire globe. In general, this variable varies with time; its time evolution is governed by the amount of energy coming in from the Sun (in the form of ultraviolet radiation) and the amount of energy leaving the Earth (in the form of infrared radiation). The mathematical challenge is to find expressions for the incoming and outgoing energy that are consistent with the observed current state of the climate system on Earth, that corresponds to the average temperature on Earth.

Informal Description: This module introduces the student to the mathematical modeling process by showing how to build a zero-dimensional energy balance model for the Earth's climate system. The process is an iterative one and generates various versions of the model. Successive versions include more physics to better match the observations. The emphasis in the module is on the *process*, rather than the models derived in the process, because the process is universal and independent of the complexity of the model. The process is illustrated in Figure 1.

The mathematical modeling process starts in the "real world" with a physical system and some observations or an experiment. We assume that the behavior of the system is governed by the laws of nature—Newton's law of motion, Fourier's law of heat conduction, etc. When these laws are formulated in mathematical terms, we obtain what we call a "mathematical model"—a set of mathematical equations that describe the state of the physical system as it evolves in time. In the next step of the modeling process, we "analyze" the model—that is, we apply our mathematical knowledge to extract information from the model, to see whether we understand and can explain what we see in the real world. In the third step we use the model to make predictions about what we will see in additional experiments and observations. We then return to the real world to test these predictions by running the experiments or collecting more observations, and either accept the model if we find that the outcome matches our predictions, or refine the model if we find that improvements are needed. Typically, we go around this modeling cycle many times, building progressively better models, thus improving our understanding of the physical system and increasing our ability to make predictions about its behavior.

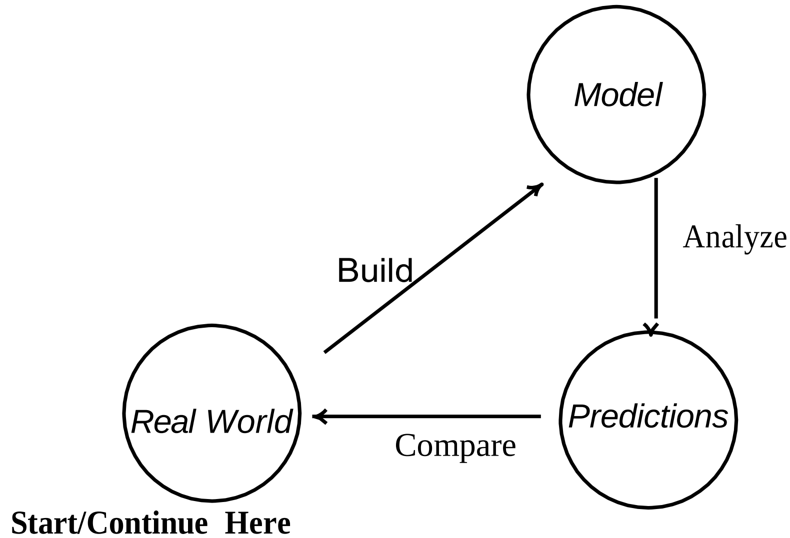

***Figure 1: The modeling cycle*.**

In this module, the physical system of interest is the Earth's climate system—a proto- typical "complex system" that has many components: the atmosphere, oceans, lakes and other bodies of water, snow and ice, land surface, all living things, and so on. The components interact and influence each other in ways that we don't always understand, so it is difficult to see how the system as a whole evolves, let alone why it evolves the way it does. For some complex system it is possible to build a physical model and observe what happens if the environment changes. This is the case, for example, for a school of fish whose behavior we can study in an aquarium.

It is also true for certain aspects of human behavior, which we can study in a social network. But in climate science this is not possible; we have only one Earth, and we cannot perform a controlled real-life experiment. The best we can do if we want to gain insight into what might have happened to the Earth's climate system in the past, or what might happen to it in the future, is to build mathematical models and "play" with them. Mathematical models are the climate scientist's only experimental tools.

The modeling process—building and testing a series of imperfect models—is the most essential brick in the foundation of climate science and an indispensable tool to evaluate the arguments for or against climate change. Models are never perfect—at best, they provide some understanding and some ability to test "what-if" scenarios. Especially in an area as complex as the Earth's climate, we cannot and should not expect perfection. Recognizing and identifying imperfection and uncertainty are key parts of all modeling and, especially, climate modeling.

Mathematical models of the Earth's climate system come in many flavors. They can be simple—simple enough that we can use them for back-of-the-envelope calculations, or they can be so complicated that we need a supercomputer to learn what we want to know. But whatever kind of models we use, we should always keep in mind that they are *simplified representations* of the real world, they are not the "real world," and they are made for a purpose, namely to better understand what is driving our climate system.

The present module looks at the simplest possible description of the Earth's climate system. In the following models, the state of the climate system is characterized by a single variable—the temperature of the Earth's surface, averaged over the entire globe (referred to as "*zero-dimensional energy balance*" models in physics). An *energy balance equation* is a formal statement of the fact that the temperature of the Earth increases if the Earth receives more energy from the Sun than it re-emits into space, and that it decreases if the opposite is the case. The module shows how to construct energy balance models by finding mathematical expressions for the incoming and outgoing energy. The models are tested against "real-world" data and improved in successive steps of the iterative modeling process to better match the available data.

In this module, the focus is on the physics, but we emphasize that modeling the Earth's climate system is fundamentally an interdisciplinary activity. Understanding the Earth's climate requires knowledge, skills, and perspectives from multiple disciplines. For example, atmospheric chemistry explains why much of the incoming energy from the Sun (largely in the ultraviolet and visible regions of the spectrum) passes through the atmosphere and reaches the Earth's surface, but much of the black-body radiation emitted by the Earth (largely in the infrared regions of the spectrum) is trapped by greenhouse gases like water vapor and carbon dioxide. Similarly, the life sciences help us understand the part played by the biosphere in the Earth's climate system—the effects of the biosphere on the Earth's albedo and the interactions between atmospheric chemistry and plant and animal life.

Target Audience: This module is suitable Lab for undergraduate students in an introductory differential equation class.

Prerequisites: Basic knowledge of the concept of derivatives and ordinary differential equations.

Mathematical Fields: Ordinary differential equations.

Applications Areas: Geophysics and climate science.

Goals and Objectives.

- Teach the process of "mathematical modeling.
- Show how a simple model like a single variable energy balance model can provide insight into aspects of climate dynamics.

1 Model #1: The Simplest

We consider the Earth with its atmosphere, oceans, and all other components of the climate system as a homogeneous solid sphere, ignoring differences in the atmosphere's composition (clouds!), differences among land and oceans, differences in topography (altitude), and many other things.

1.1 Observation

The climate system is powered by the Sun, which emits radiation in the ultraviolet (UV) regime. This energy reaches the Earth's surface, where it is converted by physical, chemical, and biological processes to radiation in the infrared (IR) regime. This IR radiation is then re-emitted into space. If the Earth's climate is in equilibrium (steady state), the average temperature of the Earth's surface does not change, so the amount of energy received must equal the amount of energy re-emitted.

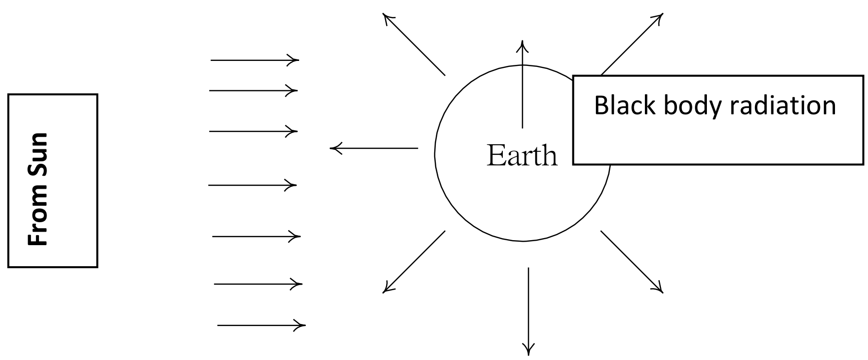

Figure 2. Simplest Climate Model

1.2 Modeling

Units: Length in meter (m), time in seconds (s), energy in joules (J), power (energy per unit time) in watts (W), and temperature in Kelvin (K). Kelvin (or absolute) temperature is obtained from Celsius (C) temperature by adding 273.15. The average temperature on Earth is about 15 C, or about 288 K.

Variable: T, the temperature of the Earth's surface averaged over the entire globe.

Building the model.

- At any given time, the Earth receives incoming solar radiation over its cross-sectional areThe area πR^2, where R is the radius of the Earth.
- S, known as the solar constant, is the average solar electromagnetic power flowing through a flat surface of area $1\,\mathrm{m}^2$ at a distance of one astronomical unit from the Sun. S atellite observations give an approximate value of $S = 1370\,\mathrm{Wm}^{-2}$. It is convenient below to define $Q = S/4$ and use $Q = 342.5$ instead of S.
- The amount of power flowing into the disk (i.e., reaching the Earth) is

$$\text{Incoming power: } P_{in} = \pi R^2 S = 4\pi R^2 Q.$$

- All bodies radiate power in the form of electromagnetic radiation. The amount of power radiated out depends on the temperature of the body.
- In physics, the Stefan–Boltzmann law gives the power per unit surface area (in units of Wm^{-2}) for "black-body radiation", as σT^4, where σ (Greek, pronounced "sigma"), is the Stefan–Boltzmann constant; its value is $\sigma = 5.67 \cdot 10^{-8}\ \mathrm{Wm}^{-2}\mathrm{K}^{-4}$.
- The area of the Earth's surface is $4\pi R^2$.
- The amount of power radiated out by the Earth is

$$\text{Outgoing power: } P_{out} = 4\pi R^2 \sigma T^4.$$

1.3 Analysis

If the incoming power P_{in} is greater than the outgoing power P_{out}, the Earth's temperature increases. If the incoming power is less than the outgoing power, the Earth's temperature decreases. If $P_{\text{in}} = P_{\text{out}}$, the Earth's temperature remains constant; the planet is said to be in *thermal equilibrium*, or in steady state. Our mathematical model the thermal equilibriu temperature T gives the equations

$$\begin{aligned} 4\pi R^2 Q &= 4\pi R^2 \sigma T^4 \\ Q &= \sigma T^4 \\ T &= (Q/\sigma)^{\frac{1}{4}} \end{aligned}$$

Exaericse 1: Use $\sigma = 5.67 \cdot 10^{-8}$ and $Q = 342.5$ to find the Kelvin and Celsius temperatures at thermal equilibrium. [Answer: $T \approx 278.78\ K$ or 5.63 C]

Conclusion. Model #1 gives the average temperature at equilibrium below the observed value of about 15 C or 288 K.

2 Model #2: Adjusting for reflectivity

We seek to improve on Model #1, which omitted a number of important factors

2.1 Observation

The first factor we want to add involves *reflection*—some of the incoming energy from the Sun is reflected back out into space. Snow, ice, and clouds, for example, reflect a great deal of the incoming light from the Sun. A fraction, called *albedo* measures the reflectivity of a surface. Albedo ranges from 0 (no reflectance) to 1 (complete reflectance).

2.2 Modeling

Additional physical constant: Earth's albedo $\alpha = 0.32$. Roughly 32% of the incoming energy from the Sun is reflected back into space, with the remaining 68% absorbed by the Earth.

Building the model.

- The amount of power absorbed by the Earth is now

$$\text{Incoming power: } P_{in} = 4\pi R^2 Q(1-\alpha).$$

- The amount of power radiated out by the Earth remains

$$\text{Outgoing pwer: } P_{out} = 4\pi R^2 \sigma T^4.$$

2.3 Analysis

At thermal equilibrium, with $P_{in} = P_{out}$, this second mathematical model leads to

$$Q(1-\alpha) = \sigma T^4.$$

Exercise 2: (a) Solve for T. [Answer: $T = [\frac{Q(1-\alpha)}{\sigma}]^{\frac{1}{4}}$.]

(b) By comparing your expression in (a) to the Model #1 expression $T = (Q/\sigma)^{\frac{1}{4}}$, will the Model #2 value of T be greater or lesser than that of Model #1? [Answer: T (in K) in Model #2 is less by a factor of $(1-\alpha)^{\frac{1}{4}}$.]

With α = 0.32, σ = $5.67{\cdot}10^{-8}$, and Q = 342.5, $T \approx 253.16$ K or -19.99 C.

Conclusion. Although Model #2 includes more physics, its prediction of the temperature is worse than the prediction of Model #1.

3 Model #3: The Greenhouse Effect

It is somewhat disconcerting that we construct a better model and get a result that is not as good as that of an earlier model. But once we accept the mathematical model, we must accept the result. The only option is to look where we might have overlooked something in the model. In this third model, we focus on the outgoing radiation.

3.1 Observation

Greenhouse gases like carbon dioxide, methane, and water, as well as dust and aerosols, have a significant effect on the properties of the atmosphere. The effect on the outgoing radiation is difficult to model, but the simplest approach is to reduce the Stefan–Boltzmann law by some factor. The underlying assumption is that Eath behaves as a “gray-body” rather than a “black-body” radiator, as atmospheric greenhouse particles emit radiation *back* to the Earth’s surface.

3.2 Modeling

Additional physical parameter: ε, greenhouse factor. This *parameter* is introduced in an attempt to model the effect that the greenhouse effect has in reducing the net amount of power radiated into space; its value will be estimated below using the observed global average temperature. ε ranges from 0 (no radiation emitted) to 1 (black body).

Building the model

- As in Model #2, the amount of power absorbed by the Earth is

$$\text{Incoming power: } P_{in} = 4\pi R^2 Q(1-\alpha).$$

- The amount of power radiated out by the Earth is adjusted by a factor ε

$$\text{Outgoing pwer: } P_{out} = 4\pi R^2 \varepsilon\sigma T^4.$$

3.3 Analysis

At thermal equilibrium, with $P_{in} = P_{out}$, this third mathematical model leads to

$$Q(1-\alpha) = \varepsilon\sigma T^4. \tag{2}$$

Exercise 3: (a) Solve for T. [Answer: $T = [\frac{Q(1-\alpha)}{\varepsilon\sigma}]^{\frac{1}{4}}$.]
(b) What value of ε gives a climate model that correctly predicts the current global average temperature $T^* \approx 288$ K? (Take $\alpha = 0.32$, $\sigma = 5.67\cdot10^{-8}$, and $Q = 342.5$ as before). [Answer: $\varepsilon = \frac{Q(1-\alpha)}{\sigma T^4} = 0.60$.]
(c) What happens if the combined effects of greenhouse gases, dust, and aerosols reduce the parameter ε to 0.5? [Answer: The equilibrium temperature T increases to 301.06 K or 27.91 C.]

Conclusion. We can match the current climate state by taking into account the effect of greenhouse gases. Our climate model predicts that, if the amount of greenhouse gasses in the Earth's atmosphere increases, then the Earth will warm up. This is the well-known *greenhouse gas effect*. However, this model is certainly too simple to predict the state of our planet with any great accuracy, so we should interpret this finding with great care.
An interesting question is what actually happens when the balance of incoming and outgoing energy is perturbed. Perhaps a volcanic eruption throws dust into the atmosphere, or humans release increasing amounts of CO_2 or other greenhouse gases into the atmosphere. Greenhouse gases affect the Earth's climate by absorbing some of the outgoing radiation.

Exercise 3 (continued). (d) What do you expect to happen to the Earth's temperature if $P_{\text{in}} > P_{\text{out}}$? What if $P_{\text{out}} > P_{\text{in}}$? [Answer: The temperature increases if $P_{\text{in}} > P_{\text{out}}$, decreases if $P_{\text{out}} > P_{\text{in}}$]

We can ask more questions. In the case when temperature increases, will the temperature continue to increase or will it level off at a higher equilibrium? What does the difference $P_{\text{in}} - P_{\text{out}}$ represent? How fast will the temperature change? To answer these questions, we need a fancier model.

4. Model #4: Differential Equation

4.1 Modeling

The fancier model uses a result from thermodyamics in which the rate of change of internal heating is proportional to the energy imbalance, as represented by the difference between the incoming and outgoing power densities (power per unit area). The *temperature evolution equation* is

$$C\frac{dT}{dt} = Q(1-\alpha) - \varepsilon\sigma T^4 \tag{1}$$

This is an *ordinary differential equation* (ODE) for the temperature T as a function of time t. The constant of proportionality $C = 2.08\cdot 10^8$ JK^{-1}m^{-2} is the *effective heat capacity* of the Earth's surface.

The equation above is an ODE of the type $\frac{dT}{dt} = f(T)$. A visual representation helps us to understand how the Earth’s temperature changes when the balance of the incoming and outgoing power is perturbed.

Figure 3 below shows the graph of $f(T) = [Q(1-\alpha) - \varepsilon\sigma T^4]/C$ for $\alpha = 0.32$, $\sigma = 5.67\cdot 10^{-8}$, $Q = 342.5$, and $\varepsilon = 0.60$. Since we are only interested in the solution of $f(T) = 0$, we use $C = 1$.

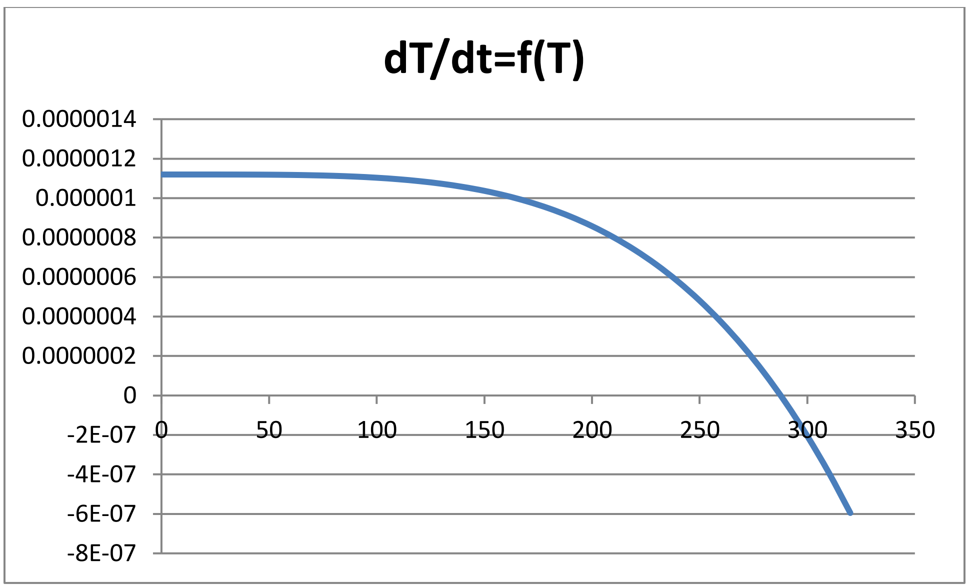

Figure 3: Graph of f(T).

Exercise 4: **(a)** What does the vertical axis represent in the physical world and what are the units? [Answer: dT/dt, the rate at which the temperature changes in Kelvin per second.]
(b) What is the zero of $f(T)$ in the range between 250 K and 350 K? Where have we seen this value before? What does it represent? [Answer: $f(T) = 0$ for $T = 288$ K. This equilibrium solution of the ODE is the same as the current global average temperature T^* that was used im Model #3 to

compute ε.]
(c) If the temperature is 300 K, do you expect the temperature to increase, decrease, or remain the same? Use the graph to help you. [Answer: At T = 300 K, the value of $dT/dt = f(T) < 0$; the temperature should decrease.[
(d) If the temperature is 250 K, do you expect the temperature to increase, decrease, or remain the same? Use the graph to help you. [Answer: At T = 250 K, the value of $dT/dt = f(T) > 0$; the temperature should increase.]
(e) Repeat (b)-(d) for $\varepsilon = 0.5$, and compare your findings in the two cases.
[Answer: As in the previous section, the equilibrium temperature increases to 301.06 K, so the graph in Figure 3 will cross the horizontal axis at that temperature. At T = 250 K and 300 K, the value of $dT/dt = f(T) > 0$; the temperature should increase.]

4.2 Analysis

The graph of $\boldsymbol{f(T)}$ contains qualitative information about the dynamics of the system. In the case $\alpha = 0.32$ and $\varepsilon = 0.60$, we found an equilibrium at $T^* = 288$ K. If the average temperature T is less than T^*, the Earth's surface will warm up; on the other hand, if T is greater than T^*, the Earth will cool down. If T is exactly equal to T^*, the average temperature will stay the same. Thus, after any small perturbation, the average temperature tends to be restored to its equilibrium value T^*. In mathematics, we say that T^* corresponds to a *stable* equilibrium.

Conclusion. We can use a differential equation to model global temperature when there is an energy imbalance. Our model indicates that the current climate state is stable.

Exercise 4 (continued) **(f)** Is the equilibrium you found for $\varepsilon = 0.5$ stable? [Answer: Yes.]

5 Summary and Further Exploration

The global temperatures at equilibrium in the first three models are determined algrebarically on the basis of energy balance, in which the incoming and outgoing power are equal. To account for an energy imbalance, we introduce the ODE in Equation (1). An ODE is a widely-used mathematical tool for modeling how physical quantities change over time. In this module, we solve algebraically for the equilibrium soluton of the ODE and analyze its stability. As our focus is on the modeling process, solving the ODE for $T(t)$ is beyond our scope. The references propose an approximation, in this care the linearization $\varepsilon\sigma T^4 \approx A + BT$, to simplify the ODE.

The models in this module can be refined further. The exercise below explores how melting sea ice would affect albedo (measuring reflectivity) and temperature.

Exercise 5: (a) Suppose that average global temperature was to rise so that sea ice becomes ocean water. Expalin how the following would change (increase or decrease): Earth's average abledo α, equilibrium temperatures found in Models 2-4, the rate dT/dt in Equation (1).
[Answer: Sea ice has higher albedo (α = 0.5 to 0.7) than ocean water (α = 0.06), so as sea ice melts to become ocean water, the Earth's albedo decreases. The temperatures have a factor of $(1-\alpha)^{\frac{1}{4}}$, which would increase. The first term, $Q(1-\alpha)$, in Equation (1) would increase, so dT/dt would also increase.]

The exercise above shows that a better model replace the constant α by a function $\alpha(T)$ that models how the albedo might depend on temperature. The next part examines the greenhouse effect in more detail.

Exercise 5: (b) Speculate on the effect (increase or decrease) of an increase in the amount of greenhouse gases, dust, aerosols in the atmosphere on the following quantities in Model #4: Earth's average abledo α, incoming power density $Q(1-\alpha)$, outgoing power density $\varepsilon\sigma T^4$, dT/dt in Equation (1)?
[Answer: α would decrease as reflectivity is reduced, thus increasing the incoming power density. The greenhouse factor ε, and therefore $\varepsilon\sigma T^4$ would decrease. dT/dt would increase.]

6 References

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